Some particular direct-sum decompositions and direct-product decompositions

Alberto Facchini
University of Padova, Italy

Lille, 25 June 2014

Rings and their Jacobson ideal

The rings R in this talk are rings with an identity 1, not necessarily commutative.

Rings and their Jacobson ideal

The rings R in this talk are rings with an identity 1, not necessarily commutative.

The *Jacobson radical* of R is the intersection of all maximal right ideals of R

Rings and their Jacobson ideal

The rings R in this talk are rings with an identity 1, not necessarily commutative.

The *Jacobson radical* of R is the intersection of all maximal right ideals of R ($=$ the intersection of all maximal left ideals of R.)

Local Rings

Proposition

The following conditions are equivalent for a ring R:
(i) *The ring R has a unique maximal right ideal.*
(ii) *The Jacobson radical $J(R)$ is a maximal right ideal.*
(iii) *The sum of two elements of R that are not right invertible is not right invertible.*
(iv) $J(R) = \{ r \in R \mid rR \neq R \}$.
(v) $R/J(R)$ *is a division ring.*
(vi) $J(R) = \{ r \in R \mid r \text{ is not invertible in } R \}$.
(vii) *The sum of two non-invertible elements of R is non-invertible.*
(viii) *For every $r \in R$, either r is invertible or $1 - r$ is invertible.*

Local Rings

The rings that satisfy the equivalent conditions of the previous proposition are called *local rings*.

Local Rings

The rings that satisfy the equivalent conditions of the previous proposition are called *local rings*.

For instance:
(i) Division rings are local rings.

Local Rings

The rings that satisfy the equivalent conditions of the previous proposition are called *local rings*.

For instance:
(i) Division rings are local rings.
(ii) If the endomorphism ring $\mathrm{End}(M_R)$ of a module M_R is local, then M_R is an indecomposable module.

Local Rings

The rings that satisfy the equivalent conditions of the previous proposition are called *local rings*.

For instance:
(i) Division rings are local rings.
(ii) If the endomorphism ring $\mathrm{End}(M_R)$ of a module M_R is local, then M_R is an indecomposable module.
(iii) The endomorphism ring $\mathrm{End}(E_R)$ of an indecomposable injective module E_R is local.

Local Rings

The rings that satisfy the equivalent conditions of the previous proposition are called *local rings*.

For instance:
(i) Division rings are local rings.
(ii) If the endomorphism ring $\mathrm{End}(M_R)$ of a module M_R is local, then M_R is an indecomposable module.
(iii) The endomorphism ring $\mathrm{End}(E_R)$ of an indecomposable injective module E_R is local.
(iv) The endomorphism ring $\mathrm{End}(M_R)$ of an indecomposable module M_R of finite composition length is local.

Krull-Schmidt-Azumaya Theorem, 1950

Theorem

Let M be a module that is a direct sum of modules with local endomorphism rings. Then M is a direct sum of indecomposable modules in an essentially unique way in the following sense. If

$$M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j,$$

where all the M_i's ($i \in I$) and all the N_j's ($j \in J$) are indecomposable modules, then there exists a bijection $\varphi \colon I \to J$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I$.

In general, there is not uniqueness

In general, there is not uniqueness

R a ring with identity

In general, there is not uniqueness

R a ring with identity

M_R be a unital right R-module.

In general, there is not uniqueness

R a ring with identity

M_R be a unital right R-module.

Our aim: describe direct-sum decompositions of M_R as a direct sum $M_R = M_1 \oplus \cdots \oplus M_n$ of finitely many direct summands.

In general, there is not uniqueness

The best algebraic way to describe direct-sum decompositions of a module M_R is making use of commutative monoids

In general, there is not uniqueness

The best algebraic way to describe direct-sum decompositions of a module M_R is making use of commutative monoids (semigroups with a binary operation that is associative, commutative and has an identity element).

In general, there is not uniqueness

The best algebraic way to describe direct-sum decompositions of a module M_R is making use of commutative monoids (semigroups with a binary operation that is associative, commutative and has an identity element).

In this talk, all monoids S will be commutative and additive.

In general, there is not uniqueness

The best algebraic way to describe direct-sum decompositions of a module M_R is making use of commutative monoids (semigroups with a binary operation that is associative, commutative and has an identity element).

In this talk, all monoids S will be commutative and additive.

A monoid S is *reduced* if $s, t \in S$ and $s + t = 0$ implies $s = t = 0$.

The reduced monoid $V(\mathcal{C})$

Classes of right R-modules will be seen as full subcategories of the category Mod-R of all right R-modules.

The reduced monoid $V(\mathcal{C})$

Classes of right R-modules will be seen as full subcategories of the category $\mathrm{Mod}\text{-}R$ of all right R-modules.

Let $\mathcal{C}$ be a category

The reduced monoid $V(\mathcal{C})$

Classes of right R-modules will be seen as full subcategories of the category $\mathrm{Mod}\text{-}R$ of all right R-modules.

Let $\mathcal{C}$ be a category and $V(\mathcal{C})$ denote a *skeleton* of $\mathcal{C}$, that is, a class of representatives of the objects of $\mathcal{C}$ modulo isomorphism.

The reduced monoid $V(\mathcal{C})$

Classes of right R-modules will be seen as full subcategories of the category Mod-R of all right R-modules.

Let $\mathcal{C}$ be a category and $V(\mathcal{C})$ denote a *skeleton* of $\mathcal{C}$, that is, a class of representatives of the objects of $\mathcal{C}$ modulo isomorphism. For every object A in $\mathcal{C}$, there is a unique object $\langle A \rangle$ in $V(\mathcal{C})$ isomorphic to A.

The reduced monoid $V(\mathcal{C})$

Classes of right R-modules will be seen as full subcategories of the category $\mathrm{Mod}\text{-}R$ of all right R-modules.

Let $\mathcal{C}$ be a category and $V(\mathcal{C})$ denote a *skeleton* of $\mathcal{C}$, that is, a class of representatives of the objects of $\mathcal{C}$ modulo isomorphism. For every object A in $\mathcal{C}$, there is a unique object $\langle A \rangle$ in $V(\mathcal{C})$ isomorphic to A. Thus there is a mapping $\mathrm{Ob}(\mathcal{C}) \to V(\mathcal{C})$, $A \mapsto \langle A \rangle$, that associates to every object A of $\mathcal{C}$ the unique object $\langle A \rangle$ in $V(\mathcal{C})$ isomorphic to A.

The reduced monoid $V(\mathcal{C})$

Classes of right R-modules will be seen as full subcategories of the category $\mathrm{Mod}\text{-}R$ of all right R-modules.

Let $\mathcal{C}$ be a category and $V(\mathcal{C})$ denote a *skeleton* of $\mathcal{C}$, that is, a class of representatives of the objects of $\mathcal{C}$ modulo isomorphism. For every object A in $\mathcal{C}$, there is a unique object $\langle A\rangle$ in $V(\mathcal{C})$ isomorphic to A. Thus there is a mapping $\mathrm{Ob}(\mathcal{C}) \to V(\mathcal{C})$, $A \mapsto \langle A\rangle$, that associates to every object A of $\mathcal{C}$ the unique object $\langle A\rangle$ in $V(\mathcal{C})$ isomorphic to A.
Assume that a product $A \times B$ exists in $\mathcal{C}$ for every pair A, B of objects of $\mathcal{C}$.

The reduced monoid $V(\mathcal{C})$

Classes of right R-modules will be seen as full subcategories of the category $\mathrm{Mod}\text{-}R$ of all right R-modules.

Let $\mathcal{C}$ be a category and $V(\mathcal{C})$ denote a *skeleton* of $\mathcal{C}$, that is, a class of representatives of the objects of $\mathcal{C}$ modulo isomorphism. For every object A in $\mathcal{C}$, there is a unique object $\langle A\rangle$ in $V(\mathcal{C})$ isomorphic to A. Thus there is a mapping $\mathrm{Ob}(\mathcal{C}) \to V(\mathcal{C})$, $A \mapsto \langle A\rangle$, that associates to every object A of $\mathcal{C}$ the unique object $\langle A\rangle$ in $V(\mathcal{C})$ isomorphic to A.
Assume that a product $A \times B$ exists in $\mathcal{C}$ for every pair A, B of objects of $\mathcal{C}$. Define an addition $+$ in $V(\mathcal{C})$ by $A + B := \langle A \times B\rangle$ for every $A, B \in V(\mathcal{C})$.

Lemma

Let $\mathcal{C}$ be a category with a terminal object and in which a product $A \times B$ exists for every pair A, B of objects of $\mathcal{C}$. Then $V(\mathcal{C})$ is a large reduced commutative monoid.

Bergman and Dicks, 1974–1978

Theorem

Let k be a field and let M be a commutative reduced monoid. Then there exists a class $\mathcal{C}$ of finitely generated projective right modules over a right and left hereditary k-algebra R such that $M \cong V(\mathcal{C})$.

Uniserial modules

A module U_R is *uniserial* if the lattice $\mathcal{L}(U_R)$ of its submodules is linearly ordered under inclusion.

Uniserial modules

A module U_R is *uniserial* if the lattice $\mathcal{L}(U_R)$ of its submodules is linearly ordered under inclusion.

The endomorphism ring of a uniserial module has at most two maximal right (left) ideals:

Uniserial modules and their endomorphism rings

Theorem

[F., T.A.M.S. 1996] *Let* U_R *be a uniserial module over a ring* R*,*

Uniserial modules and their endomorphism rings

Theorem

[F., T.A.M.S. 1996] *Let* U_R *be a uniserial module over a ring* R*,*
$E := \mathrm{End}(U_R)$ *its endomorphism ring,*

Uniserial modules and their endomorphism rings

Theorem

[F., T.A.M.S. 1996] *Let* U_R *be a uniserial module over a ring* R*,* $E := \mathrm{End}(U_R)$ *its endomorphism ring,* $I := \{ f \in E \mid f$ *is not injective* $\}$

Uniserial modules and their endomorphism rings

Theorem

[F., T.A.M.S. 1996] *Let* U_R *be a uniserial module over a ring* R*,* $E := \mathrm{End}(U_R)$ *its endomorphism ring,* $I := \{\, f \in E \mid f \text{ is not injective} \,\}$ *and* $K := \{\, f \in E \mid f \text{ is not surjective} \,\}$*.*

Uniserial modules and their endomorphism rings

Theorem

[F., T.A.M.S. 1996] *Let* U_R *be a uniserial module over a ring* R*,* $E := \mathrm{End}(U_R)$ *its endomorphism ring,* $I := \{ f \in E \mid f \text{ is not injective} \}$ *and* $K := \{ f \in E \mid f \text{ is not surjective} \}$*. Then* I *and* K *are two two-sided completely prime ideals of* E*,*

Uniserial modules and their endomorphism rings

Theorem

[F., T.A.M.S. 1996] *Let U_R be a uniserial module over a ring R, $E := \mathrm{End}(U_R)$ its endomorphism ring, $I := \{ f \in E \mid f$ is not injective$\}$ and $K := \{ f \in E \mid f$ is not surjective$\}$. Then I and K are two two-sided completely prime ideals of E, and every proper right ideal of E and every proper left ideal of E is contained either in I or in K.*

Uniserial modules and their endomorphism rings

Theorem

[F., T.A.M.S. 1996] *Let* U_R *be a uniserial module over a ring* R*,* $E := \mathrm{End}(U_R)$ *its endomorphism ring,* $I := \{ f \in E \mid f$ *is not injective*$\}$ *and* $K := \{ f \in E \mid f$ *is not surjective*$\}$*. Then* I *and* K *are two two-sided completely prime ideals of* E*, and every proper right ideal of* E *and every proper left ideal of* E *is contained either in* I *or in* K*. Moreover,*
(a) *either* E *is a local ring with maximal ideal* $I \cup K$*, or*

Uniserial modules and their endomorphism rings

Theorem

[F., T.A.M.S. 1996] *Let U_R be a uniserial module over a ring R, $E := \mathrm{End}(U_R)$ its endomorphism ring, $I := \{ f \in E \mid f$ is not injective$\}$ and $K := \{ f \in E \mid f$ is not surjective$\}$. Then I and K are two two-sided completely prime ideals of E, and every proper right ideal of E and every proper left ideal of E is contained either in I or in K. Moreover,*

(a) *either E is a local ring with maximal ideal $I \cup K$, or*

(b) *E/I and E/K are division rings, and $E/J(E) \cong E/I \times E/K$.*

Monogeny class, epigeny class

Two modules U and V are said to have

1. *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$;

Monogeny class, epigeny class

Two modules U and V are said to have

1. *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$;
2. *the same epigeny class*, denoted $[U]_e = [V]_e$, if there exist an epimorphism $U \to V$ and an epimorphism $V \to U$.

Monogeny class, epigeny class

Two modules U and V are said to have

1. *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$;
2. *the same epigeny class*, denoted $[U]_e = [V]_e$, if there exist an epimorphism $U \to V$ and an epimorphism $V \to U$.

For instance, two injective modules have the same monogeny class if and only if they are isomorphic (Bumby's Theorem).

Weak Krull-Schmidt Theorem

Theorem

[F., T.A.M.S. 1996] *Let* $U_1, \ldots, U_n, V_1, \ldots, V_t$ *be* $n+t$ *non-zero uniserial right modules over a ring* R*. Then the direct sums* $U_1 \oplus \cdots \oplus U_n$ *and* $V_1 \oplus \cdots \oplus V_t$ *are isomorphic* R*-modules if and only if* $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \ldots, n\}$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i = 1, 2, \ldots, n$*.*

Cyclically presented modules over local rings

The behavior of uniserial modules is enjoyed by other classes of modules.

Cyclically presented modules over local rings

The behavior of uniserial modules is enjoyed by other classes of modules.

First example [B. Amini, A. Amini and A. Facchini, J. Algebra 2008].

Cyclically presented modules over local rings

The behavior of uniserial modules is enjoyed by other classes of modules.

First example [B. Amini, A. Amini and A. Facchini, J. Algebra 2008].

A right module over a ring R is *cyclically presented* if it is isomorphic to R/aR for some element $a \in R$.

Cyclically presented modules over local rings

The behavior of uniserial modules is enjoyed by other classes of modules.

First example [B. Amini, A. Amini and A. Facchini, J. Algebra 2008].

A right module over a ring R is *cyclically presented* if it is isomorphic to R/aR for some element $a \in R$. For any ring R, we will denote with $U(R)$ the group of all invertible elements of R.

Cyclically presented modules over local rings

If R/aR and R/bR are cyclically presented modules over a local ring R, we say that R/aR and R/bR *have the same lower part*, and write $[R/aR]_l = [R/bR]_l$, if there exist $u, v \in U(R)$ and $r, s \in R$ with $au = rb$ and $bv = sa$.

Cyclically presented modules over local rings

If R/aR and R/bR are cyclically presented modules over a local ring R, we say that R/aR and R/bR *have the same lower part*, and write $[R/aR]_l = [R/bR]_l$, if there exist $u, v \in U(R)$ and $r, s \in R$ with $au = rb$ and $bv = sa$.

(Two cyclically presented modules over a local ring have the same lower part if and only if their Auslander-Bridger transposes have the same epigeny class.)

Cyclically presented modules and idealizer

The endomorphism ring $\mathrm{End}_R(R/aR)$ of a non-zero cyclically presented module R/aR is isomorphic to E/aR, where $E := \{\, r \in R \mid ra \in aR \,\}$ is the *idealizer* of aR.

Cyclically presented modules over local rings

$E := \{ r \in R \mid ra \in aR \}$ is the *idealizer* of aR.

Cyclically presented modules over local rings

$E := \{ r \in R \mid ra \in aR \}$ is the *idealizer* of aR.

Theorem

Let a be a non-zero non-invertible element of an arbitrary local ring R, let E be the idealizer of aR, and let E/aR be the endomorphism ring of the cyclically presented right R-module R/aR.

Cyclically presented modules over local rings

$E := \{ r \in R \mid ra \in aR \}$ is the *idealizer* of aR.

Theorem

Let a be a non-zero non-invertible element of an arbitrary local ring R, let E be the idealizer of aR, and let E/aR be the endomorphism ring of the cyclically presented right R-module R/aR. Set $I := \{ r \in R \mid ra \in aJ(R) \}$ and $K := J(R) \cap E$. Then I and K are two two-sided completely prime ideals of E containing aR, the union $(I/aR) \cup (K/aR)$ is the set of all non-invertible elements of E/aR, and every proper right ideal of E/aR and every proper left ideal of E/aR is contained either in I/aR or in K/aR.

Cyclically presented modules over local rings

$E := \{ r \in R \mid ra \in aR \}$ is the *idealizer* of aR.

Theorem

Let a be a non-zero non-invertible element of an arbitrary local ring R, let E be the idealizer of aR, and let E/aR be the endomorphism ring of the cyclically presented right R-module R/aR. Set $I := \{ r \in R \mid ra \in aJ(R) \}$ and $K := J(R) \cap E$. Then I and K are two two-sided completely prime ideals of E containing aR, the union $(I/aR) \cup (K/aR)$ is the set of all non-invertible elements of E/aR, and every proper right ideal of E/aR and every proper left ideal of E/aR is contained either in I/aR or in K/aR. Moreover, exactly one of the following two conditions holds:
(a) *Either I and K are comparable (that is, $I \subseteq K$ or $K \subseteq I$), in which case E/aR is a local ring, or*
(b) *I and K are not comparable, and in this case E/I and E/K are division rings, $J(E/aR) = (I \cap K)/aR$, and $(E/aR)/J(E/aR)$ is canonically isomorphic to the direct product $E/I \times E/K$.*

Weak Krull-Schmidt Theorem for cyclically presented modules over local rings

Theorem

(Weak Krull-Schmidt Theorem) *Let $a_1, \ldots, a_n, b_1, \ldots, b_t$ be $n + t$ non-invertible elements of a local ring R. Then the direct sums $R/a_1R \oplus \cdots \oplus R/a_nR$ and $R/b_1R \oplus \cdots \oplus R/b_tR$ are isomorphic right R-modules if and only if $n = t$ and there exist two permutations σ, τ of $\{1, 2, \ldots, n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i = 1, 2, \ldots, n$.*

Equivalence of matrices

The Weak Krull-Schmidt Theorem for cyclically presented modules has an immediate consequence as far as equivalence of matrices is concerned. Recall that two $m \times n$ matrices A and B with entries in a ring R are said to be *equivalent* matrices, denoted $A \sim B$, if there exist an $m \times m$ invertible matrix P and an $n \times n$ invertible matrix Q with entries in R (that is, matrices invertible in the rings $M_m(R)$ and $M_n(R)$, respectively) such that $B = PAQ$.

Equivalence of matrices

The Weak Krull-Schmidt Theorem for cyclically presented modules has an immediate consequence as far as equivalence of matrices is concerned. Recall that two $m \times n$ matrices A and B with entries in a ring R are said to be *equivalent* matrices, denoted $A \sim B$, if there exist an $m \times m$ invertible matrix P and an $n \times n$ invertible matrix Q with entries in R (that is, matrices invertible in the rings $M_m(R)$ and $M_n(R)$, respectively) such that $B = PAQ$. We denote by $\mathrm{diag}(a_1, \ldots, a_n)$ the $n \times n$ diagonal matrix whose (i, i) entry is a_i and whose other entries are zero.

Equivalence of matrices

If R is a *commutative* local ring and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are elements of R, then $\mathrm{diag}(a_1, \ldots, a_n) \sim \mathrm{diag}(b_1, \ldots, b_n)$ if and only if there exists a permutation σ of $\{1, 2, \ldots, n\}$ with a_i and $b_{\sigma(i)}$ associate elements of R for every $i = 1, 2, \ldots, n$. Here $a, b \in R$ are *associate elements* if they generate the same principal ideal of R.

Equivalence of matrices

If R is a *commutative* local ring and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are elements of R, then $\mathrm{diag}(a_1, \ldots, a_n) \sim \mathrm{diag}(b_1, \ldots, b_n)$ if and only if there exists a permutation σ of $\{1, 2, \ldots, n\}$ with a_i and $b_{\sigma(i)}$ associate elements of R for every $i = 1, 2, \ldots, n$. Here $a, b \in R$ are *associate elements* if they generate the same principal ideal of R.
If the ring R is local, but non-necessarily commutative, we have the following result:

Proposition

Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements of a local ring R. Then $\mathrm{diag}(a_1, \ldots, a_n) \sim \mathrm{diag}(b_1, \ldots, b_n)$ if and only if there exist two permutations σ, τ of $\{1, 2, \ldots, n\}$ with

$$[R/a_iR]_l = [R/b_{\sigma(i)}R]_l \quad \textit{and} \quad [R/a_iR]_e = [R/b_{\tau(i)}R]_e$$

for every $i = 1, 2, \ldots, n$.

Kernels of morphisms between indecomposable injective modules

For a right module A_R over a ring R, let $E(A_R)$ denote the injective envelope of A_R. We say that two modules A_R and B_R *have the same upper part*, and write $[A_R]_u = [B_R]_u$, if there exist a homomorphism $\varphi\colon E(A_R) \to E(B_R)$ and a homomorphism $\psi\colon E(B_R) \to E(A_R)$ such that $\varphi^{-1}(B_R) = A_R$ and $\psi^{-1}(A_R) = B_R$.

Kernels of morphisms between indecomposable injective modules

A standard technique of homological algebra to extend a morphism between two modules to their injective resolutions.

Kernels of morphisms between indecomposable injective modules

A standard technique of homological algebra to extend a morphism between two modules to their injective resolutions.

Notation. Assume that E_0, E_1, E_0', E_1' are indecomposable injective right modules over a ring R, and that $\varphi\colon E_0 \to E_1, \varphi'\colon E_0' \to E_1'$ are two right R-module morphisms. A morphism $f\colon \ker\varphi \to \ker\varphi'$ extends to a morphism $f_0\colon E_0 \to E_0'$. Now f_0 induces a morphism $\widetilde{f_0}\colon E_0/\ker\varphi \to E_0'/\ker\varphi'$, which extends to a morphism $f_1\colon E_1 \to E_1'$.

Kernels of morphisms between indecomposable injective modules

A standard technique of homological algebra to extend a morphism between two modules to their injective resolutions.

Notation. Assume that E_0, E_1, E_0', E_1' are indecomposable injective right modules over a ring R, and that $\varphi\colon E_0 \to E_1, \varphi'\colon E_0' \to E_1'$ are two right R-module morphisms. A morphism $f\colon \ker\varphi \to \ker\varphi'$ extends to a morphism $f_0\colon E_0 \to E_0'$. Now f_0 induces a morphism $\widetilde{f_0}\colon E_0/\ker\varphi \to E_0'/\ker\varphi'$, which extends to a morphism $f_1\colon E_1 \to E_1'$. Thus we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker\varphi & \longrightarrow & E_0 & \xrightarrow{\varphi} & E_1 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ 0 & \longrightarrow & \ker\varphi' & \longrightarrow & E_0' & \xrightarrow[\varphi']{} & E_1'. \end{array} \tag{1}$$

The morphisms f_0 and f_1 are not uniquely determined by f.

Kernels of morphisms between indecomposable injective modules

Theorem

Let E_0 and E_1 be indecomposable injective right modules over a ring R, and let $\varphi\colon E_0 \to E_1$ be a non-zero non-injective morphism. Let $S := \mathrm{End}_R(\ker\varphi)$ denote the endomorphism ring of $\ker\varphi$. Set $I := \{\, f \in S \mid$ the endomorphism f of $\ker\varphi$ is not a monomorphism $\}$ and $K := \{\, f \in S \mid$ the endomorphism f_1 of E_1 is not a monomorphism $\} = \{\, f \in S \mid \ker\varphi \subset f_0^{-1}(\ker\varphi)\,\}$.

Kernels of morphisms between indecomposable injective modules

Theorem

Let E_0 and E_1 be indecomposable injective right modules over a ring R, and let $\varphi\colon E_0 \to E_1$ be a non-zero non-injective morphism. Let $S := \mathrm{End}_R(\ker\varphi)$ denote the endomorphism ring of $\ker\varphi$. Set $I := \{\, f \in S \mid$ the endomorphism f of $\ker\varphi$ is not a monomorphism$\,\}$ and $K := \{\, f \in S \mid$ the endomorphism f_1 of E_1 is not a monomorphism$\,\} = \{\, f \in S \mid \ker\varphi \subset f_0^{-1}(\ker\varphi)\,\}$. Then I and K are two two-sided completely prime ideals of S, and every proper right ideal of S and every proper left ideal of S is contained either in I or in K.

Kernels of morphisms between indecomposable injective modules

Theorem

Let E_0 and E_1 be indecomposable injective right modules over a ring R, and let $\varphi\colon E_0 \to E_1$ be a non-zero non-injective morphism. Let $S := \mathrm{End}_R(\ker\varphi)$ denote the endomorphism ring of $\ker\varphi$. Set $I := \{\, f \in S \mid$ the endomorphism f of $\ker\varphi$ is not a monomorphism $\}$ and $K := \{\, f \in S \mid$ the endomorphism f_1 of E_1 is not a monomorphism $\} = \{\, f \in S \mid \ker\varphi \subset f_0^{-1}(\ker\varphi)\,\}$. Then I and K are two two-sided completely prime ideals of S, and every proper right ideal of S and every proper left ideal of S is contained either in I or in K. Moreover, exactly one of the following two conditions holds:

(a) *Either I and K are comparable (that is, $I \subseteq K$ or $K \subseteq I$), in which case S is a local ring with maximal ideal $I \cup K$, or*

(b) *I and K are not comparable, and in this case S/I and S/K are division rings and $S/J(S) \cong S/I \times S/K$.*

Kernels of morphisms between indecomposable injective modules

Theorem

(Weak Krull-Schmidt Theorem) *Let* $\varphi_i \colon E_{i,0} \to E_{i,1}$ $(i = 1, 2, \dots, n)$ *and* $\varphi'_j \colon E'_{j,0} \to E'_{j,1}$ $(j = 1, 2, \dots, t)$ *be* $n + t$ *non-injective morphisms between indecomposable injective right modules* $E_{i,0}, E_{i,1}, E'_{j,0}, E'_{j,1}$ *over an arbitrary ring* R*. Then the direct sums* $\oplus_{i=0}^{n} \ker \varphi_i$ *and* $\oplus_{j=0}^{t} \ker \varphi'_j$ *are isomorphic* R*-modules if and only if* $n = t$ *and there exist two permutations* σ, τ *of* $\{1, 2, \dots, n\}$ *such that* $[\ker \varphi_i]_m = [\ker \varphi'_{\sigma(i)}]_m$ *and* $[\ker \varphi_i]_u = [\ker \varphi'_{\tau(i)}]_u$ *for every* $i = 1, 2, \dots, n$*.*

Other classes of modules with the same behaviour

(1) Couniformly presented modules.

Other classes of modules with the same behaviour

(1) Couniformly presented modules.

(2) Biuniform modules (modules of Goldie dimension one and dual Goldie dimension one).

Other classes of modules with the same behaviour

(1) Couniformly presented modules.

(2) Biuniform modules (modules of Goldie dimension one and dual Goldie dimension one).

(3) Another class of modules that can be described via two invariants is that of Auslander-Bridger modules. For Auslander-Bridger modules, the two invariants are epi-isomorphism and lower-isomorphism.

A general pattern

A general pattern

Let $\mathcal{C}$ be a full subcategory of the category Mod-R for some ring R and assume that every object of $\mathcal{C}$ is an indecomposable right R-module.

A general pattern

Let $\mathcal{C}$ be a full subcategory of the category Mod-R for some ring R and assume that every object of $\mathcal{C}$ is an indecomposable right R-module. Define a *completely prime ideal* $\mathcal{P}$ of $\mathcal{C}$ as an assignement of a subgroup $\mathcal{P}(A, B)$ of the additive abelian group $\mathrm{Hom}_R(A, B)$ to every pair (A, B) of objects of $\mathcal{C}$ with the following two properties: (1) for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$, every $f \colon A \to B$ and every $g \colon B \to C$, one has that $gf \in \mathcal{P}(A, C)$ if and only if either $f \in \mathcal{P}(A, B)$ or $g \in \mathcal{P}(B, C)$; (2) $\mathcal{P}(A, A)$ is a proper subgroup of $\mathrm{Hom}_R(A, A)$ for every object $A \in \mathrm{Ob}(\mathcal{C})$.

A general pattern

Let $\mathcal{C}$ be a full subcategory of the category $\mathrm{Mod}\text{-}R$ for some ring R and assume that every object of $\mathcal{C}$ is an indecomposable right R-module. Define a *completely prime ideal* $\mathcal{P}$ of $\mathcal{C}$ as an assignement of a subgroup $\mathcal{P}(A, B)$ of the additive abelian group $\mathrm{Hom}_R(A, B)$ to every pair (A, B) of objects of $\mathcal{C}$ with the following two properties: (1) for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$, every $f\colon A \to B$ and every $g\colon B \to C$, one has that $gf \in \mathcal{P}(A, C)$ if and only if either $f \in \mathcal{P}(A, B)$ or $g \in \mathcal{P}(B, C)$; (2) $\mathcal{P}(A, A)$ is a proper subgroup of $\mathrm{Hom}_R(A, A)$ for every object $A \in \mathrm{Ob}(\mathcal{C})$.

Let $\mathcal{P}$ be a completely prime ideal of $\mathcal{C}$. If A, B are objects of $\mathcal{C}$, we say that A and B *have the same* $\mathcal{P}$ *class*, and write $[A]_{\mathcal{P}} = [B]_{\mathcal{P}}$, if $\mathcal{P}(A, B) \neq \mathrm{Hom}_R(A, B)$ and $\mathcal{P}(B, A) \neq \mathrm{Hom}_R(B, A)$.

A general pattern

Theorem

[F.-Příhoda, Algebr. Represent. Theory 2011] *Let* $\mathcal{C}$ *be a full subcategory of* $\mathrm{Mod}\text{-}R$ *and* $\mathcal{P}, \mathcal{Q}$ *be two completely prime ideals of* $\mathcal{C}$. *Assume that all objects of* $\mathcal{C}$ *are indecomposable right* R*-modules and that, for every* $A \in \mathrm{Ob}(\mathcal{C})$, $f \colon A \to A$ *is an automorphism of* A *if and only if* $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. *Then, for every* $A_1, \dots, A_n, B_1, \dots, B_t \in \mathrm{Ob}(\mathcal{C})$, *the modules* $A_1 \oplus \cdots \oplus A_n$ *and* $B_1 \oplus \cdots \oplus B_t$ *are isomorphic if and only if* $n = t$ *and there exist two permutations* σ, τ *of* $\{1, 2, \dots, n\}$ *such that* $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ *and* $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ *for all* $i = 1, \dots, n$.

General pattern

For the classes $\mathcal{C}$ of modules described until now, the fact that the weak form of the Krull-Schmidt Theorem holds can be described saying that the corresponding monoid $V(\mathcal{C})$ is a subdirect product of two free monoids.

Direct sums of infinite families of uniserial modules

Let's go back to the case of $\mathcal{C} = \{\, \text{uniserial modules} \,\}$.

Direct sums of infinite families of uniserial modules

Let's go back to the case of $\mathcal{C} = \{ \text{uniserial modules} \}$. Until now we have considered direct sums of *finite* families of uniserial modules.

Direct sums of infinite families of uniserial modules

Let's go back to the case of $\mathcal{C} = \{\, \text{uniserial modules} \,\}$. Until now we have considered direct sums of *finite* families of uniserial modules. What happens for *infinite* families of uniserial modules?

Direct sums of infinite families of uniserial modules

Theorem

[F.-Dung, J. Algebra 1997] *Let* $\{ A_i \mid i \in I \}$ *and* $\{ B_j \mid j \in J \}$ *be two families of uniserial right R-modules. Assume that there exist two bijections* $\sigma, \tau \colon I \to J$ *such that* $[A_i]_m = [B_{\sigma(i)}]_m$ *and* $[A_i]_e = [B_{\tau(i)}]_e$ *for every* $i \in I$*. Then*

$$\oplus_{i \in I} A_i \cong \oplus_{j \in J} B_j.$$

Quasismall modules

A module N_R is *quasismall* if for every set $\{ M_i \mid i \in I \}$ of R-modules such that N_R is isomorphic to a direct summand of $\bigoplus_{i \in I} M_i$, there exists a finite subset F of I such that N_R is isomorphic to a direct summand of $\bigoplus_{i \in F} M_i$.

Quasismall modules

For instance:

Quasismall modules

For instance:

(1) Every finitely generated module is quasismall.

Quasismall modules

For instance:
(1) Every finitely generated module is quasismall.
(2) Every module with local endomorphism ring is quasismall.

Quasismall modules

For instance:
(1) Every finitely generated module is quasismall.
(2) Every module with local endomorphism ring is quasismall.
(3) Every uniserial module is either quasismall or countably generated.

Quasismall modules

For instance:
(1) Every finitely generated module is quasismall.
(2) Every module with local endomorphism ring is quasismall.
(3) Every uniserial module is either quasismall or countably generated.
(4) There exist uniserial modules that are not quasismall (Puninski 2001).

Direct sums of infinite families of uniserial modules

Theorem

[Příhoda 2006] *Let* $\{ U_i \mid i \in I \}$ *and* $\{ V_j \mid j \in J \}$ *be two families of uniserial modules over an arbitrary ring* R*. Let* I' *be the sets of all indices* $i \in I$ *with* U_i *quasismall, and similarly for* J'*. Then* $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$ *if and only if there exist a bijection* $\sigma \colon I \to J$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and a bijection* $\tau \colon I' \to J'$ *such that* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i \in I'$*.*

Direct products of infinite families of uniserial modules

Until now: direct sums.

Direct products of infinite families of uniserial modules

Until now: direct sums.

What about direct products?

Direct products of infinite families of uniserial modules

Theorem

[Alahmadi-F. 2014] *Let* $\{ U_i \mid i \in I \}$ *and* $\{ V_j \mid j \in J \}$ *be two families of uniserial modules over an arbitrary ring* R. *Assume that there exist two bijections* $\sigma, \tau \colon I \to J$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i \in I$. *Then* $\prod_{i \in I} U_i \cong \prod_{j \in J} V_j$.

General pattern

A full subcategory $\mathcal{C}$ of Mod-R is said to *satisfy Condition (DSP)* (direct summand property) if whenever A, B, C, D are right R-modules with $A \oplus B \cong C \oplus D$ and $A, B, C \in \mathrm{Ob}(\mathcal{C})$, then also $D \in \mathrm{Ob}(\mathcal{C})$.

General pattern

Theorem

Let $\mathcal{C}$ be a full subcategory of Mod-R in which all objects are indecomposable right R-modules and let $\mathcal{P}, \mathcal{Q}$ be two completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, an endomorphism $f\colon A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A,A) \cup \mathcal{Q}(A,A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP).

General pattern

Theorem

Let $\mathcal{C}$ be a full subcategory of Mod-R in which all objects are indecomposable right R-modules and let $\mathcal{P}, \mathcal{Q}$ be two completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, an endomorphism $f \colon A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP). Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$. Assume that there exist two bijections $\sigma, \tau \colon I \to J$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i \in I$. Then the R-modules $\prod_{i \in I} A_i$ and $\prod_{j \in J} B_j$ are isomorphic.

Cyclically presented modules

Theorem

Let R be a local ring and $\{ U_i \mid i \in I \}$ and $\{ V_j \mid j \in J \}$ be two families of cyclically presented right R-modules. Suppose that there exist two bijections $\sigma, \tau \colon I \to J$ such that $[U_i]_l = [V_{\sigma(i)}]_l$ and and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I$. Then $\prod_{i \in I} U_i \cong \prod_{j \in J} V_j$.

Kernels of morphisms between indecomposable injective modules

Theorem

Let R be a ring and $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of right R-modules that are all kernels of non-injective morphisms between indecomposable injective modules. Suppose that there exist bijections $\sigma, \tau \colon I \to J$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ and $[A_i]_u = [B_{\tau(i)}]_u$ for every $i \in I$. Then $\prod_{i \in I} A_i \cong \prod_{j \in J} B_j$.

Another example

Let R be a ring and let S_1, S_2 be two fixed non-isomorphic simple right R-modules.

Another example

Let R be a ring and let S_1, S_2 be two fixed non-isomorphic simple right R-modules. Let $\mathcal{C}$ be the full subcategory of Mod-R whose objects are all artinian right R-modules A_R with $\mathrm{soc}(A_R) \cong S_1 \oplus S_2$. Set $\mathcal{P}_i(A, B) := \{ f \in \mathrm{Hom}_R(A, B) \mid f(\mathrm{soc}_{S_i}(A)) = 0 \}$.

Another example

Let R be a ring and let S_1, S_2 be two fixed non-isomorphic simple right R-modules. Let $\mathcal{C}$ be the full subcategory of Mod-R whose objects are all artinian right R-modules A_R with $\mathrm{soc}(A_R) \cong S_1 \oplus S_2$. Set $\mathcal{P}_i(A, B) := \{ f \in \mathrm{Hom}_R(A, B) \mid f(\mathrm{soc}_{S_i}(A)) = 0 \}$.

Theorem

Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$. Suppose that there exist two bijections $\sigma_k \colon I \to J$, $k = 1, 2$, such that $[A_i]_k = [B_{\sigma_k(i)}]_k$ for both $k = 1, 2$. Then $\prod_{i \in I} A_i \cong \prod_{j \in J} B_j$.

Reversing the main result

Reversing the main result

Is it possible to invert our result?

Reversing the main result

Is it possible to invert our result?
For example,

Reversing the main result

Is it possible to invert our result?
For example, does a direct product of uniserial modules determine the monogeny classes and the epigeny classes of the factors?

Negative example 1

$R =$ localization of the ring $\mathbb{Z}$ of integers at a maximal ideal (p),

Negative example 1

$R =$ localization of the ring $\mathbb{Z}$ of integers at a maximal ideal (p),
$\mathbb{Q} \oplus (\mathbb{Z}(p^\infty))^{\mathbb{N}^*} \cong (\mathbb{Z}(p^\infty))^{\mathbb{N}^*}$,

Negative example 1

$R =$ localization of the ring $\mathbb{Z}$ of integers at a maximal ideal (p), $\mathbb{Q} \oplus (\mathbb{Z}(p^\infty))^{\mathbb{N}^*} \cong (\mathbb{Z}(p^\infty))^{\mathbb{N}^*}$, all the factors are uniserial R-modules with a local endomorphism ring,

Negative example 1

$R =$ localization of the ring $\mathbb{Z}$ of integers at a maximal ideal (p), $\mathbb{Q} \oplus (\mathbb{Z}(p^\infty))^{\mathbb{N}^*} \cong (\mathbb{Z}(p^\infty))^{\mathbb{N}^*}$, all the factors are uniserial R-modules with a local endomorphism ring, but there are no bijections preserving the monogeny classes and the epigeny classes.

Negative example 2

$R = \mathbb{Z}$,

Negative example 2

$R = \mathbb{Z}$, $\mathcal{C}$ be the full subcategory of Mod-R whose objects are all injective indecomposable R-modules. If A and B are objects of $\mathcal{C}$, let $\mathcal{P}(A, B)$ be the group of all morphisms $A \to B$ that are not automorphisms, so that $\mathcal{P}$ is a completely prime ideal of $\mathcal{C}$,

Negative example 2

$R = \mathbb{Z}$, $\mathcal{C}$ be the full subcategory of $\mathrm{Mod}\text{-}R$ whose objects are all injective indecomposable R-modules. If A and B are objects of $\mathcal{C}$, let $\mathcal{P}(A, B)$ be the group of all morphisms $A \to B$ that are not automorphisms, so that $\mathcal{P}$ is a completely prime ideal of $\mathcal{C}$, $\mathbb{Q} \oplus \prod_p \mathbb{Z}(p^\infty) \cong \prod_p \mathbb{Z}(p^\infty)$,

Negative example 2

$R = \mathbb{Z}$, $\mathcal{C}$ be the full subcategory of $\mathrm{Mod}\text{-}R$ whose objects are all injective indecomposable R-modules. If A and B are objects of $\mathcal{C}$, let $\mathcal{P}(A, B)$ be the group of all morphisms $A \to B$ that are not automorphisms, so that $\mathcal{P}$ is a completely prime ideal of $\mathcal{C}$, $\mathbb{Q} \oplus \prod_p \mathbb{Z}(p^\infty) \cong \prod_p \mathbb{Z}(p^\infty)$, but there does not exist a bijection σ preserving the $\mathcal{P}$ classes.

Negative example 3

$p =$ prime number,

Negative example 3

p = prime number, $\widehat{\mathbb{Z}_p}$ = ring of p-adic integers,

Negative example 3

$p =$ prime number, $\widehat{\mathbb{Z}_p} =$ ring of p-adic integers, so that $\mathbb{Z}/p^n\mathbb{Z}$ is a module over $\widehat{\mathbb{Z}_p}$ for every integer $n \geq 1$.

Negative example 3

$p =$ prime number, $\widehat{\mathbb{Z}_p} =$ ring of p-adic integers, so that $\mathbb{Z}/p^n\mathbb{Z}$ is a module over $\widehat{\mathbb{Z}_p}$ for every integer $n \geq 1$. Then $\widehat{\mathbb{Z}_p} \oplus \prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z} \cong \prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z}$. In these direct products, all the factors $\widehat{\mathbb{Z}_p}$ and $\mathbb{Z}/p^n\mathbb{Z}$ $(n \geq 1)$ are pair-wise non-isomorphic uniserial $\widehat{\mathbb{Z}_p}$-modules, have distinct monogeny classes and distinct epigeny classes $\Rightarrow$ there cannot be bijections σ and τ preserving the monogeny and the epigeny classes in the two direct-product decompositions.

But. . . slender modules.

$R =$ a ring,

But. . . slender modules.

$R =$ a ring,
$R^\omega = \prod_{n<\omega} e_n R$ right R-module that is the direct product of countably many copies of the right R-module R_R, where e_n is the element of R^ω with support $\{n\}$ and equal to 1 in n.

But... slender modules.

$R =$ a ring,
$R^\omega = \prod_{n<\omega} e_n R$ right R-module that is the direct product of countably many copies of the right R-module R_R, where e_n is the element of R^ω with support $\{n\}$ and equal to 1 in n.

A right R-module M_R is *slender* if, for every homomorphism $f\colon R^\omega \to M$ there exists $n_0 < \omega$ such that $f(e_n) = 0$ for all $n \geq n_0$.

But. . . slender modules.

$R =$ a ring,
$R^\omega = \prod_{n<\omega} e_n R$ right R-module that is the direct product of countably many copies of the right R-module R_R, where e_n is the element of R^ω with support $\{n\}$ and equal to 1 in n.

A right R-module M_R is *slender* if, for every homomorphism $f\colon R^\omega \to M$ there exists $n_0 < \omega$ such that $f(e_n) = 0$ for all $n \geq n_0$.

Theorem

A module M_R is slender if and only if for every countable family $\{\, P_n \mid n \geq 0 \,\}$ of right R-modules and any homomorphism $f\colon \prod_{n\geq 0} P_n \to M_R$ there exists $m \geq 0$ such that $f(\prod_{n\geq m} P_n) = 0$.

Measurable cardinals

A cardinal α is *measurable* if it is an uncountable cardinal with an α-complete, non-principal ultrafilter.

Measurable cardinals

A cardinal α is *measurable* if it is an uncountable cardinal with an α-complete, non-principal ultrafilter.

α-complete $=$ the intersection of any *strictly less than* α-many sets in the ultrafilter is also in the ultrafilter.

Measurable cardinals

A cardinal α is *measurable* if it is an uncountable cardinal with an α-complete, non-principal ultrafilter.

α-complete $=$ the intersection of any *strictly less than* α-many sets in the ultrafilter is also in the ultrafilter.

If a cardinal is not measurable, then neither are all smaller cardinals.

Measurable cardinals

A cardinal α is *measurable* if it is an uncountable cardinal with an α-complete, non-principal ultrafilter.

α-complete $=$ the intersection of any *strictly less than* α-many sets in the ultrafilter is also in the ultrafilter.

If a cardinal is not measurable, then neither are all smaller cardinals. Thus if there exists a measurable cardinal, then there is a smallest one and all larger cardinals are measurable.

Measurable cardinals

A cardinal α is *measurable* if it is an uncountable cardinal with an α-complete, non-principal ultrafilter.

α-complete $=$ the intersection of any *strictly less than* α-many sets in the ultrafilter is also in the ultrafilter.

If a cardinal is not measurable, then neither are all smaller cardinals. Thus if there exists a measurable cardinal, then there is a smallest one and all larger cardinals are measurable.

It is not known whether ZFC $\Rightarrow \exists$ a measurable cardinal.

Slender modules

Theorem

If M_R is slender and $\{ P_i \mid i \in I \}$ is a family of right R-modules with $|I|$ non-measurable, then $\mathrm{Hom}(\prod_{i \in I} P_i, M_R) \cong \bigoplus_{i \in I} \mathrm{Hom}(P_i, M_R)$.

Slender modules

Theorem

If M_R is slender and $\{ P_i \mid i \in I \}$ is a family of right R-modules with $|I|$ non-measurable, then $\mathrm{Hom}(\prod_{i\in I} P_i, M_R) \cong \bigoplus_{i\in I} \mathrm{Hom}(P_i, M_R)$.

Every submodule of a slender module is a slender module.

Slender modules

Theorem

If M_R is slender and $\{ P_i \mid i \in I \}$ is a family of right R-modules with $|I|$ non-measurable, then $\mathrm{Hom}(\prod_{i \in I} P_i, M_R) \cong \bigoplus_{i \in I} \mathrm{Hom}(P_i, M_R)$.

Every submodule of a slender module is a slender module.

Theorem

A $\mathbb{Z}$-module is slender if and only if it does not contains a copy of $\mathbb{Q}$, $\mathbb{Z}^\omega$, $\mathbb{Z}/p\mathbb{Z}$ or $\widehat{\mathbb{Z}_p}$ for any prime p.

Theorem

Let $\mathcal{C}$ be a full subcategory of $\mathrm{Mod}\text{-}R$ *in which all objects are indecomposable slender right R-modules and let $\mathcal{P}, \mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, $f \colon A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP). Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$ with $|I|$ and $|J|$ non-measurable.*

Theorem

Let $\mathcal{C}$ be a full subcategory of Mod-R in which all objects are indecomposable slender right R-modules and let $\mathcal{P}, \mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, $f : A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP). Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$ with $|I|$ and $|J|$ non-measurable. Assume that:

(a) *In both families, there are at most countably many modules in each $\mathcal{P}$ class.*

(b) *In both families, there are at most countably many modules in each $\mathcal{Q}$ class.*

(c) *The R-modules $\prod_{i \in I} A_i$ and $\prod_{j \in J} B_j$ are isomorphic.*

Theorem

Let $\mathcal{C}$ be a full subcategory of $\mathrm{Mod}\text{-}R$ in which all objects are indecomposable slender right R-modules and let $\mathcal{P}, \mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, $f\colon A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP). Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$ with $|I|$ and $|J|$ non-measurable. Assume that:

(a) *In both families, there are at most countably many modules in each $\mathcal{P}$ class.*

(b) *In both families, there are at most countably many modules in each $\mathcal{Q}$ class.*

(c) *The R-modules $\prod_{i\in I} A_i$ and $\prod_{j\in J} B_j$ are isomorphic.*

Then there exist two bijections $\sigma, \tau\colon I \to J$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i \in I$.

Corollary

Let $\mathcal{C}$ be a full subcategory of Mod-R in which all objects are indecomposable slender right R-modules and let $\mathcal{P}, \mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, $f : A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP).

Corollary

Let $\mathcal{C}$ be a full subcategory of Mod-R in which all objects are indecomposable slender right R-modules and let $\mathcal{P}, \mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, $f : A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP). Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two countable families of objects of $\mathcal{C}$. Assume that $\prod_{i \in I} A_i \cong \prod_{j \in J} B_j$.

Corollary

Let $\mathcal{C}$ be a full subcategory of Mod-R in which all objects are indecomposable slender right R-modules and let $\mathcal{P}, \mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, $f \colon A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP). Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two countable families of objects of $\mathcal{C}$. Assume that $\prod_{i \in I} A_i \cong \prod_{j \in J} B_j$. Then there exist two bijections $\sigma, \tau \colon I \to J$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i \in I$.

Theorem

Let $\mathcal{C}$ be a full subcategory of $\mathrm{Mod}\text{-}R$ *in which all objects are slender right R-modules and let $\mathcal{P}$ be a completely prime ideal of $\mathcal{C}$. Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$ with $|I|$ and $|J|$ non-measurable.*

Theorem

Let $\mathcal{C}$ be a full subcategory of Mod-R in which all objects are slender right R-modules and let $\mathcal{P}$ be a completely prime ideal of $\mathcal{C}$. Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$ with $|I|$ and $|J|$ non-measurable. Assume that:

(a) *For every object A of $\mathcal{C}$, $\mathcal{P}(A, A)$ is a maximal right ideal of $\mathrm{End}_R(A)$.*

(b) *There are at most countably many modules in each $\mathcal{P}$ class in both families $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$.*

(c) *The R-modules $\prod_{i \in I} A_i$ and $\prod_{j \in J} B_j$ are isomorphic.*

Theorem

Let $\mathcal{C}$ be a full subcategory of Mod-R in which all objects are slender right R-modules and let $\mathcal{P}$ be a completely prime ideal of $\mathcal{C}$. Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$ with $|I|$ and $|J|$ non-measurable. Assume that:
(a) *For every object A of $\mathcal{C}$, $\mathcal{P}(A, A)$ is a maximal right ideal of $\mathrm{End}_R(A)$.*
(b) *There are at most countably many modules in each $\mathcal{P}$ class in both families $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$.*
(c) *The R-modules $\prod_{i\in I} A_i$ and $\prod_{j\in J} B_j$ are isomorphic.*
Then there is a bijection $\sigma_{\mathcal{P}} \colon I \to J$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma_{\mathcal{P}}(i)}]_{\mathcal{P}}$ for every $i \in I$.

Corollary

[Franetič, 2014] *Let R be a ring and $\{ A_i \mid i \in I \}$ be a family of slender right R-modules with local endomorphism rings. Let $\{ B_j \mid j \in J \}$ be a family of indecomposable slender right R-modules. Assume that:*

(a) *$|I|$ and $|J|$ are non-measurable cardinals.*

(b) *There are at most countably many mutually isomorphic modules in each of the two families $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$.*

(c) *The R-modules $\prod_{i \in I} A_i$ and $\prod_{j \in J} B_j$ are isomorphic.*

Then there exists a bijection $\sigma \colon I \to J$ such that $A_i \cong B_{\sigma(i)}$ for every $i \in I$.