Modules of the affine Temperley-Lieb algebra

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Tours, January 2026

Based on

- YI and A. Morin-Duchesne, SciPost Phys. 12, 030 (2022)
- YI and A. Morin-Duchesne, SciPost Phys. 17, 132 (2024)
- YI and A. Morin-Duchesne, *Fusion in the periodic Temperley-Lieb algebra: general definition of a bifunctor*, arXiv:2509.11756

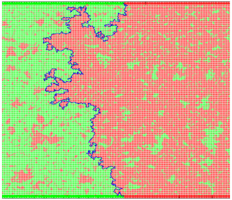
Outline

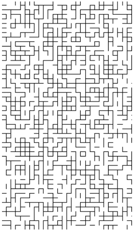
1. Temperley-Lieb algebras in 2d Statistical Mechanics
2. A tensor product for affine TL representations
3. Scaling limit of the dense O(n) model

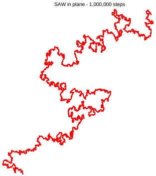
1. Temperley-Lieb algebras in 2d Statistical Mechanics

Critical random curves

- Classical 2d critical system $\rightarrow$ random fractal curves
- Examples: domain walls of a ferromagnet, contours of percolation clusters, configuration of a self-avoiding walk







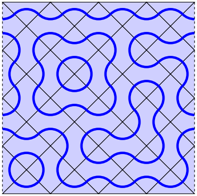
pictures courtesy of Tom Kennedy (University of Arizona)

- At criticality: non-trivial fractal dimensions
  [# disks of radius ε needed to cover the curve : $N_\varepsilon \propto 1/\varepsilon^{d_f}$]

  $d_{\text{Ising-DW}} = 11/8 \quad d_{\text{perco-contour}} = 7/4 \quad d_{\text{SAW}} = 4/3$

The dense O(n) loop model

[Nienhuis 1982]

- Configurations: 

- Partition function: $Z = \sum_{\text{config}} n^{\#\text{ closed loops}}$ $\quad$ $n = -2\cos \pi g$, $\quad \frac{1}{2} < g < 1$

- Thermal average: $\langle X \rangle = \frac{1}{Z} \sum_{\text{config}} n^{\#\text{ closed loops}} X$

- Correlation functions:

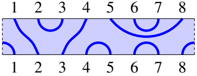
  $\mathbb{P}(r_1, r_2, r_3$ sit on same loop$)$, $\quad \mathbb{P}(r_1, r_2$ separated by j loops$)$, $\quad \dots$

- Scaling limit = CFT $\quad c = 1 - \frac{6(1-g)^2}{g} \quad h_{em} = \frac{(e-mg)^2-(1-g)^2}{4g}$

- Example: fractal dim of a loop $d_f = 2 - 2h_{01} = 1 + \frac{1}{2g}$

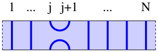
The affine Temperley-Lieb algebra $\mathbf{T}^a(N)$

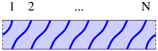
- Parameter= loop weight $\quad n = -q - q^{-1}, \quad q \in \mathbb{C}^{\times}$
- Diagrams of $\mathbf{T}^a(N)$= non-intersecting pairings inside annulus

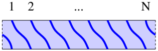
Example: 

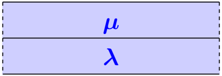
$\in \mathbf{T}^a(8)$

- Generators= $\quad \{e_1, \dots, e_N, \Omega, \Omega^{-1}\}$

$e_j =$ 

$\Omega =$ 

$\Omega^{-1} =$ 

- Composition: $\qquad \lambda \cdot \mu =$ 

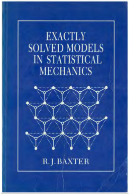
- Relations:

$$e_j^2 = n\, e_j\,, \qquad e_j e_{j\pm1} e_j = e_j\,, \qquad e_i e_j = e_j e_i \quad |i-j| > 1\,,$$

$$\Omega e_j \Omega^{-1} = e_{j-1}\,, \quad \Omega\, \Omega^{-1} = \Omega^{-1}\, \Omega = \mathbf{1}\,, \quad e_{N-1} \dots e_2 e_1 = \Omega^2 e_1$$

Other applications

- Statistical Mechanics in 2d



  - Solid-On-Solid (Andrews-Baxter-Forrester/Pasquier)
  - Ising and Potts models
  - Fortuin-Kasteleyn clusters, percolation
  - dense polymers ...

- Quantum integrability $\check{R}_j(z) = (qz - \frac{1}{qz})\mathbf{1} + (z - \frac{1}{z})e_j$

$$\check{R}_j(z)\check{R}_{j+1}(zw)\check{R}_j(w) = \check{R}_{j+1}(w)\check{R}_j(zw)\check{R}_{j+1}(z)$$

- XXZ spin chain, $\Delta = (q + q^{-1})/2$

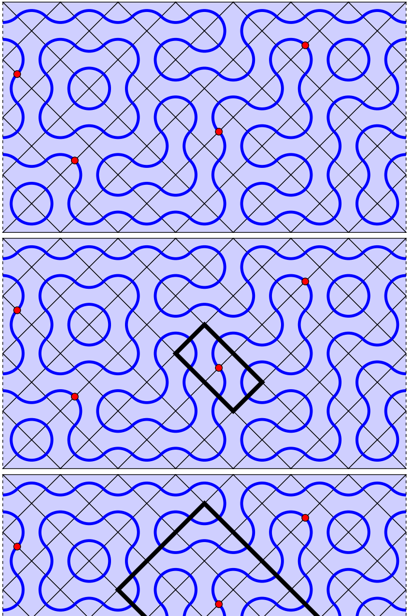
$$H = \sum_{j=1}^{N}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z) = \sum_{j=1}^{N} e_j$$

$$e_j = \mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes(|0\rangle\langle 0|)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}$$

$|0\rangle = \sqrt{-q}|\uparrow\downarrow\rangle + \sqrt{-1/q}|\downarrow\uparrow\rangle$ q-deformed singlet

Connectivity correlations

- Example: $\mathbb{P}(r_1, r_2, r_3, r_4 \text{ sit on same loop})$



Representations of the affine Temperley-Lieb algebra

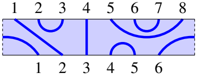
- $\dim \mathbf{T}^a(N) = \infty \quad \Rightarrow$ non-trivial representation theory
- Irreducible reps classified in [Graham-Lehrer '98]
- Fusion (or tensor) product of reps: $\quad (A, B) \mapsto A \otimes B$?
  $\to$ Several proposals: [Rasmussen-Pearce '07] [Gainutdinov-Vasseur '13] [Gainutdinov-Saleur '16] [Gainutdinov-Jacobsen-Saleur '17] [Belletête,Saint-Aubin '18] [YI,Morin-Duchesne '22-25]
- Properties of $\otimes \quad \longrightarrow \quad$ braided monoidal category
- Fusion rules between irreps
- Scaling limit

2. A tensor product for affine TL representations

The affine Temperley-Lieb category

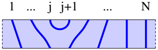
[Graham-Lehrer 1998]

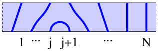
- Generalise $\mathbf{T}^a(N)$ to $\mathbf{T}^a(N, N')$
- Diagrams of $\mathbf{T}^a(N, N')$= non-intersecting pairings inside annulus

Example: 

$\in \mathbf{T}^a(6, 8)$

- Generators= $\{c_0, \ldots, c_{N-1}, c_0^\dagger, \ldots c_{N+1}^\dagger\}$

$c_j =$ 

$c_j^\dagger =$ 

- Composition: $\begin{cases} \mathbf{T}^a(N, N') \times \mathbf{T}^a(N', N'') & \rightarrow & \mathbf{T}^a(N, N'') \\ (\lambda, \mu) & \mapsto & \lambda \cdot \mu \end{cases}$
- Relations: $c_j c_{j+1}^\dagger = \mathbf{1}\,, \quad c_j c_j^\dagger = n\,\mathbf{1}\,, \quad \ldots$

Modules over the affine TL category

[Graham-Lehrer 1998]

- A $\mathbf{T}^a$ module A is a pair of maps [functor]

$$A : \begin{cases} \mathbb{N} & \rightarrow & \mathrm{Vect}(\mathbb{C}) \\ N & \mapsto & A(N) \end{cases} \qquad \begin{cases} \mathbf{T}^a(N, N') & \rightarrow & \mathcal{L}\big(A(N'), A(N)\big) \\ \lambda & \mapsto & A(\lambda) \end{cases}$$

  such that $A(\lambda \cdot \mu) = A(\lambda) \cdot A(\mu)$.

- $A \equiv$ collection of representations $\{A(0), A(1), A(2) \dots\}$, with a linear action $\mathbf{T}^a(N, N') : A(N') \rightarrow A(N)$.

- Example: $\Lambda_{N_1}(N) = \mathbf{T}^a(N, N_1)$ defines a $\mathbf{T}^a$ module.

- A morphism of $\mathbf{T}^a$ modules $\phi : A \rightarrow B$ is a collection of linear maps $\phi_N : A(N) \rightarrow B(N)$ such that

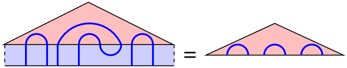
$$\forall \lambda \in \mathbf{T}^a(N, N'),\ u \in A(N'), \qquad \phi_N(\lambda \cdot u) = \lambda \cdot \phi_{N'}(u)\,.$$

  [natural transformation]

Link-state modules

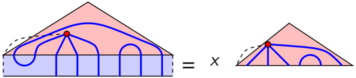
[Graham-Lehrer 1998]

- Vacuum module V: simple link states. Example:

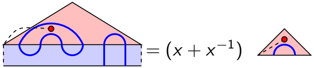


- Standard module $W_{k,x}$: link states with $2k$ defects attached to a marked point ($k \in \mathbb{N}/2$).

$W_{2,x}$:



$W_{0,x}$:



- **Theorem.** Let $m \in k + \mathbb{Z}_{>0}$ and $\varepsilon, \sigma \in \{-1, +1\}$. Then $W_{k,\varepsilon q^{\sigma m}}$ has a submodule isomorphic to $W_{m,\varepsilon q^{\sigma k}}$. Otherwise, $W_{k,x}$ is irreducible. All irreducible $\mathbf{T}^a$ modules are of the form $L_{k,x} = W_{k,x} \big/ R_{k,x}$.
- Example: $k = 0, m = 1 \;\Rightarrow\; V = W_{0,-q} \big/ W_{1,-1}$. If q is not a root of 1, then V is irreducible.

Tensor product of $\mathbf{T}^a$ modules

[YI,Morin-Duchesne '25]

- If A, B are two $\mathbf{T}^a$ modules, then in general $A \otimes_{\mathbb{C}} B$ is not a $\mathbf{T}^a$ module... (≠ Lie algebras)

- Diagrams in annulus: category $\mathbf{T}^a$
- Diagrams in a rectangle: subcategory $\mathbf{T}$
  Horizontal composition
  $\otimes : \mathbf{T}(N_1, N_1') \times \mathbf{T}(N_2, N_2') \to \mathbf{T}(N_1 + N_2, N_1' + N_2')$

$$\boxed{\boldsymbol{\lambda}} \otimes \boxed{\boldsymbol{\mu}} = \boxed{\boldsymbol{\lambda} \quad \boldsymbol{\mu}}$$

- **Definition:** A, B two $\mathbf{T}^a$ modules

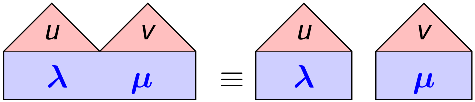
$$(A \otimes B)(N) = \sum_{N_A, N_B} \mathbf{T}^a(N, N_A + N_B) \cdot \left(A(N_A) \otimes_{\mathbb{C}} B(N_B)\right) \Big/ \equiv$$

with

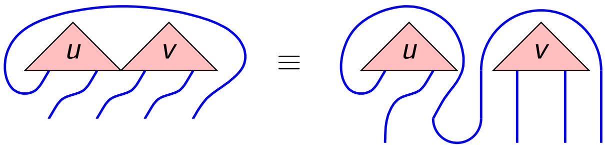
$$\begin{cases} (\lambda \otimes \mu) \cdot (u \otimes v) \equiv \lambda\, u \otimes \mu\, v & \forall\ \lambda \in \mathbf{T}(N_A', N_A), \mu \in \mathbf{T}(N_B', N_B) \\ \Omega \cdot (u \otimes v) \equiv c_{N_A} \cdot (\Omega\, u \otimes c_0^\dagger v) \end{cases}$$

Rules of the tensor product

- $(\lambda \otimes \mu) \cdot (u \otimes v) \equiv \lambda\, u \otimes \mu\, v \quad \forall\, \lambda \in \mathbf{T}(N'_A, N_A),\ \mu \in \mathbf{T}(N'_B, N_B)$



- $\Omega \cdot (u \otimes v) \equiv c_{N_A} \cdot (\Omega\, u \otimes c_0^\dagger v)$



Properties of the tensor product

- If A, B are $\mathbf{T}^a$ modules, then $A \otimes B$ is a $\mathbf{T}^a$ module.
- Respects composition of morphisms. [bifunctor]
  If $\phi : A \to A'$ and $\psi : B \to B'$ are morphisms then the map

$$\phi \otimes \psi : \begin{cases} A \otimes B & \to \quad A' \otimes B' \\ \lambda \cdot (u \otimes v) & \mapsto \quad \lambda \cdot \big(\phi(u) \otimes \psi(v)\big) \end{cases}$$

  is a morphism. Moreover, $\mathbf{1}_A \otimes \mathbf{1}_B = \mathbf{1}_{A \otimes B}$.
- Associative: $\begin{cases} A \otimes (B \otimes C) & \to \quad (A \otimes B) \otimes C \\ u \otimes (v \otimes w) & \mapsto \quad (u \otimes v) \otimes w \end{cases}$
  (proof is non-trivial!)
- Commutative: $\begin{cases} A \otimes B & \to \quad B \otimes A \\ u \otimes v & \mapsto \quad \Omega^{N_B} \cdot (\Omega^{-N_B} v \otimes u) \end{cases}$
- Unit element = Vacuum module V: $\begin{cases} A & \to \quad A \otimes V \\ u & \mapsto \quad u \otimes v_0 \end{cases}$
  v_0 = unique link state of $V(0)$.

Coherence equations

- Structure maps:

$$\alpha_{ABC} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$
$$\ell_A : V \otimes A \to A$$
$$r_A : A \otimes V \to A$$

- Triangle and pentagon equations:

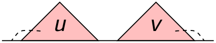
$$\begin{tikzcd} & (A \otimes V) \otimes B \arrow[dr, "r_A \otimes \mathbf{1}_B"] & \\ A \otimes (V \otimes B) \arrow[ur, "\alpha_{AVB}"] \arrow[rr, "\mathbf{1}_A \otimes \ell_B"'] & & A \otimes B \end{tikzcd}$$

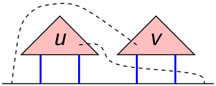
$$\begin{tikzcd} & A \otimes ((B \otimes C) \otimes D) \arrow[dr] & \\ A \otimes (B \otimes (C \otimes D)) \arrow[ur] \arrow[d] & & (A \otimes (B \otimes C)) \otimes D \arrow[d] \\ (A \otimes B) \otimes (C \otimes D) \arrow[rr] & & ((A \otimes B) \otimes C) \otimes D \end{tikzcd}$$

- $\Rightarrow$ [$\mathbf{T}^a$ modules with $\otimes, \alpha, r, \ell$] = monoidal category

Braiding of $\mathbf{T}^a$ modules

- Isomorphisms $\quad \sigma_{AB} : \begin{cases} A \otimes B & \to & B \otimes A \\ u \otimes v & \mapsto & \Omega^{N_B} \cdot (\Omega^{-N_B} v \otimes u) \end{cases}$

$u \otimes v =$ 

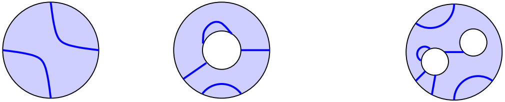
$\sigma(u \otimes v) =$ 

- Hexagon equations
$$\alpha_{CAB} \circ \sigma_{A\otimes B,C} \circ \alpha_{ABC} = (\sigma_{AC} \otimes \mathbf{1}_B) \circ \alpha_{ACB} \circ (\mathbf{1}_A \otimes \sigma_{BC})$$
$$\alpha^{-1}_{BCA} \circ \sigma_{A,B\otimes C} \circ \alpha^{-1}_{ABC} = (\mathbf{1}_B \otimes \sigma_{AC}) \circ \alpha^{-1}_{BAC} \circ (\sigma_{AB} \otimes \mathbf{1}_C)$$
- $\Rightarrow \sigma$ satisfies braid relations (=Yang-Baxter)
- Monodromy: $M_{AB} = \sigma_{BA} \circ \sigma_{AB} = \theta_{A\otimes B} \circ (\theta_A^{-1} \otimes \theta_B^{-1})$
$\theta_{A(N)} = \left[u \mapsto \Omega^N u \right] =$ twist automorphism
- $[\mathbf{T}^a$ modules with $\otimes, \alpha, r, \ell, \sigma] =$ braided monoidal category

Fusion of $\mathbf{T}^a$ diagrams

- We recover the planar algebras [Jones '99]

$$V, \qquad \Lambda_{N_1}(N) = \mathbf{T}^a(N, N_1), \quad \Lambda_{N_1,N_2} = \Lambda_{N_1} \otimes \Lambda_{N_2} \ldots$$



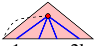
- Fusion of $\mathbf{T}^a$ modules

$$A \otimes B \simeq \sum_{N_A,N_B} \Lambda_{N_A,N_B} \cdot \big(A(N_A) \otimes_{\mathbb{C}} B(N_B)\big)$$

$$A \otimes (B \otimes C) \simeq \sum_{N_A,N_B,N_C} \Lambda_{N_A,N_B,N_C} \cdot \big(A(N_A) \otimes_{\mathbb{C}} B(N_B) \otimes_{\mathbb{C}} C(N_C)\big)$$

$\ldots$

Fusion of standard modules

- Standard modules $W_{k,x}$ (one marked point, $2k$ defects)
  - $W_{k,x}(N) = \mathbf{T}^a(N,2k)\cdot u_k \qquad u_k =$ 

    $\dim\ W_{k,x}(N) = \begin{bmatrix} N \\ \frac{N}{2}-k \end{bmatrix}$ for $N \in 2k+2\mathbb{N}$
  - Generic case: if $x \notin \pm q^{\mathbb{Z}+k}$ then $W_{k,x}$ irreducible [GL '98]
  - Braid transfer matrices ($\neq$ braiding σ): $F, \bar{F} \in$ center of $\mathbf{T}^a(N)$

    In $W_{k,x}$: $\quad F = (q^k x + \frac{1}{q^k x})\mathbf{1} = f_{k,x} \quad \bar{F} = (\frac{q^k}{x} + \frac{x}{q^k})\mathbf{1} = \bar{f}_{k,x}$
- $(W_{k,x} \otimes W_{\ell,y})$ = link diagrams with two marked points
- $\dim\ (W_{k,x} \otimes W_{\ell,y})(N) = \infty \qquad [\Omega^p \cdot (u_k \otimes u_\ell), p \in \mathbb{Z}]$
- If $z \notin \pm q^{\mathbb{Z}+m}$ then $(W_{k,x} \otimes W_{\ell,y})\Big/_{\{(F,\bar{F})\equiv(f_{m,z},\bar{f}_{m,z})\}} \simeq W_{m,z}$
- Surjective morphism $\left\{ \begin{array}{ccc} W_{k,x} \otimes W_{\ell,y} & \to & W_{m,z} \\ u \otimes v & \mapsto & \sum_j w_j \langle w_j, u \otimes v \rangle \end{array} \right.$

  $\{w_j\}$ = o.n. basis of $W_{m,z}(N)$

  $\langle\, ,\, \rangle$ = generalised Gram form

Fusion with quotient modules

when q is not a root of unity

- Quotient modules: $Q_{\ell,-q^m} = W_{\ell,-q^m} \big/ W_{m,-q^\ell} \quad m \in \ell + \mathbb{Z}_{>0}$
- Fusion rules

$$W_{k,x} \otimes Q_{\ell,-q^m} \simeq \bigoplus_{i=-\frac{m+\ell-1}{2}}^{\frac{m+\ell-1}{2}} \bigoplus_{j=-\frac{m-\ell-1}{2}}^{\frac{m-\ell-1}{2}} W_{k+i+j,\, x\, q^{i-j}}$$

$$Q_{k,-q^n} \otimes Q_{\ell,-q^m} \simeq \bigoplus_{i=\frac{|\mu-\nu|+1}{2}}^{\frac{\mu+\nu-1}{2}} \bigoplus_{j=\frac{|\bar{\mu}-\bar{\nu}|+1}{2}}^{\frac{\bar{\mu}+\bar{\nu}-1}{2}} Q_{i-j,-q^{i+j}} \qquad \begin{cases} \mu=m+\ell,\ \nu=n+k \\ \bar{\mu}=m-\ell,\ \bar{\nu}=n-k \end{cases}$$

- Examples
  - $W_{k,x} \otimes Q_{0,-q} \simeq W_{k,x} \qquad [Q_{0,-q} = V]$
  - $W_{k,x} \otimes Q_{0,-q^2} \simeq W_{k,xq} \oplus W_{k,x/q} \oplus W_{k+1,x} \oplus W_{k-1,x}$
  - In $W_{k,x} \otimes Q_{0,-q^2}$: $\begin{cases} F^2 + n\, f_{k,x}\, F + f_{k,x}^2 + (q-q^{-1})^2 \mathbf{1} = 0 \\ \bar{F}^2 + n\, \bar{f}_{k,x}\, \bar{F} + \bar{f}_{k,x}^2 + (q-q^{-1})^2 \mathbf{1} = 0 \end{cases}$

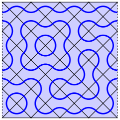
3. Scaling limit of the dense $O(n)$ model

Scaling limit

- Dense O(n) loop model

  $n = -2\cos\pi g \qquad g \notin \mathbb{Q}$

  $q = \exp(-i\pi g)$



- Scaling limit = CFT $\quad c = 1 - \frac{6(1-g)^2}{g} \quad h_{rs} = \frac{(r-sg)^2-(1-g)^2}{4g}$
- Vir. characters: $\chi(h) = \frac{e^{2i\pi\tau(h-\frac{c}{24})}}{\eta(\tau)}, \quad \chi_{rs} = \chi(h_{rs}) - \chi(h_{r,-s})$
- Lattice characters: $\chi_N(A) = \mathrm{Tr}_{A(N)}\left(\Omega^{\lfloor N\,\mathrm{Re}\,\tau\rfloor}\, T^{\lfloor N\,\mathrm{Im}\,\tau\rfloor}\right)$
  - Std modules: $\chi_N(W_{k,\exp(i\pi\alpha)}) \to \sum_{p\in\mathbb{Z}} \chi(h_{\alpha+p,k})\,\bar{\chi}(h_{\alpha+p,-k})$
  - Quotient modules:

$$\chi_N(Q_{\ell,-q^m}) = \chi_N(W_{\ell,-q^m}) - \chi_N(W_{m,-q^\ell}) \to \sum_{p=1}^{\infty} \chi_{p,m+\ell}\,\bar{\chi}_{p,m-\ell}$$

Fusion rules

- Scaling limits

$$W_{k,e^{i\pi\alpha}} \to [h_{\alpha,k}]\,\overline{[h_{\alpha,-k}]} + \dots \qquad W_{k+i+j,e^{i\pi\alpha}q^{i-j}} \to [h_{\alpha,k+2i}]\,\overline{[h_{\alpha,-k-2j}]} + \dots$$

$$Q_{\ell,-q^m} \to \phi_{1,m+\ell}\,\bar{\phi}_{1,m-\ell} + \dots \qquad Q_{i-j,-q^{i+j}} \to \phi_{1,2i}\,\bar{\phi}_{1,2j} + \dots$$

- Virasoro fusion rules:

$$[h_{\alpha,k}]\otimes\phi_{1,\mu} \simeq \bigoplus_{i=-\frac{\mu-1}{2}}^{\frac{\mu-1}{2}} [h_{\alpha,k+2i}] \qquad \phi_{1,\nu}\otimes\phi_{1,\mu} \simeq \bigoplus_{i=\frac{|\mu-\nu|+1}{2}}^{\frac{\mu+\nu-1}{2}} \phi_{1,2i}$$

- $\mathbf{T}^a$ fusion rules

$$W_{k,x}\otimes Q_{\ell,-q^m} \simeq \bigoplus_{i=-\frac{m+\ell-1}{2}}^{\frac{m+\ell-1}{2}} \bigoplus_{j=-\frac{m-\ell-1}{2}}^{\frac{m-\ell-1}{2}} W_{k+i+j,\,x\,q^{i-j}}$$

$$Q_{k,-q^n}\otimes Q_{\ell,-q^m} \simeq \bigoplus_{i=\frac{|\mu-\nu|+1}{2}}^{\frac{\mu+\nu-1}{2}} \bigoplus_{j=\frac{|\bar{\mu}-\bar{\nu}|+1}{2}}^{\frac{\bar{\mu}+\bar{\nu}-1}{2}} Q_{i-j,-q^{i+j}} \qquad \begin{cases} \mu=m+\ell, & \nu=n+k \\ \bar{\mu}=m-\ell, & \bar{\nu}=n-k \end{cases}$$

Conclusion

- Summary
  - Constructed $\otimes$ and structure maps $\Rightarrow$ ($\mathbf{T}^a$ modules)=braided monoidal category, similar to fusion categories of CFT
  - Computed $W \otimes W$, $W \otimes Q$, $Q \otimes Q$
  - In scaling limit, $\mathbf{T}^a$ fusion rules $\to$ Virasoro fusion rules
- Open questions
  - Lattice analogs of conformal blocks ? 6j-symbols, etc ?
  - Bootstrap of lattice correlation functions ?
  - Modular structure ?
  - Case when q is a root of 1 ?

Thank you!