Energy-consistent modeling and simulation of stents in arterial tissues

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from joint works with:

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ANR JCJC Project "MaNStarT" (2025–2029)

MaNStarT: *Mathematical, mechanical, and Numerical modeling of Stents in arterial Tissues*

Members

- **Mikaël BARBOTEU** (PR, Université de Perpignan) : modeling, simulation, domain decomposition techniques
- **Francesco BONALDI** (MCF HDR, PI, Université de Perpignan) : modeling, 3D simulations
- **Serge DUMONT** (PR, Université de Perpignan) : modeling, 3D simulations
- **Mircea SOFONEA** (PR, Université de Perpignan) : theoretical analysis, existence, uniqueness
- **Rawane MANSOUR** (PhD student, since Jan 2025) : adhesion, plasticity, 2D/3D simulations
- **Vo Anh Thuong NGUYEN** (Post-doc, since Feb 2026) : active set method, domain decomposition, HPC

External collaborators

- **Sophie BRUGE** (*R&D Director*, Sim&Cure, Montpellier) – Validation of numerical results
- **Christophe CHNAFA** (*Strategy Officer*, Sim&Cure, Montpellier) – Validation of numerical results
- **Franck JOURDAN** (Professeur, LMGC, Université de Montpellier) – Validation of numerical results

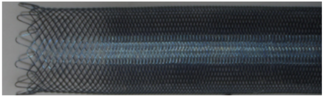
Project “MAPHAS” (2024–2027)

MAPHAS : *Un Modèle de contact avec Adhésion/décollement pour des Problèmes Hyperélastiques en dynamique : Application au déploiement d’un Stent en contact avec un tissu artériel*

PhD thesis of Thach-Hoang NGUYEN

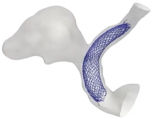
Advisors

- **Mikaël BARBOTEU** (PR, PI, Univ. Perpignan)
- **Francesco BONALDI** (MCF HDR, Univ. Perpignan)
- **Serge DUMONT** (PR, Univ. Perpignan)
- **Franck JOURDAN** (PR, Univ. Montpellier)

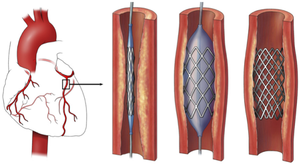


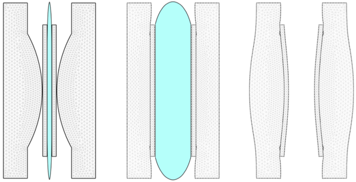
External collaborators

- **Gianmarco BERNAVA** (HUG Genève – experiments)
- **Paolo MACHI** (HUG Genève – experiments)
- **Beatrice BISIGHINI** (CIS Saint Etienne – numerical aspects)
- **Miquel AGUIRRE FONT** (UPC Barcelona – numerical aspects)



Context and goals





Goals

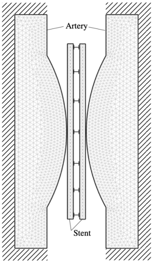
- Devise a **digital twin** of a **stent-artery** system
- Develop numerical schemes consistent with the **energy** of the system

Timeline

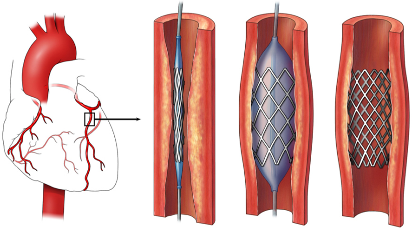
- Mechanical modeling of elastic continua undergoing **large strains**
- Application to **stents** and to **soft biological tissues**
- **Validation** of numerical results (LMGC and Sim&Cure, Montpellier)

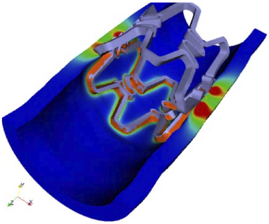
Scientific challenges

- **Strong nonlinearities** (viscoelasticity in large strains, superelasticity)
- **Strong non-smoothness** (frictional contact, adhesion, plasticity)
- **Numerical simulations in 3D** (HPC techniques)



Stents

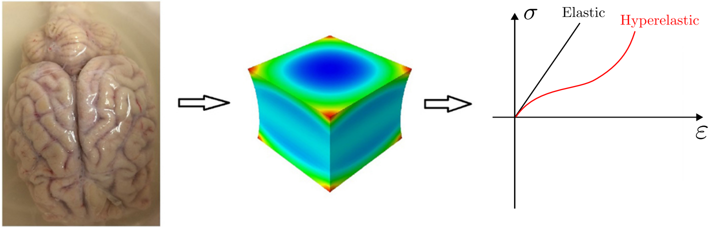




- Must be capable of being squeezed into wire-like structures
- Recover original shape depending on the material they are made of
- ***Hyperelastic*** (stainless steel, silicone) or ***superelastic*** (nitinol)

♣ [Auricchio & Taylor, ’97; Auricchio *et al.*, ’14]

Soft biological tissues



- Mechanical properties determined by collagen and elastin proteins
- Can be stretched up to 15% without damage
- Typically characterized by a *hyper-viscoelastic* behavior

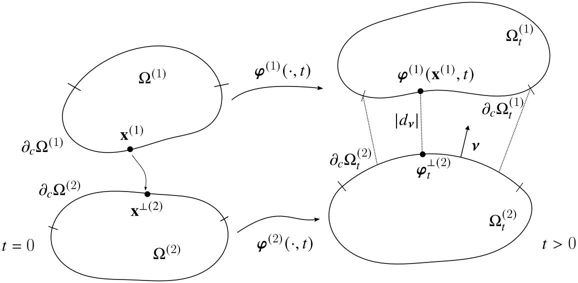
♣ [Pioletti & Rakotomanana, '00; Hu & Desai, '04; Destrade *et al.*, '14]

Outline

1. Model: dynamic hyperelasticity with viscosity and contact
2. Energy-consistent discretization
3. Numerical experiments
   - Benchmark test case
   - Real-life application: stent in an arterial tissue
4. Extension to superelasticity/plasticity
   - Main features of the model
   - Stent–artery simulation with plasticity
5. Conclusions and perspectives

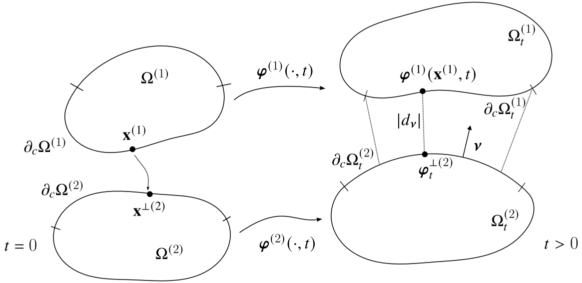
Mechanical model

Mechanical model I



$$\mathbf{x}^{\perp(2)} := \arg\min_{\mathbf{x}^{(2)}\in\partial_c\Omega^{(2)}} \left\|\varphi^{(1)}(\mathbf{x}^{(1)}, t) - \varphi^{(2)}(\mathbf{x}^{(2)}, t)\right\| = \left(\varphi^{(2)}\right)^{-1}(\varphi_t^{\perp(2)}, t)$$

Mechanical model I



$$\mathbf{x}^{\perp(2)} := \arg\min_{\mathbf{x}^{(2)}\in\partial_c\Omega^{(2)}} \left\| \varphi^{(1)}(\mathbf{x}^{(1)},t) - \varphi^{(2)}(\mathbf{x}^{(2)},t) \right\| = \left(\varphi^{(2)}\right)^{-1}(\varphi_t^{\perp(2)},t)$$

Normal distance (gap):

$$d_\nu := \nu \cdot \left(\varphi^{(1)}(\mathbf{x}^{(1)},t) - \varphi^{(2)}(\mathbf{x}^{\perp(2)},t) \right)$$

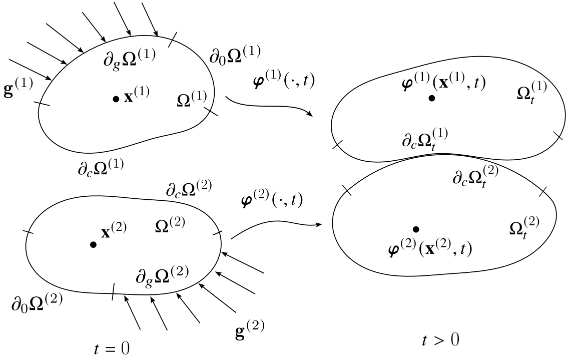
Normal and tangential relative velocities:

$$\mathring{d}_\nu := \nu \cdot \left(\dot{\varphi}^{(1)}(\mathbf{x}^{(1)},t) - \dot{\varphi}^{(2)}(\mathbf{x}^{\perp(2)},t) \right)$$

$$\mathring{\mathbf{d}}_\tau := (\mathbf{I} - \nu \otimes \nu) \left(\dot{\varphi}^{(1)}(\mathbf{x}^{(1)},t) - \dot{\varphi}^{(2)}(\mathbf{x}^{\perp(2)},t) \right)$$

♣ [Curnier & Pietrzak, ’99, ...]

Mechanical model II



Contact law: $\begin{cases} d_\nu \geq 0 \\ \lambda_\nu \leq 0 \\ \lambda_\nu\, d_\nu = 0 \end{cases}$ Friction law: $\begin{cases} \mathring{\mathbf{d}}_{\boldsymbol{\tau}} = \|\mathring{\mathbf{d}}_{\boldsymbol{\tau}}\| \frac{\boldsymbol{\lambda}_{\boldsymbol{\tau}}}{\|\boldsymbol{\lambda}_{\boldsymbol{\tau}}\|} \\ \|\boldsymbol{\lambda}_{\boldsymbol{\tau}}\| + \mu\lambda_\nu \leq 0 \\ \|\mathring{\mathbf{d}}_{\boldsymbol{\tau}}\|(\|\boldsymbol{\lambda}_{\boldsymbol{\tau}}\| + \mu\lambda_\nu) = 0 \end{cases}$

Persistence condition: $\lambda_\nu\, \mathring{d}_\nu = 0$

♣ [Moreau, ’99; Laursen, ’03]

Mechanical model III

Let $\boldsymbol{\varphi}^{(i)}(\mathbf{x},t) = \mathbf{x} + \mathbf{u}^{(i)}(\mathbf{x},t)$ and $\mathbf{F}^{(i)} = \nabla\boldsymbol{\varphi}^{(i)} = \mathbf{I} + \nabla\mathbf{u}^{(i)}$ be the deformation gradient.

Find the displacement field $\mathbf{u} = \left(\mathbf{u}^{(i)}\right)_{i=1,2}$ s.t.

Momentum balance, constitutive law, BC and IC:

$$\begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{Div}\boldsymbol{\Pi}(\mathbf{F},\dot{\mathbf{F}}) = \mathbf{f} & \text{in} \quad \Omega^{(i)} \times (0,T), \\ \boldsymbol{\Pi}(\mathbf{F},\dot{\mathbf{F}}) = \boldsymbol{\Pi}^e(\mathbf{F}) + \boldsymbol{\Pi}^v(\mathbf{F},\dot{\mathbf{F}}) & \text{in} \quad \Omega^{(i)} \times (0,T), \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \partial_0\Omega^{(i)} \times (0,T), \\ \boldsymbol{\Pi}\boldsymbol{\nu} = \mathbf{g} & \text{on} \quad \partial_g\Omega^{(i)} \times (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0,\ \dot{\mathbf{u}}(0) = \mathbf{u}_1 & \text{in} \quad \Omega^{(i)}; \end{cases}$$

Mechanical model III

Let $\boldsymbol{\varphi}^{(i)}(\mathbf{x},t) = \mathbf{x} + \mathbf{u}^{(i)}(\mathbf{x},t)$ and $\mathbf{F}^{(i)} = \nabla\boldsymbol{\varphi}^{(i)} = \mathbf{I} + \nabla\mathbf{u}^{(i)}$ be the deformation gradient.

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Persistent contact:

$$\begin{cases} d_{\boldsymbol{\nu}}^{-} > 0; \quad \lambda_{\boldsymbol{\nu}} = 0 \\ d_{\boldsymbol{\nu}}^{-} = 0; \quad \overset{\circ}{d}_{\boldsymbol{\nu}} \geq 0,\ \lambda_{\boldsymbol{\nu}} \leq 0,\ \lambda_{\boldsymbol{\nu}}\overset{\circ}{d}_{\boldsymbol{\nu}} = 0 \end{cases} \quad \text{on } \partial_c\Omega \times (0,T);$$

Mechanical model III

Let $\boldsymbol{\varphi}^{(i)}(\mathbf{x},t) = \mathbf{x} + \mathbf{u}^{(i)}(\mathbf{x},t)$ and $\mathbf{F}^{(i)} = \nabla\boldsymbol{\varphi}^{(i)} = \mathbf{I} + \nabla\mathbf{u}^{(i)}$ be the deformation gradient.

Find the displacement field $\mathbf{u} = \left(\mathbf{u}^{(i)}\right)_{i=1,2}$ s.t.

Momentum balance, constitutive law, BC and IC:

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Persistent contact:

$$\begin{cases} d_\nu^- > 0; \quad \lambda_\nu = 0 \\ d_\nu^- = 0; \quad \mathring{d}_\nu \geq 0,\ \lambda_\nu \leq 0,\ \lambda_\nu \mathring{d}_\nu = 0 \end{cases} \quad \text{on } \partial_c\Omega \times (0,T);$$

Friction:

$$\begin{cases} \mathring{\mathbf{d}}_\tau = \|\mathring{\mathbf{d}}_\tau\| \frac{\boldsymbol{\lambda}_\tau}{\|\boldsymbol{\lambda}_\tau\|}, \\ \|\boldsymbol{\lambda}_\tau\| + \mu\lambda_\nu \leq 0, \\ \|\mathring{\mathbf{d}}_\tau\|(\|\boldsymbol{\lambda}_\tau\| + \mu\lambda_\nu) = 0 \end{cases} \quad \text{on } \partial_c\Omega \times (0,T);$$

where

- $\boldsymbol{\Pi}(\mathbf{F},\dot{\mathbf{F}})$ = Piola–Kirchhoff stress tensor
- $\boldsymbol{\Pi}^e(\mathbf{F}) = \partial_{\mathbf{F}} W^e(\mathbf{F})$ = elastic stress, $\boldsymbol{\Pi}^v(\mathbf{F},\dot{\mathbf{F}}) = \partial_{\dot{\mathbf{F}}} W^v(\mathbf{F},\dot{\mathbf{F}})$ = viscous stress

Mechanical model IV

Let $\Omega := \Omega^{(1)} \cup \Omega^{(2)}$, $V^{(i)} := \{\mathbf{v}^{(i)} \in H^1(\Omega^{(i)})^d : \mathbf{v}^{(i)} = \mathbf{0} \text{ on } \partial_0\Omega^{(i)}\}$, $V := V^{(1)} \times V^{(2)}$.

For $t \in (0,T)$, find $\mathbf{u} \in V$, $\lambda_\nu \in H^{-1/2}(\partial_c\Omega)$, and $\boldsymbol{\lambda}_\tau \in H^{-1/2}(\partial_c\Omega)^{d-1}$ s.t.

$$\int_\Omega \rho\ddot{\mathbf{u}} \cdot \mathbf{v}\,\mathrm{d}x + \int_\Omega \boldsymbol{\Pi} : \boldsymbol{\nabla}\mathbf{v}\,\mathrm{d}x + \mathrm{P}_{c+f}(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}x + \int_{\partial_g\Omega} \mathbf{g} \cdot \mathbf{v}\,\mathrm{d}a, \quad \forall \mathbf{v} \in V,$$

$$\mathrm{P}_{c+f}(\mathbf{u},\mathbf{v}) := \int_{\partial_c\Omega} (\lambda_\nu\,\partial d_\nu + \boldsymbol{\lambda}_\tau \cdot \partial\mathbf{d}_\tau)\,\mathrm{d}a,$$

with

$$\partial d_\nu := \boldsymbol{\nu} \cdot \left(\mathbf{v}^{(1)} - \mathbf{v}^{\perp(2)}\right),$$

$$\partial\mathbf{d}_\tau := (\mathbf{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu})\left(\mathbf{v}^{(1)} - \mathbf{v}^{\perp(2)}\right)$$

Energy-consistent discretization

Time integration scheme I

Let $\dot{\mathbf{u}}_{i+1} = \frac{2}{\Delta t}(\mathbf{u}_{i+1} - \mathbf{u}_i) - \dot{\mathbf{u}}_i$ and $\mathbf{C}_i = \mathbf{F}_i^\top \mathbf{F}_i$. Find $\mathbf{u}_{i+1} \in V$ s.t., for all $\mathbf{v} \in V$,

$$\frac{1}{\Delta t}\int_\Omega \rho(\dot{\mathbf{u}}_{i+1} - \dot{\mathbf{u}}_i)\cdot \mathbf{v}\,\mathrm{d}x + \int_\Omega \mathbf{\Pi}^e_{\text{algo}} : \nabla \mathbf{v}\;\mathrm{d}x + \int_\Omega \mathbf{\Pi}^v_{i+\frac{1}{2}} : \nabla \mathbf{v}\;\mathrm{d}x$$
$$- \int_\Omega \mathbf{f}_{i+\frac{1}{2}} \cdot \mathbf{v}\;\mathrm{d}x - \int_{\partial_g\Omega} \mathbf{g}_{i+\frac{1}{2}} \cdot \mathbf{v}\;\mathrm{d}a$$
$$+ \int_{\partial_c\Omega} (\lambda_{\boldsymbol{\nu}_{i+\frac{1}{2}}}\, \partial d_{\boldsymbol{\nu}_{i+\frac{1}{2}}} + \boldsymbol{\lambda}_{\boldsymbol{\tau}_{i+\frac{1}{2}}} \cdot \partial \mathbf{d}_{\boldsymbol{\tau}_{i+\frac{1}{2}}})\,\mathrm{d}a = 0$$

Time integration scheme I

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$$\frac{1}{\Delta t}\int_\Omega \rho(\dot{\mathbf{u}}_{i+1} - \dot{\mathbf{u}}_i)\cdot \mathbf{v}\,\mathrm{d}x + \int_\Omega \boldsymbol{\Pi}^e_{\text{algo}} : \nabla \mathbf{v}\ \mathrm{d}x + \int_\Omega \boldsymbol{\Pi}^v_{i+\frac{1}{2}} : \nabla \mathbf{v}\ \mathrm{d}x$$
$$- \int_\Omega \mathbf{f}_{i+\frac{1}{2}} \cdot \mathbf{v}\ \mathrm{d}x - \int_{\partial_g \Omega} \mathbf{g}_{i+\frac{1}{2}} \cdot \mathbf{v}\ \mathrm{d}a$$
$$+ \int_{\partial_c \Omega} (\lambda_{\boldsymbol{\nu}_{i+\frac{1}{2}}} \partial d_{\boldsymbol{\nu}_{i+\frac{1}{2}}} + \boldsymbol{\lambda}_{\boldsymbol{\tau}_{i+\frac{1}{2}}} \cdot \partial \mathbf{d}_{\boldsymbol{\tau}_{i+\frac{1}{2}}})\,\mathrm{d}a = 0$$

Gonzalez discrete elastic stress

$$\begin{cases} \boldsymbol{\Pi}^e_{\text{algo}} = \mathbf{F}_{i+\frac{1}{2}} \boldsymbol{\Sigma}_{\text{algo}} \\ \boldsymbol{\Sigma}_{\text{algo}} = 2\dfrac{\partial W^e}{\partial \mathbf{C}}(\mathbf{C}_{i+\frac{1}{2}}) + 2\left(W^e(\mathbf{C}_{i+1}) - W^e(\mathbf{C}_i) - \dfrac{\partial W^e}{\partial \mathbf{C}}(\mathbf{C}_{i+\frac{1}{2}}) : \Delta \mathbf{C}_i\right) \dfrac{\Delta \mathbf{C}_i}{\Delta \mathbf{C}_i : \Delta \mathbf{C}_i} \end{cases}$$

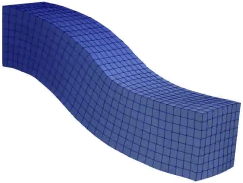
with $\Delta \mathbf{C}_i \coloneqq \mathbf{C}_{i+1} - \mathbf{C}_i$

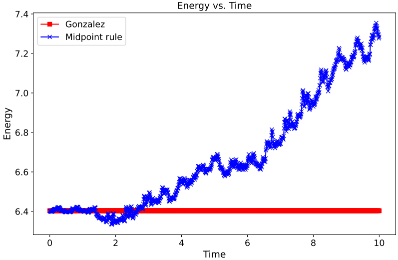
♣ [Gonzalez, '00]

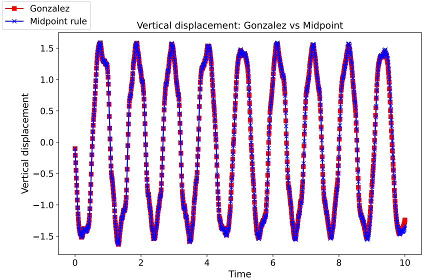
Gonzalez scheme vs Midpoint rule

Mechanical energy at $t = t_i$

$$E_i \coloneqq \frac{1}{2}\int_\Omega \rho \dot{\mathbf{u}}_i \cdot \dot{\mathbf{u}}_i \,\mathrm{d}x + \int_\Omega W^e(\mathbf{C}_i)\,\mathrm{d}x$$







Time integration scheme II

Let $d_{\boldsymbol{\nu}_{i-\frac{1}{2}}}^{\text{Leapfrog}} := d_{\boldsymbol{\nu}_i} + \frac{\Delta t}{2}\frac{d_{\boldsymbol{\nu}_i} - d_{\boldsymbol{\nu}_{i-1}}}{\Delta t}$, and define the discrete contact velocities

$$\delta\overset{\circ}{d}_{\boldsymbol{\nu}_{i+\frac{1}{2}}} := \frac{d_{\boldsymbol{\nu}_{i+1}} - d_{\boldsymbol{\nu}_i}}{\Delta t},$$

$$\delta\overset{\circ}{\mathbf{d}}_{\boldsymbol{\tau}_{i+\frac{1}{2}}} := \frac{\mathbf{d}_{\boldsymbol{\tau}_{i+1}} - \mathbf{d}_{\boldsymbol{\tau}_i}}{\Delta t}$$

Contact & Friction: **Midpoint with Leapfrog procedure**

$$\begin{cases} \text{if } d_{\boldsymbol{\nu}_{i-\frac{1}{2}}}^{\text{Leapfrog}} > 0: & \lambda_{\boldsymbol{\nu}_{i+\frac{1}{2}}} = 0,\ \boldsymbol{\lambda}_{\boldsymbol{\tau}_{i+\frac{1}{2}}} = \mathbf{0}; \\ \text{if } d_{\boldsymbol{\nu}_{i-\frac{1}{2}}}^{\text{Leapfrog}} = 0: & \delta\overset{\circ}{d}_{\boldsymbol{\nu}_{i+\frac{1}{2}}} \geq 0,\ \lambda_{\boldsymbol{\nu}_{i+\frac{1}{2}}} \leq 0,\ \lambda_{\boldsymbol{\nu}_{i+\frac{1}{2}}}\delta\overset{\circ}{d}_{\boldsymbol{\nu}_{i+\frac{1}{2}}} = 0; \\ & \delta\overset{\circ}{\mathbf{d}}_{\boldsymbol{\tau}_{i+\frac{1}{2}}} = \|\delta\overset{\circ}{\mathbf{d}}_{\boldsymbol{\tau}_{i+\frac{1}{2}}}\| \frac{\boldsymbol{\lambda}_{\boldsymbol{\tau}_{i+\frac{1}{2}}}}{\|\boldsymbol{\lambda}_{\boldsymbol{\tau}_{i+\frac{1}{2}}}\|},\ \|\boldsymbol{\lambda}_{\boldsymbol{\tau}_{i+\frac{1}{2}}}\| + \mu\lambda_{\boldsymbol{\nu}_{i+\frac{1}{2}}} \leq 0, \\ & \|\delta\overset{\circ}{\mathbf{d}}_{\boldsymbol{\tau}_{i+\frac{1}{2}}}\|\left(\|\boldsymbol{\lambda}_{\boldsymbol{\tau}_{i+\frac{1}{2}}}\| + \mu\lambda_{\boldsymbol{\nu}_{i+\frac{1}{2}}}\right) = 0 \end{cases}$$

Space-time discretization

- $\mathbb{P}_k$–Finite Elements

$$V^h := \{\mathbf{v}^h \in [C(\overline{\Omega})]^d : \mathbf{v}^h|_K \in \mathbb{P}_k(K)^d \ \forall K \in \mathcal{T}^h,\ \mathbf{v}^h = \mathbf{0} \text{ on } \partial_0\Omega\}$$

- Find $\mathbf{u}^{h\Delta t} = \{\mathbf{u}_i^h\}_{i=0}^n \subset V^h$, $\lambda_\nu^{h\Delta t} = \{\lambda_{\nu\,i}^h\}_{i=0}^n \subset (V^h)^\star$, and $\boldsymbol{\lambda}_{\boldsymbol{\tau}}^{h\Delta t} = \{\boldsymbol{\lambda}_{\boldsymbol{\tau}\,i}^h\}_{i=0}^n \subset (V^h)^\star$ s.t., for all $i = 0, \dots, n$,

$$\frac{1}{\Delta t}\int_\Omega \rho(\mathbf{u}_{i+1}^h - \mathbf{u}_i^h)\cdot\mathbf{v}^h \,\mathrm{d}x + \int_\Omega \mathbf{\Pi}_{i+\frac{1}{2}} : \nabla\mathbf{v}^h$$

$$- \int_\Omega \mathbf{f}_{i+\frac{1}{2}}\cdot\mathbf{v}^h \,\mathrm{d}x - \int_{\partial_g\Omega} \mathbf{g}_{i+\frac{1}{2}}\cdot\mathbf{v}^h \,\mathrm{d}a$$

$$+ \int_{\partial_c\Omega} \left(\lambda_{\nu_{i+\frac{1}{2}}}^h \partial d_{\nu_{i+\frac{1}{2}}}^h + \boldsymbol{\lambda}_{\boldsymbol{\tau}_{i+\frac{1}{2}}}^h \cdot \partial\mathbf{d}_{\boldsymbol{\tau}_{i+\frac{1}{2}}}^h\right) \mathrm{d}a = 0, \quad \forall \mathbf{v}^h \in V^h,$$

with $\mathbf{\Pi}_{i+\frac{1}{2}} = \mathbf{\Pi}_{\text{algo}}^e + \mathbf{\Pi}_{i+\frac{1}{2}}^v$ and frictional contact conditions

Space discretization of persistent contact conditions

Let $p \in \mathcal{S} \subset \partial_c \Omega^h$ be a mesh boundary node. Then:

$$
\begin{aligned}
\text{if } \; d^-_{\boldsymbol{\nu},p} > 0 : & \qquad \lambda_{\boldsymbol{\nu},p} = 0, \; \boldsymbol{\lambda}_{\boldsymbol{\tau},p} = \mathbf{0}; \\
\text{if } \; d^-_{\boldsymbol{\nu},p} = 0 : & \qquad \overset{\circ}{d}_{\boldsymbol{\nu},p} \geq 0, \\
& \qquad \lambda_{\boldsymbol{\nu},p} \leq 0, \\
& \qquad \lambda_{\boldsymbol{\nu},p} \, \overset{\circ}{d}_{\boldsymbol{\nu},p} = 0,
\end{aligned}
$$

along with the discrete friction conditions:

$$
\begin{cases}
\overset{\circ}{\mathbf{d}}_{\boldsymbol{\tau},p} = \|\overset{\circ}{\mathbf{d}}_{\boldsymbol{\tau},p}\| \dfrac{\boldsymbol{\lambda}_{\boldsymbol{\tau},p}}{\|\boldsymbol{\lambda}_{\boldsymbol{\tau},p}\|}, \\
\|\boldsymbol{\lambda}_{\boldsymbol{\tau},p}\| + \mu \lambda_{\boldsymbol{\nu},p} \leq 0, \\
\|\overset{\circ}{\mathbf{d}}_{\boldsymbol{\tau},p}\| (\|\boldsymbol{\lambda}_{\boldsymbol{\tau},p}\| + \mu \lambda_{\boldsymbol{\nu},p}) = 0
\end{cases}
$$

Complementarity functions

Let $c_{\boldsymbol{\nu}}, c_{\boldsymbol{\tau}} > 0$. Persistent contact and friction conditions can be recast as

$$\mathcal{R}^{\lambda}_{\boldsymbol{\nu}}(\mathring{d}_{\boldsymbol{\nu},p}, \lambda_{\boldsymbol{\nu},p}) = -\lambda_{\boldsymbol{\nu},p} - \max(0, -\lambda_{\boldsymbol{\nu},p} - c_{\boldsymbol{\nu}}\, \mathring{d}_{\boldsymbol{\nu},p}) = 0,$$

$$\mathcal{R}^{\lambda}_{\boldsymbol{\tau}}(\mathring{\mathbf{d}}_{\boldsymbol{\tau},p}, \lambda_{\boldsymbol{\nu},p}, \boldsymbol{\lambda}_{\boldsymbol{\tau},p}) = \max(-\mu\lambda_{\boldsymbol{\nu},p}, \|\boldsymbol{\lambda}_{\boldsymbol{\tau},p} + c_{\boldsymbol{\tau}}\, \mathring{\mathbf{d}}_{\boldsymbol{\tau},p}\|)\boldsymbol{\lambda}_{\boldsymbol{\tau},p} + \mu\lambda_{\boldsymbol{\nu},p}(\boldsymbol{\lambda}_{\boldsymbol{\tau},p} + c_{\boldsymbol{\tau}}\, \mathring{\mathbf{d}}_{\boldsymbol{\tau},p}) = \mathbf{0}$$

$\rightsquigarrow$ **Primal–Dual Active Set method**

♣ [Hintermuller *et al.*, ’03, Wohlmuth *et al.*, ’08]

Primal–Dual Active Set method

Let $k \in \mathbb{N}$ be an iteration index. Define

$$\mathcal{A}_{\nu}^{k+1} = \{p \in \mathcal{S} : \lambda_{\nu,p}^{(k)} + c_{\nu} \overset{\circ}{d}{}_{\nu,p}^{(k)} < 0\} = \{\text{contact nodes}\},$$

$$\mathcal{I}_{\nu}^{k+1} = \{p \in \mathcal{S} : \lambda_{\nu,p}^{(k)} + c_{\nu} \overset{\circ}{d}{}_{\nu,p}^{(k)} \geq 0\} = \{\text{gap nodes}\},$$

$$\mathcal{A}_{\tau}^{k+1} = \{p \in \mathcal{A}_{\nu}^{k+1} : \|\boldsymbol{\lambda}_{\tau,p}^{(k)} + c_{\tau} \overset{\circ}{\mathbf{d}}{}_{\tau,p}^{(k)}\| + \mu\lambda_{\nu,p}^{(k)} \geq 0\} = \{\text{sliding nodes}\},$$

$$\mathcal{I}_{\tau}^{k+1} = \{p \in \mathcal{A}_{\nu}^{k+1} : \|\boldsymbol{\lambda}_{\tau,p}^{(k)} + c_{\tau} \overset{\circ}{\mathbf{d}}{}_{\tau,p}^{(k)}\| + \mu\lambda_{\nu,p}^{(k)} < 0\} = \{\text{sticking nodes}\},$$

and solve the following nonlinear system at each time step:

$$\mathcal{R}(\mathbf{u},\boldsymbol{\lambda}) = \begin{pmatrix} \mathcal{R}^{\mathbf{u}}(\mathbf{u},\boldsymbol{\lambda}) = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{A}(\mathbf{u},\dot{\mathbf{u}}) + \boldsymbol{\lambda}_{\nu} + \boldsymbol{\lambda}_{\tau} - \mathbf{f} - \mathbf{g} \\ \\ \mathcal{R}_{\nu}^{\lambda}(\mathbf{u},\boldsymbol{\lambda}) = -\lambda_{\nu,p} - \max(0, -\lambda_{\nu,p} - c_{\nu} \overset{\circ}{d}_{\nu,p}) \\ \\ \mathcal{R}_{\tau}^{\lambda}(\mathbf{u},\boldsymbol{\lambda}) = \max(-\mu\lambda_{\nu,p}, \|\boldsymbol{\lambda}_{\tau,p} + c_{\tau} \overset{\circ}{\mathbf{d}}_{\tau,p}\|)\boldsymbol{\lambda}_{\tau,p} + \mu\lambda_{\nu,p}(\boldsymbol{\lambda}_{\tau,p} + c_{\tau} \overset{\circ}{\mathbf{d}}_{\tau,p}) \end{pmatrix} = \mathbf{0}$$

Numerical experiments

Hyper-viscoelastic ring on rigid a foundation I

Ciarlet–Geymonat hyperelastic energy (compressible material):

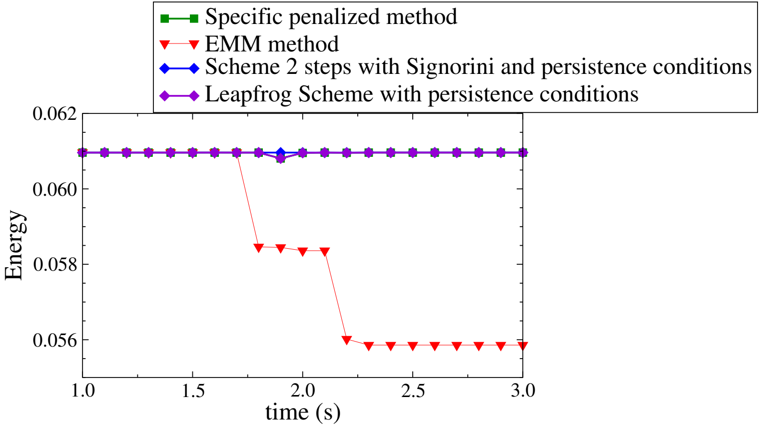
$$W^e(\mathbf{C}) = C_1(I_1 - 3) + C_2(I_2 - 3) + D(I_3 - 1) - (C_1 + 2C_2 + D)\ln I_3,$$

$$\mathbf{C} = \mathbf{F}^\top \mathbf{F}, \qquad I_1 = \mathrm{tr}(\mathbf{C}), \qquad I_2 = \frac{(\mathrm{tr}(\mathbf{C}))^2 - \mathrm{tr}(\mathbf{C}^2)}{2}, \qquad I_3 = \det(\mathbf{C})$$

Curnier's viscous power density:

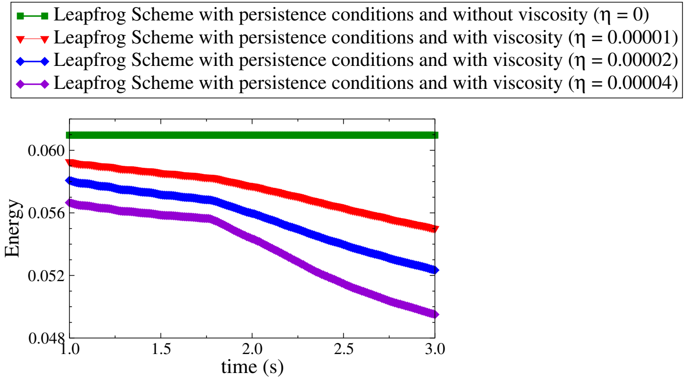
$$W^v(\dot{\mathbf{C}}) = \frac{\eta}{2}\mathrm{tr}(\dot{\mathbf{C}}^2)$$

Hyper-viscoelastic ring on a rigid foundation II



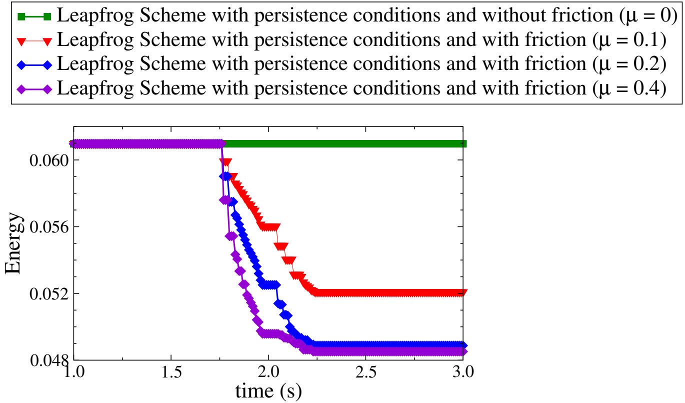
Discrete energy behavior of selected time integration schemes during impact ($\eta = 0$)

Hyper-viscoelastic ring on a rigid foundation III



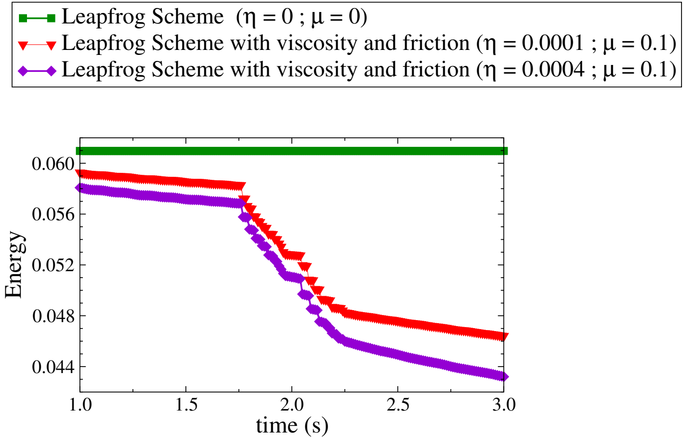
Discrete energy behavior with the PDAS–Leapfrog scheme with frictionless persistence and viscosity (η) conditions

Hyper-viscoelastic ring on a rigid foundation IV



Discrete energy behavior with the PDAS–Leapfrog scheme with persistence and friction (μ) conditions without viscosity

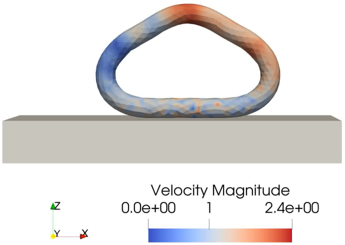
Hyper-viscoelastic ring on a rigid foundation V

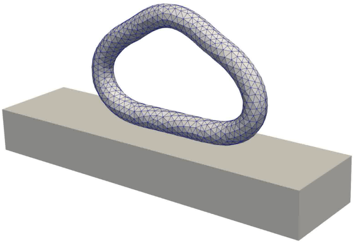


Discrete energy behavior with the PDAS–Leapfrog scheme with persistence, friction (μ) and viscosity (η) conditions

Hyperelastic ring in 3D on a rigid foundation

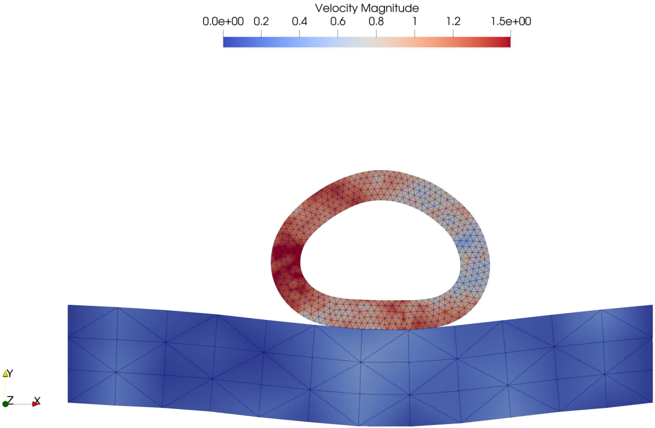
Simulation: Torus 3D





Credits: V.A.T. Nguyen (postdoc at LAMPS)

Hyperelastic ring in 2D on a deformable foundation



Credits: V.A.T. Nguyen (postdoc at LAMPS)

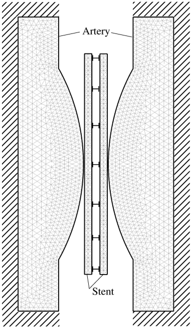
Stent in an arterial tissue I

Ciarlet–Geymonat hyperelastic energy (compressible material):

$$W^e(\mathbf{C}) = C_1(I_1 - 3) + C_2(I_2 - 3) + D(I_3 - 1) - (C_1 + 2C_2 + D)\ln I_3,$$

Curnier's viscous power density:

$$W^v(\dot{\mathbf{C}}) = \frac{\eta}{2}\mathrm{tr}(\dot{\mathbf{C}}^2)$$



Initial stent–artery assembly mesh

$$\rho = 1000\,\mathrm{kg/m^3}, \quad \Delta t = \tfrac{1}{50}\,\mathrm{s},$$

$$C_1^{\mathrm{artery}} = 0.5\,\mathrm{MPa}, \quad C_2^{\mathrm{artery}} = 0.5 \cdot 10^{-2}\,\mathrm{MPa},$$

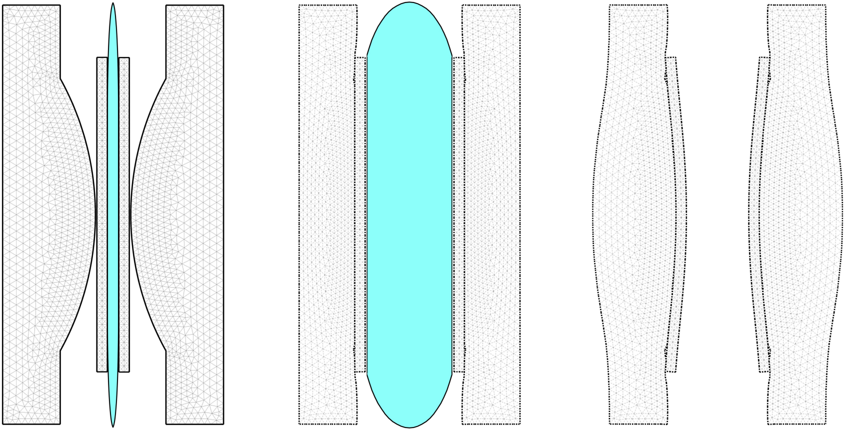
$$C_1^{\mathrm{stent}} = 2 \cdot 10^2\,\mathrm{MPa}, \quad C_2^{\mathrm{stent}} = 2\,\mathrm{MPa},$$

$$D^{\mathrm{artery}} = 0.35\,\mathrm{MPa}, \quad D^{\mathrm{stent}} = 140\,\mathrm{MPa},$$

$$c_{\boldsymbol{\nu}} = 10000\,\mathrm{MN{\cdot}m/s}, \quad c_{\boldsymbol{\tau}} = 100\,\mathrm{MN{\cdot}m/s}$$

Stent in an arterial tissue II

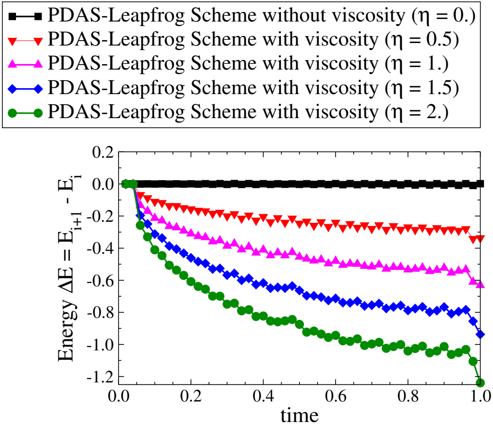
Step 0: Insertion of the balloon   Step 1: Inflation of the balloon   Step 2: Extraction of the balloon



Distorted meshes of the stent–artery assembly

Stent in an arterial tissue III

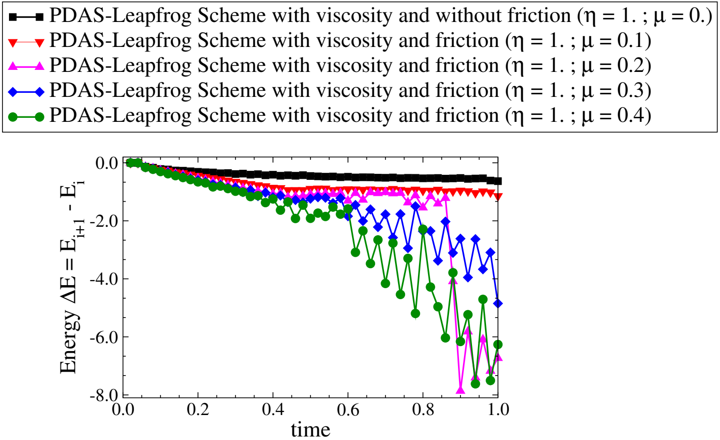
$$E_i := \frac{1}{2} \int_\Omega \rho \dot{\mathbf{u}}_i \cdot \dot{\mathbf{u}}_i \, \mathrm{d}x + \int_\Omega W^e(\mathbf{C}_i) \, \mathrm{d}x$$



Behavior of the discrete energy difference between the times t_{i+1} and t_i for the PDAS–Leapfrog scheme with contact conditions, without friction and with viscosity (η)

Stent in an arterial tissue IV

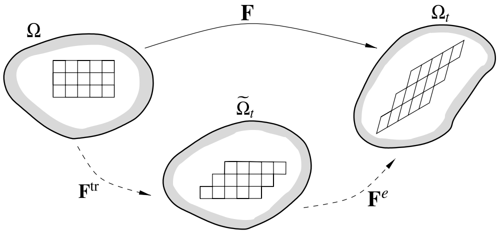
$$E_i := \frac{1}{2}\int_\Omega \rho \dot{\mathbf{u}}_i \cdot \dot{\mathbf{u}}_i \; \mathrm{d}x + \int_\Omega W^e(\mathbf{C}_i)\,\mathrm{d}x$$



Discrete energy difference between the times t_{i+1} and t_i for the PDAS–Leapfrog scheme with conditions of contact, friction (μ), and viscosity (η)

Extension to superelasticity/plasticity

Large-strain kinematics for plasticity



$$\mathbf{F} = \nabla\boldsymbol{\varphi} = \mathbf{I} + \nabla\mathbf{u},$$

$$\boxed{\mathbf{F} = \mathbf{F}^e\mathbf{F}^{\mathrm{tr}}}$$

- $\mathbf{F}^e$ = elastic deformation gradient
- $\mathbf{F}^{\mathrm{tr}}$ = inelastic/plastic deformation gradient

- Inelastic strain spatial velocity gradient:

$$\mathbf{L}^{\mathrm{tr}} = \dot{\mathbf{F}}^{\mathrm{tr}}\mathbf{F}^{\mathrm{tr}-1} = \underbrace{\mathbf{D}^{\mathrm{tr}}}_{\mathrm{sym}\,\mathbf{L}^{\mathrm{tr}}} + \underbrace{\mathbf{W}^{\mathrm{tr}}}_{\mathrm{skw}\,\mathbf{L}^{\mathrm{tr}}}$$

- For isotropic transformations (no kinematic hardening), $\mathbf{W}^{\mathrm{tr}} = \mathbf{0}$

Mandel's tensor

To analyze the evolution of stress in the intermediate configuration $\widetilde{\Omega}_t$, we introduce Mandel's tensor:

$$\bar{\mathbf{M}} = 2\mathbf{C}^e \frac{\partial W^e}{\partial \mathbf{C}^e}(\mathbf{C}^e),$$

satisfying the dissipation condition

$$\bar{\mathbf{M}} : \mathbf{D}^{\mathrm{tr}} \geq 0.$$

It allows to express the transformation condition as

$$f^{\mathrm{tr}}(\bar{\mathbf{M}}) = \sqrt{\bar{\mathbf{M}} : \mathbb{H} : \bar{\mathbf{M}}} - Y^{\mathrm{tr}} = \|\bar{\mathbf{M}}\|_{\mathbb{H}} - Y^{\mathrm{tr}} \leq 0,$$

$$\mathbb{H} := \frac{3}{2}\left(\mathbb{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}\right) = \text{Hill's tensor}$$

♣ [Popp *et al.*, '15]

Plastic flow rule

Let $\bar{\mathbf{M}}_{\text{dev}} = \bar{\mathbf{M}} - \frac{1}{3}(\operatorname{tr}\bar{\mathbf{M}})\mathbf{I}$, then $\|\bar{\mathbf{M}}\|_{\mathbb{H}} = \sqrt{\frac{3}{2}\bar{\mathbf{M}}_{\text{dev}} : \bar{\mathbf{M}}_{\text{dev}}}$.

The plastic flow rule is

$$\mathbf{D}^{\text{tr}} = \dot{\mathbf{F}}^{\text{tr}}\mathbf{F}^{\text{tr}-1} = \dot{\gamma}\frac{\partial f^{\text{tr}}}{\partial \bar{\mathbf{M}}} = \dot{\gamma}\sqrt{\frac{3}{2}}\frac{\bar{\mathbf{M}}_{\text{dev}}}{\|\bar{\mathbf{M}}_{\text{dev}}\|} = \dot{\gamma}\mathbf{n}.$$

Once $\Delta\gamma = \int_{t_n}^{t_{n+1}} \dot{\gamma}\,\mathrm{d}t$ is known, the update rule for $\mathbf{F}^{\text{tr}}$ is

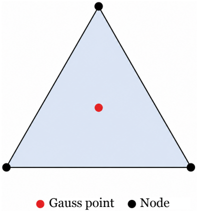
$$\mathbf{F}^{\text{tr}}_{n+1} = \mathbf{F}^{\text{tr}}_{n}\exp(\Delta\gamma\,\mathbf{n})$$

Primal–Dual Active Set method for plasticity I

Idea: Express the Karush–Kuhn–Tucker conditions

$$f^{\mathrm{tr}} \le 0, \quad \dot{\gamma} \ge 0, \quad f^{\mathrm{tr}}\dot{\gamma} = 0$$

as the roots of a suitable *complementarity function*.



Let $q \in \mathcal{G}$ be a Gauss quadrature node, and

$$\bar{\mathbf{M}}_q^{\mathrm{tr}} := \bar{\mathbf{M}}_q + c^{\mathrm{tr}}\mathbb{H}^{-1} : \Delta\mathbf{D}_q^{\mathrm{tr}}, \quad c^{\mathrm{tr}} > 0,$$

with $\Delta\mathbf{D}_q^{\mathrm{tr}} = \frac{3}{2}\Delta\gamma\frac{\bar{\mathbf{M}}_{\mathrm{dev}}}{\|\bar{\mathbf{M}}\|_{\mathbb{H}}}$ and $\dot{\gamma}_{n+1} \approx \frac{1}{\Delta t}(\gamma_{n+1} - \gamma_n) = \frac{\Delta\gamma}{\Delta t}$

Primal–Dual Active Set method for plasticity II

For any Gauss node $q \in \mathcal{G}$, the condition

$$\boxed{C_q^{\text{tr}}(\mathbf{u}, \Delta\mathbf{D}_q^{\text{tr}}) = \max\left(Y^{\text{tr}}, \|\bar{\mathbf{M}}_q^{\text{tr}}\|_{\mathbb{H}}\right)\bar{\mathbf{M}}_q - Y^{\text{tr}}\bar{\mathbf{M}}_q^{\text{tr}} = 0}$$

is equivalent to the following set of discrete KKT conditions:

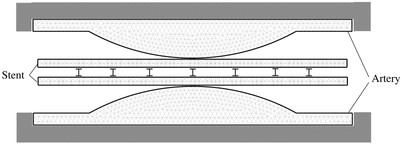
$$f^{\text{tr}} \leq 0, \quad \Delta\gamma \geq 0, \quad f^{\text{tr}}\Delta\gamma = 0, \quad \text{where} \quad \bar{\mathbf{M}}_q = Y_q^{\text{tr}} \frac{\Delta\mathbf{D}_q^{\text{tr}}}{\|\Delta\mathbf{D}_q^{\text{tr}}\|_{\mathbb{H}}}$$

We have $\mathcal{G} = \mathcal{I}_k^{\text{tr}} \cup \mathcal{A}_k^{\text{tr}}$, with

$$\mathcal{I}_k^{\text{tr}} := \left\{q \in \mathcal{G} : \|\bar{\mathbf{M}}_q^{\text{tr},k}\|_{\mathbb{H}} - Y^{\text{tr}} \leq 0\right\},$$

$$\mathcal{A}_k^{\text{tr}} := \left\{q \in \mathcal{G} : \|\bar{\mathbf{M}}_q^{\text{tr},k}\|_{\mathbb{H}} - Y^{\text{tr}} > 0\right\}.$$

Stent–artery simulation with plasticity I



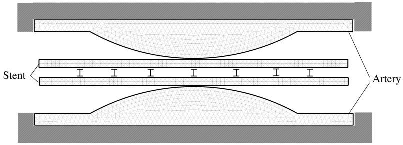
Material model:

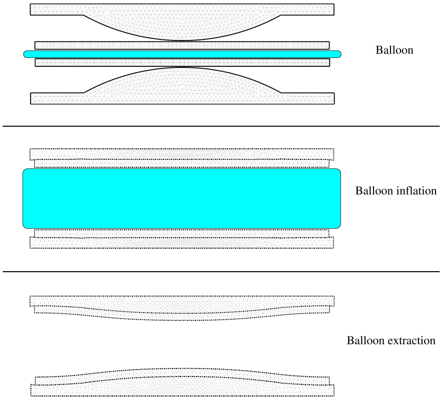
- Hyperelasticity:
  Ciarlet–Geymonat energy
- Viscosity: quadratic potential

$$W^v(\dot{\mathbf{C}}) = \frac{\eta}{2}\,\mathrm{tr}\left(\dot{\mathbf{C}}^2\right)$$

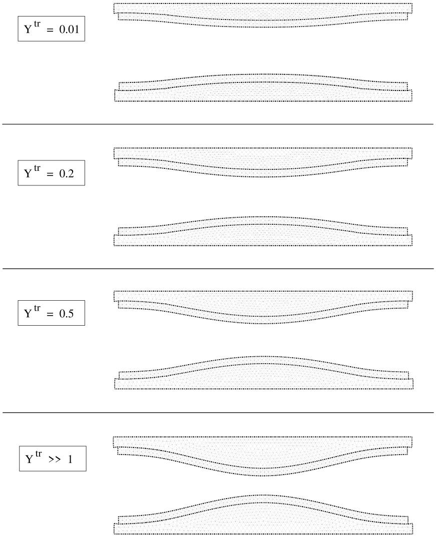
- Inelastic transformation with Y^{tr}

Stent–artery simulation with plasticity II





Stent–artery simulation with plasticity III



Y^{tr} controls transformation activity

- Small $Y^{\mathrm{tr}} \Rightarrow$ strong inelastic effects
- Large $Y^{\mathrm{tr}} \Rightarrow$ quasi-hyperelastic response

Summary and perspectives

Highlights

- Hyper-viscoelastic frictional contact model with persistence condition
- Energy-consistent time integration scheme: **midpoint–leapfrog**
- **Active Set**/Semi-smooth Newton scheme with persistence condition
- Comparison with other methods in terms of energy consistency
- Real-life scenario simulation of a stent–artery system
- Enrichment of the model to account for inelastic behaviors

Perspectives

- Extension to a 3D framework, complex stent–artery scenarios
- Domain decomposition methods & HPC techniques for 3D problems
- Validation by comparison with experiments
- Probability of stent failure, uncertainty quantification

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Merci de votre attention

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