Large orders and Resurgence in Functional Renormalization

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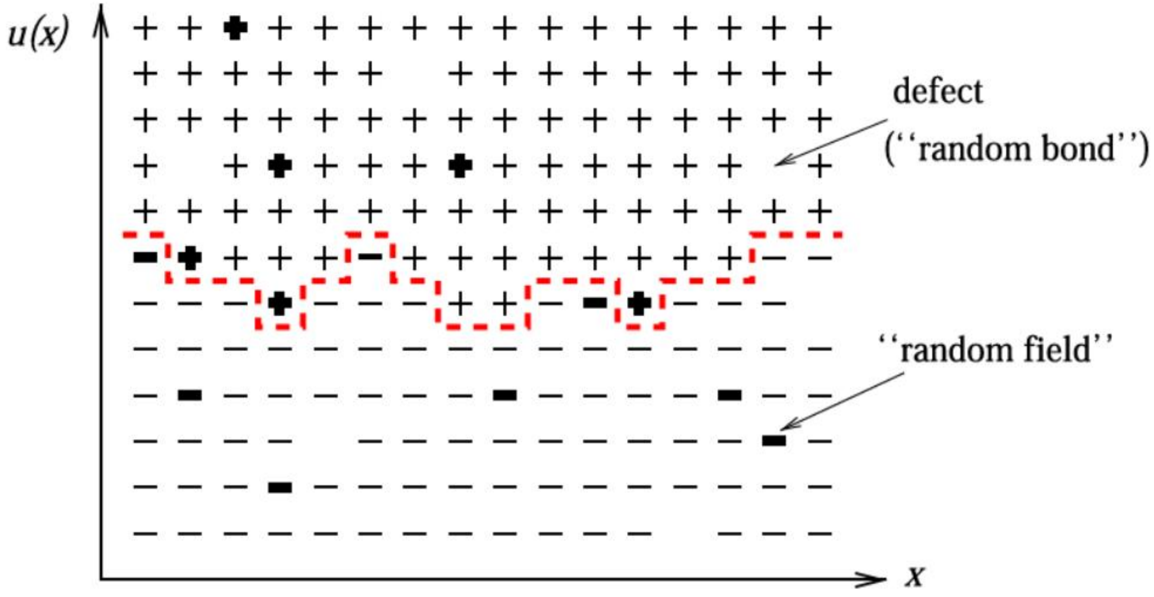
Introduction

Ising magnet with disorder



Strong disorder: disorder dominates over entropy, effectively the system is at zero temperature.

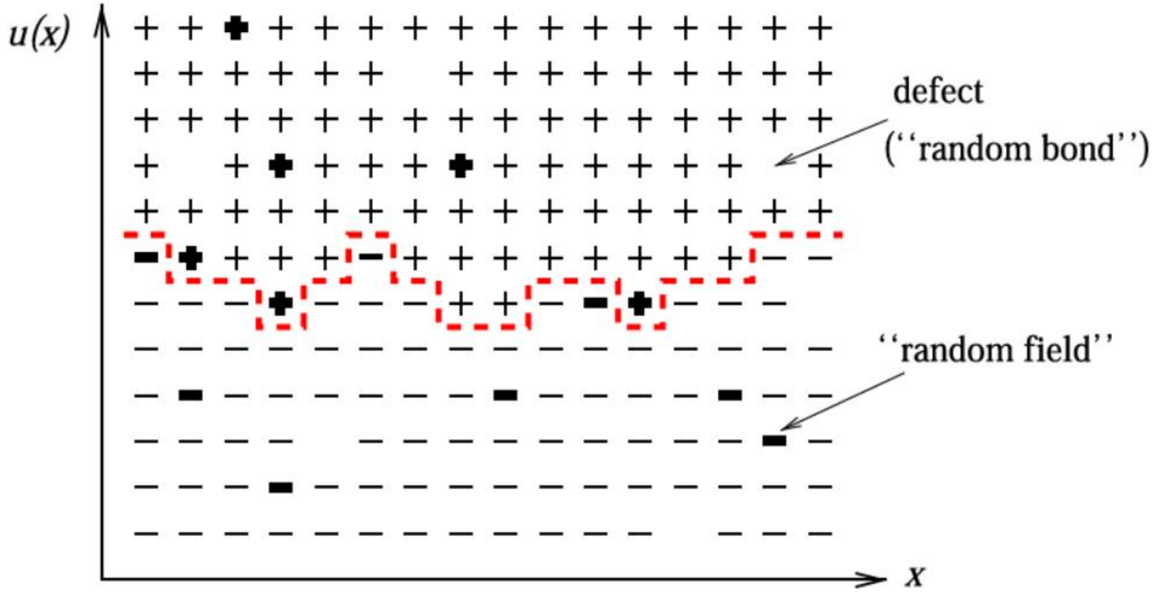
Metastable states: close in energy to the ground-state, far apart in configuration-space.

Equilibrium

If external parameters change, they change so slowly that the system has enough time to find the ground state.

Out-of-equilibrium (Depinning)

An external applied field yields jumps in the center-of-mass of the system.



Displacement: $\vec{x} \in \mathbb{R}^d \rightarrow \vec{u}(\vec{x}) \in \mathbb{R}^N$

We suppress the vector notation wherever possible, and mostly consider N = 1.

The energy s consists of three parts:

The elastic energy: $\mathcal{H}_{\text{el}}[u] = \int \mathrm{d}^d x \, \frac{1}{2}[\nabla u(x)]^2$

The confining potential: $\mathcal{H}_{\text{conf}}[u] = \int \mathrm{d}^d x \frac{m^2}{2}[u(x) - w]^2$

forbids the interface to wander off to infinity. This avoids that observables are dominated by rare events.

The disorder: $\mathcal{H}_{\text{dis}}[u] = \int \mathrm{d}^d x \, V(x, u(x))$

Suppose that fluctuations of *u* scale as: $\overline{\left\langle [u(x) - u(y)]^2 \right\rangle} \sim |x - y|^{2\zeta}$

This defines a *roughness-exponent* ζ.

Disorder potential is Gaussian: $\overline{V(x,u)V(x',u')} := R(u-u')\delta^d(x-x')$

The corresponding disorder force–force correlator can be written as: $\overline{\langle F(x,u)F(x',u')\rangle} = \Delta(u-u')\delta^d(x-x')$

$$\Delta(u) = -R''(u)$$

Replica trick

$$\overline{\mathcal{O}[u]} := \overline{\frac{\left\langle \mathcal{O}[u]\mathrm{e}^{-\mathcal{H}[u]/T}\right\rangle}{\left\langle \mathrm{e}^{-\mathcal{H}[u]/T}\right\rangle}} \xrightarrow{T\to 0} \overline{\frac{\mathcal{O}[u_{\mathrm{gs}}]\mathrm{e}^{-\mathcal{H}[u_{\mathrm{gs}}]/T}}{\mathrm{e}^{-\mathcal{H}[u_{\mathrm{gs}}]/T}}} \equiv \overline{\mathcal{O}[u_{\mathrm{gs}}]}.$$

The problem comes from the denominator: The denominator is correlated with the numerator

Integer powers Z, with $n \in \mathrm{N}$ can be obtained by using *n copies* or *replicas* of the system, $1/Z$ cannot.

$$\overline{\mathcal{O}[u]} = \overline{\frac{\left\langle \mathcal{O}[u]\mathrm{e}^{-\mathcal{H}[u]/T}\right\rangle \mathcal{Z}^{n-1}}{\mathcal{Z}^{n}}} \quad \Rightarrow \quad \overline{\mathcal{O}[u]} = \lim_{n\to 0}\overline{\left\langle \mathcal{O}[u]\mathrm{e}^{-\mathcal{H}[u]/T}\right\rangle \mathcal{Z}^{n-1}}$$

The idea is to set $n \to 0$ at the end of the calculation, eliminating the denominator.

Since thermal averages over distinct replicas factorize, we write their joint measure as:

$$\overline{\left\langle \mathcal{O}[u]\mathrm{e}^{-\mathcal{H}[u]/T}\right\rangle \mathcal{Z}^{n-1}} = \overline{\left\langle \mathcal{O}[u_1]\prod_{a=1}^{n}\mathrm{e}^{-\mathcal{H}[u_a]/T}\right\rangle} = \left\langle \mathcal{O}[u_1]\overline{\mathrm{e}^{-\frac{1}{T}\sum_{a=1}^{n}\mathcal{H}_{\mathrm{el}}[u_a]+\mathcal{H}_{\mathrm{conf}}[u_a]+\mathcal{H}_{\mathrm{dis}}[u_a]}}\right\rangle$$

$$\overline{\mathrm{e}^{-\frac{1}{T}\sum_{a=1}^{n}\mathcal{H}_{\mathrm{dis}}[u_a]}} = \overline{\exp\left(-\frac{1}{T}\int_x\sum_a V(x,u_a(x))\right)} = \exp\left(\frac{1}{2T^2}\int_x\int_y\sum_{a,b=1}^{n}\overline{V(x,u_a(x))V(y,u_b(y))}\right) = \exp\left(\frac{1}{2T^2}\int_x\sum_{a,b=1}^{n}R\left(u_a(x)-u_b(x)\right)\right)$$

$$\mathcal{S}_{\mathrm{rep}}[u] := \frac{1}{T}\sum_{a=1}^{n}\int_x\left\{\frac{1}{2}[\nabla u_a(x)]^2 + \frac{m^2}{2}[u_a(x)-w_a]^2\right\} - \frac{1}{2T^2}\int_x\sum_{a,b=1}^{n}R\left(u_a(x)-u_b(x)\right)$$

Field Theory

$$\mathcal{S}_{\text{rep}}[u] := \frac{1}{T}\sum_{a=1}^{n}\int_x \left\{\frac{1}{2}[\nabla u_a(x)]^2 + \frac{m^2}{2}[u_a(x) - w_a]^2\right\} - \frac{1}{2T^2}\int_x \sum_{a,b=1}^{n} R\left(u_a(x) - u_b(x)\right)$$

$$\langle \tilde{u}_a(-k)\tilde{u}_b(k)\rangle_0 = T\delta_{ab}\tilde{C}(k)$$

associated with a correlation length

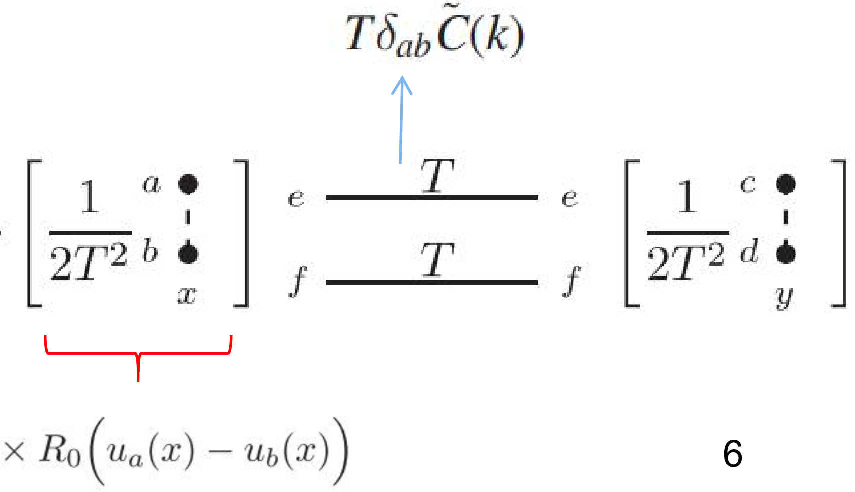
$$C(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{e}^{\mathrm{i}kx}}{k^2 + m^2} \sim \mathrm{e}^{-m|x|} \Rightarrow \xi = \frac{1}{m}$$

Wick contractions:

$$\overbrace{u_a(x)^n u_b(y)^m} = n u_a(x)^{n-1} \times m u_b(x)^{m-1} \times T\delta_{ab}C(x-y)$$

$$V\left(\overbrace{u_a(x)\right)V\left(u_b(y)}\right) = V'\left(u_a(x)\right) \times V'\left(u_b(y)\right) \times T\delta_{ab}C(x-y)$$

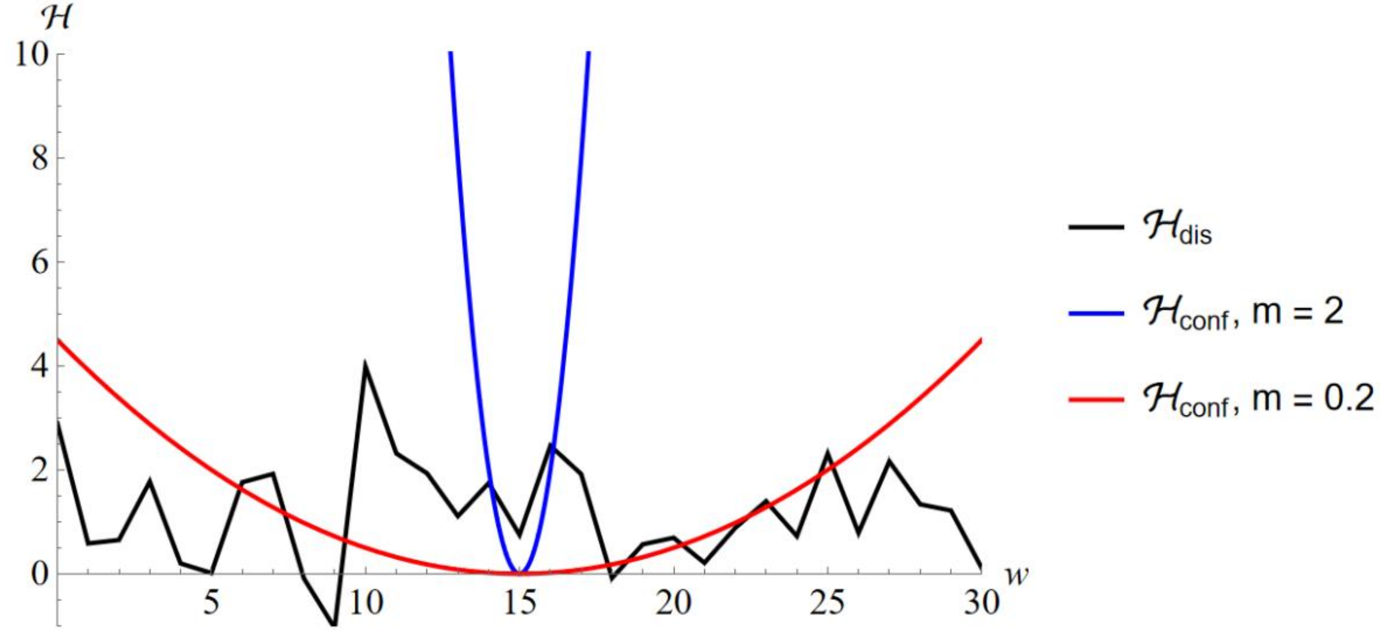
$$\frac{1}{2T^2}\delta R = \frac{1}{2!}$$



Fixed point

associated with a correlation length

$$C(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{e}^{ikx}}{k^2 + m^2} \quad \sim \mathrm{e}^{-m|x|} \quad \Rightarrow \quad \xi = \frac{1}{m}$$

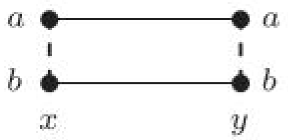


As we lower *m*, the parabola opens up, and the interface gains access to more and more disorder configurations: we move from microscopic to macroscopic scales and begin to see the whole disorder landscape.

One-loop

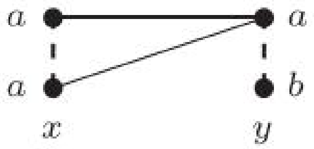
Using the rules from the previous slide , we obtain two distinct contributions:

$$\delta R^{(1)} = \frac{1}{2}$$



$$= \frac{1}{2}\int_x R_0''\big(u_a(x) - u_b(x)\big) R_0''\big(u_a(y) - u_b(y)\big) C(x-y)^2$$

$$\delta R^{(2)} =$$



$$= -\int_x R_0''\big(u_a(x) - u_a(x)\big) R_0''\big(u_a(y) - u_b(y)\big) C(x-y)^2$$

One-loop

We obtain for the effective disorder correlator at 1-loop:

$$R(u) = R_0(u) + \left[\frac{1}{2}R_0''(u)^2 - R_0''(u)R_0''(0)\right] I_1 + \dots$$

bare

$$I_1 = \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}} m^{-\epsilon} \simeq \frac{m^{-\epsilon}}{\epsilon}$$

$$\epsilon = 4 - d$$

Flow equation w.r.t *m*:

$$-m\frac{\partial}{\partial m}R(u) = \left[\frac{1}{2}R''(u)^2 - R''(u)R''(0)\right] \epsilon I_1$$

The dimensionless effective disorder, as function of the dimensionless field is defined as: $\tilde{R}(\mathbf{u}) := \epsilon I_1 m^{4\zeta} R(u = \mathbf{u} m^{-\zeta})$

Fixed-point:

$$\partial_\ell \tilde{R}(\mathbf{u}) := -m\frac{\partial}{\partial m}\tilde{R}(\mathbf{u}) = (\epsilon - 4\zeta)\tilde{R}(\mathbf{u}) + \zeta \mathbf{u}\tilde{R}'(\mathbf{u}) + \frac{1}{2}\tilde{R}''(\mathbf{u})^2 - \tilde{R}''(\mathbf{u})\tilde{R}''(0)$$

Why follow a function?

Dimensional reduction: A *d*-dimensional disordered system at zero temperature is equivalent to all orders in perturbation theory to a pure system in *d*-2 dimensions at finite temperature.

$$\frac{1}{2}\overline{[u(x)-u(y)]^2} = -R''(0)\mathcal{A}|x-y|^{4-d}$$

d=4 is an upper critical dimension and field *u(x)* is dimensionless there.

One can demostrate explicetly that at some RG-scale this result no-longer holds and ζ is no loneger 0 in *d*=4.

The natural conclusion is to follow the full function $R(u)$ under renormalization.

“Supersymmetry” and disorder

$$\overline{\mathcal{O}[u_i]} = \int \prod_{a=1}^{2} \mathcal{D}[\tilde{u}_a]\, \mathcal{D}[u_a]\, \mathcal{D}[\bar{\psi}_a]\, \mathcal{D}[\psi_a]\, \mathcal{O}[u_i]\, \overline{\exp\left[-\int_x \tilde{u}_a(x)\frac{\delta\mathcal{H}[u_a]}{\delta u_a(x)} + \bar{\psi}_a(x)\frac{\delta^2\mathcal{H}[u_a]}{\delta u_a(x)\delta u_a(y)}\psi_a(y)\right]}.$$

where ψ_a and $\bar{\psi}_a$ are Grassman fields,

$$\mathcal{H} = \int_x \left\{\frac{1}{2}\left[\nabla u(x)\right]^2 + \frac{m^2}{2}\left(u(x) - w\right)^2 + V(x, u(x))\right\},$$

In the limit of T → 0 only only configurations which minimize the energy survive: $\frac{\delta\mathcal{H}[u_a,V]}{\delta u_a(x)} = 0$

We want to insert a δ-distribution enforcing this condition into the path-integral: $\det\left[\frac{\delta^2\mathcal{H}[u_a,V]}{\delta u_a(x)\delta u_a(y)}\right]$

$$\det\left(\frac{\delta^2\mathcal{H}[u,V]}{\delta u(x)\delta u(y)}\right) = \int \mathcal{D}[\bar{\psi}_a]\mathcal{D}[\psi_a]\exp\left(-\int_x \bar{\psi}(x)\frac{\delta^2\mathcal{H}[u,V]}{\delta u(x)\delta u(y)}\psi(x)\right)$$

Averaging over disorder with the force–force correlator and changing variables:

$$\begin{aligned}\mathcal{S} = \int_x & \tilde{\phi}(x)(m^2 - \nabla^2)[\phi(x) - w] \\ & + \sum_{a=1}^{2} \bar{\psi}_a(x)(m^2 - \nabla^2)\psi_a(x) \\ & + \tilde{\phi}(x)^2 \Big[\Delta(\phi(x)) - \Delta(0)\Big] \\ & + \tilde{\phi}(x)\Delta'(\phi(x))\Big[\bar{\psi}_2(x)\psi_2(x) + \bar{\psi}_1(x)\psi_1(x)\Big] \\ & + \bar{\psi}_2(x)\psi_2(x)\bar{\psi}_1(x)\psi_1(x)\Delta''(\phi(x)) \ .\end{aligned}$$

$$\left\{\begin{aligned} \phi &:= u_1 - u_2 \\ \langle\phi\rangle &= w \\ u &:= (u_1 + u_2)/2 \end{aligned}\right.$$

Going to dimension $d = 0$, dropping the Grassmann fields, and rescaling $\tilde{\phi} \to \tilde{\phi}/m^2$ yields:

$$Z(\lambda, w) = \int \mathrm{d}\phi\, \mathrm{d}\tilde{\phi}\, \mathrm{e}^{-(\phi - w)\tilde{\phi} + \lambda\tilde{\phi}^2\left(\Delta(0) - \Delta(\phi)\right)}$$

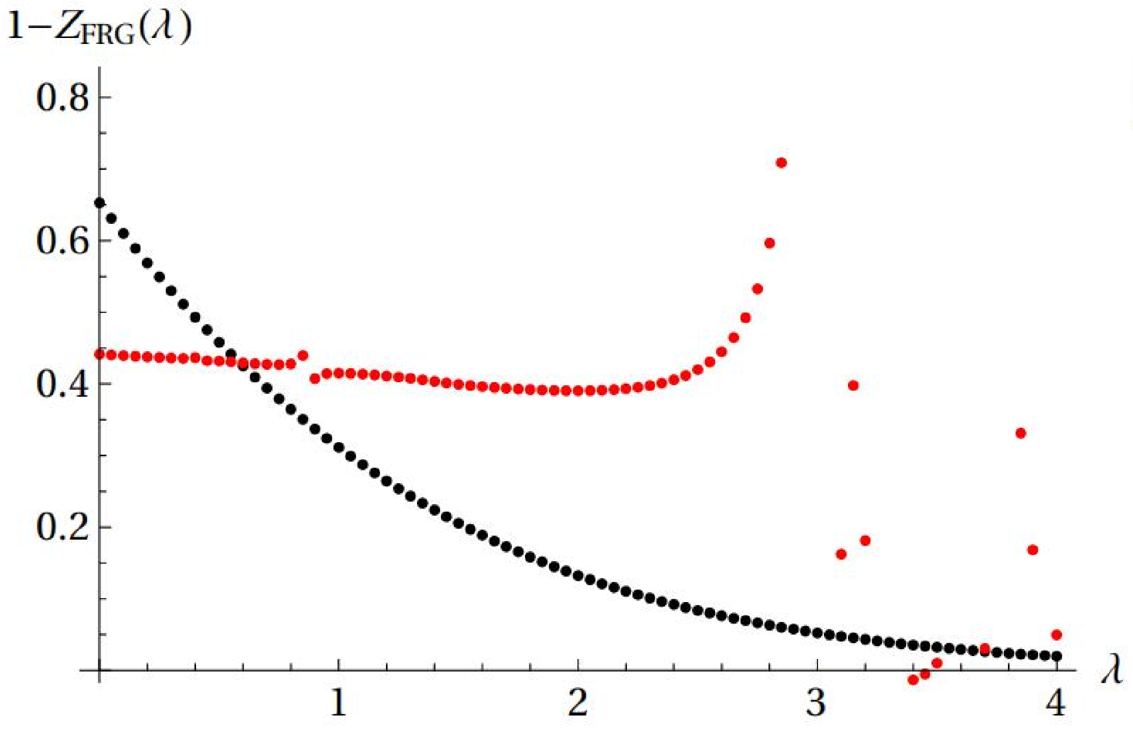
with $\lambda = \dfrac{1}{m^4}$

Our model

- The partition function: $$Z(\lambda, w) = \int \mathrm{d}\phi\, \mathrm{d}\tilde{\phi}\, \mathrm{e}^{-(\phi-w)\tilde{\phi}+\lambda\tilde{\phi}^2\left(\Delta(0)-\Delta(\phi)\right)}$$, where $\Delta(\phi) = \mathrm{e}^{-\phi}$ and $\Delta(0) = 1$.

Wick's theorem allows us to write the perturbative expansion for $w > 0$:

$$Z(\lambda, w) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\partial_{\phi}^{2n}\left(1-\Delta(w+\phi)\right)^n\Big|_{\phi=0}$$



Mikhail N. Semeikin and Kay Jörg Wiese, *Large Orders and Strong-Coupling Limit in Functional Renormalization*, Phys. Rev. E **112**, L052102 (2025).

The result of Pade-Borel resummation for $\lambda = 0.1$ (black) and $\lambda = 100$ (red).

Perturbative series	Residue theorem

$$Z(\lambda, w) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \partial_\phi^{2n} \Big(1 - \Delta(w+\phi)\Big)^n \Big|_{\phi=0}$$

$$\mathcal{Z}^{(n)}_{\mathrm{FRG}}(w) \;=\; \frac{(2n)!}{n!}\frac{1}{2\pi i}\oint \frac{\mathrm{d}\phi}{\phi} g_w(\phi)^n,$$

$$g_w(\phi) := \frac{1-\Delta(w+\phi)}{\phi^2}.$$

$$\mathcal{BI}(t) = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \oint_C \Big(t g_w(w+\phi)\Big)^n \frac{1}{2\pi i \phi} \mathrm{d}\phi$$

$$= \oint_C \frac{1}{\sqrt{1 - 4t g_w(w+\phi)}} \frac{1}{2\pi i \phi} \mathrm{d}\phi$$

Branch cut

$$\mathcal{BI}(t) = \frac{1}{\pi} \int_{\phi_0}^{\phi_1} \frac{\mathrm{d}\phi}{\sqrt{4t g_w(w+\phi)\phi^2 - \phi^2}}$$

Large coupling limit

After some non-trivial computations the inverse transform gives

$$\mathcal{BI}(t) = \frac{1}{\pi}\int_{\phi_0}^{\phi_1} \frac{\mathrm{d}\phi}{\sqrt{4tg_w(w+\phi)\phi^2 - \phi^2}} \quad \Rightarrow \quad \mathcal{Z}_{\mathrm{FRG}}(w,\lambda) = \frac{1}{\sqrt{4\pi\lambda}}\int_0^\infty \mathrm{d}\phi\, \frac{\mathrm{e}^{\frac{-(\phi-w)^2}{4\lambda[1-\Delta(\phi)]}}}{\sqrt{1-\Delta(\phi)}}$$

This allows us to extract the large-λ behavior:

$$\mathcal{Z}_{\mathrm{FRG}}(w,\lambda) \simeq \frac{1}{\sqrt{4\pi\lambda}}\int_0^\infty \mathrm{d}\phi\, \mathrm{e}^{-\frac{(\phi-w)^2}{4\lambda}} = \frac{1}{\sqrt{4\pi}}\int_0^\infty \mathrm{d}\phi\, \mathrm{e}^{-\frac{(\phi-w/\sqrt{\lambda})^2}{4}}$$

$\lim_{\lambda\to\infty} \mathcal{Z}_{\mathrm{FRG}}(w\sqrt{\lambda},\lambda)$ exists, and is given by

$$\mathcal{Z}^{\infty}_{\mathrm{FRG}}(w) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{w}{2}\right)\right].$$

By examining the perturbative series: $\Rightarrow$ $\mathcal{Z}_{\mathrm{FRG}}(w,\lambda) = 1 - \lambda\Delta''_{\mathrm{FRG}}(w,\lambda)$

$\tilde{\Delta}''_{\mathrm{FRG}}(w) := \lim_{\lambda\to\infty} \lambda^{-1}\Delta''_{\mathrm{FRG}}(w\sqrt{\lambda},\lambda) = \frac{1}{2}\mathrm{erfc}\left(\frac{w}{2}\right)$ Integrating twice yields $\tilde{\Delta}_{\mathrm{FRG}}(w) = \frac{w^2+2}{4}\mathrm{erfc}\left(\frac{w}{2}\right) - \frac{\mathrm{e}^{-\frac{w^2}{4}}w}{2\sqrt{\pi}}$

$$
\begin{aligned}
\mathcal{S} = \int_x \tilde{\phi}(x)(m^2 - \nabla^2)[\phi(x) - w] \\
&+ \sum_{a=1}^{2} \bar{\psi}_a(x)(m^2 - \nabla^2)\psi_a(x) \\
&+ \tilde{\phi}(x)^2 \Big[\Delta(\phi(x)) - \Delta(0)\Big] \\
&+ \tilde{\phi}(x)\Delta'(\phi(x)) \Big[\bar{\psi}_2(x)\psi_2(x) + \bar{\psi}_1(x)\psi_1(x)\Big] \\
&+ \bar{\psi}_2(x)\psi_2(x)\bar{\psi}_1(x)\psi_1(x)\Delta''(\phi(x)) \ .
\end{aligned}
$$

We can define force correlator: $\Delta_{\mathrm{Susy}}(0,\lambda) - \Delta_{\mathrm{Susy}}(w,\lambda) = \frac{m^4}{2} \left\langle (\phi - w)^2 \right\rangle_{\mathcal{S}}^{\mathrm{c}}$

⇩

It has a large coupling limit: $\tilde{\Delta}_{\mathrm{Susy}}(w) := \lim_{\lambda\to\infty} \Delta_{\mathrm{Susy}}(w\sqrt{\lambda}, \lambda).$

⇩

It coincides with our model ! $\tilde{\Delta}_{\mathrm{Susy}}(w) = \tilde{\Delta}_{\mathrm{FRG}}(w).$

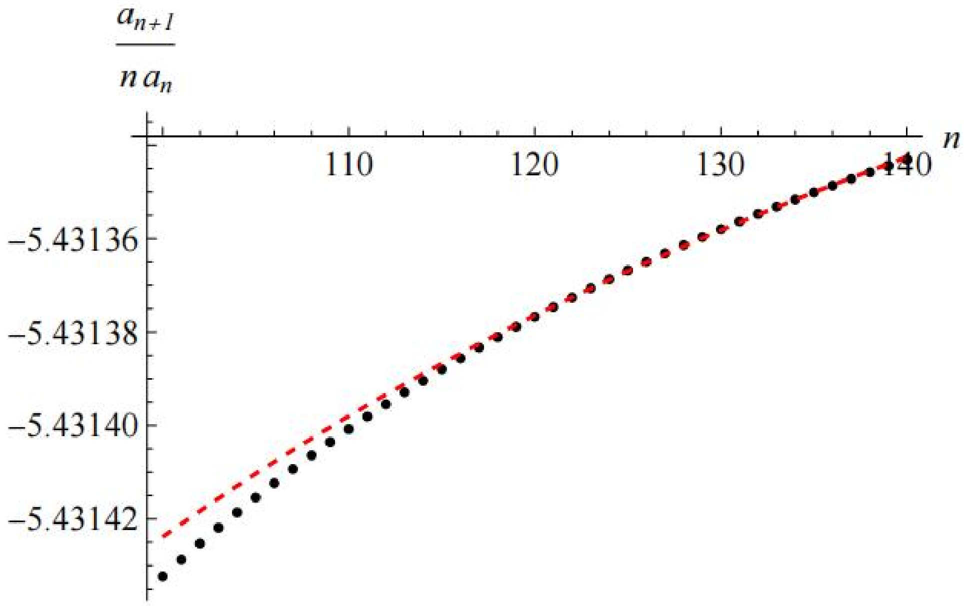
We checked both perturbatively and non-peturbatively.

Large-orders

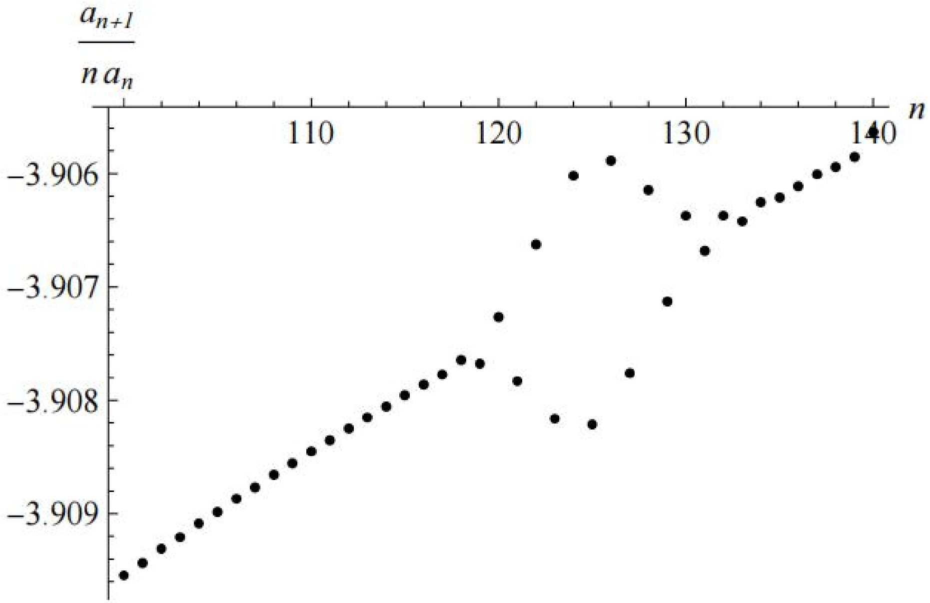
Gevrey-1: $F(g) = \sum_{n=0}^{\infty} a_n g^n$ $\qquad \mathcal{B}[F](t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$ $\qquad \mathrm{Disc}\,\mathcal{B}[F](x) \sim 2\pi i (x-A)^{-\alpha}$ for some α

$$a_n \sim S\,\Gamma(1-\alpha)\frac{n!}{A^{n+\alpha}} n^{\alpha-1}$$

Let us numerically test this asymptotic behavior: $\frac{a_{n+1}}{n a_n} \approx \frac{1}{A}(1 + \frac{\alpha}{n} + \ldots)$

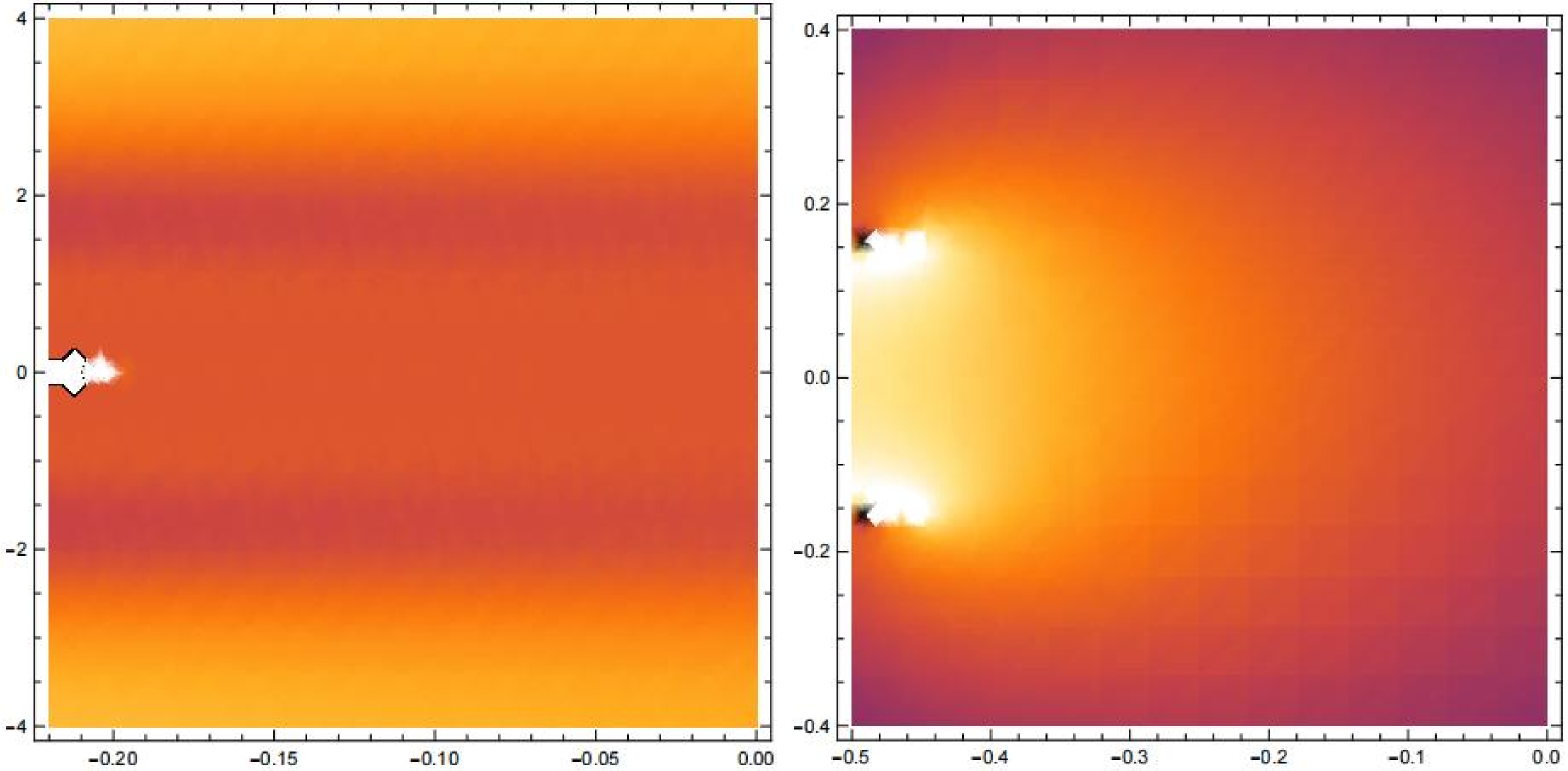


(a) Black dots are $\frac{a_{n+1}}{na_n}$ and red line is a fit $\frac{1}{A}(1+\alpha/n)$ for a fixed $w = 0.1$, where $\alpha = 0.005$ and $\frac{1}{A} = -5.431$.



(b) Black dots are $\frac{a_{n+1}}{na_n}$ for $w = 0.32$.

Borel plane



(a) $w = 0.1,\ n = 200$ (b) $w = 0.8,\ n = 200$

Numerical evaluation of Borel singularities of the asymptotic series for $w = 0.1$ and $w = 0.8$. Here the real part of $\mathrm{Re}\mathcal{B}(Z)(t)$ is plotted versus $t \in \mathbb{C}$.

Saddle point and LOB

$$Z(\lambda, w) = \int \mathrm{d}\phi\, \mathrm{d}\tilde{\phi}\, \mathrm{e}^{-(\phi-w)\tilde{\phi}+\lambda\tilde{\phi}^2(1-\mathrm{e}^{-\phi})}$$

⇩ Perturbative coefficients

$$a_n = \frac{1}{n!}\int d\tilde{\phi}\, d\phi\, \mathrm{e}^{-\tilde{\phi}(\phi-w)}\left[-(\mathrm{e}^{-\phi}-1)\,\tilde{\phi}^2\right]^n$$

⇩

To analyze the LOB as $n \to \infty$, we rescale $\tilde{\phi} \to n\,\tilde{\phi}$

⇩

$$\begin{cases} a_n = \dfrac{n}{n!}\displaystyle\int d\tilde{\phi}\, d\phi\, \mathrm{e}^{-n(\mathcal{S}+2\ln n)} \\ \mathcal{S}(\tilde{\phi}, \phi) = \tilde{\phi}(\phi - w) - \ln\left(\mathrm{e}^{-\phi}-1\right) - 2\ln\tilde{\phi} \end{cases}$$

$$\lambda \equiv \frac{1}{m^4}$$

Fixed point is reached when $m \to 0$, or $\lambda \to \infty$.

Saddle point and LOB

$$a_n = \frac{n}{n!} \int d\tilde{\phi}\, d\phi\, \mathrm{e}^{-n(\mathcal{S}+2\ln n)}$$

The saddle-point equations are:

$$\frac{\partial \mathcal{S}}{\partial \tilde{\phi}} = -(\phi - w) + \frac{2}{\tilde{\phi}} = 0 \quad \Longrightarrow \quad \tilde{\phi} = \frac{2}{\phi - w}.$$

$$\frac{\partial \mathcal{S}}{\partial \phi} = -\tilde{\phi} + \frac{\mathrm{e}^{-\phi}}{1 - \mathrm{e}^{-\phi}} = 0 \quad \Longrightarrow \quad \tilde{\phi} = \frac{1}{\mathrm{e}^{\phi} - 1}.$$

$$\tilde{\phi}_{\mathrm{sp}} = -\frac{2}{2 + W_k(-2\mathrm{e}^{-2+w})},$$

$\phi_{\mathrm{sp}} = -2 + w - W_k(-2\mathrm{e}^{-2+w})$, where $k \in \mathbb{Z}$ indexes the k-th branch of the Lambert W-function.

Saddle point and LOB

The LOB of the coefficient is given by:

$$a_n \approx \frac{(2n)!}{(n!)\sqrt{4\pi n}} C_k(w)^{-n} \frac{1}{\sqrt{h_k(w)}}$$

$$h_k(w) = W_k\left(-2e^{w-2}\right) + 1$$

$$C_k(w) = -W_k\left(-2\mathrm{e}^{w-2}\right)\left(W_k\left(-2\mathrm{e}^{w-2}\right) + 2\right)$$

we can define the Borel transform as:

$$I_k(w,t) = \frac{1}{2\pi}\frac{1}{\sqrt{h_k(w)}}\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}\sqrt{1 + \frac{4te^{-x}}{C_k(w)}}}$$

Saddle point and LOB

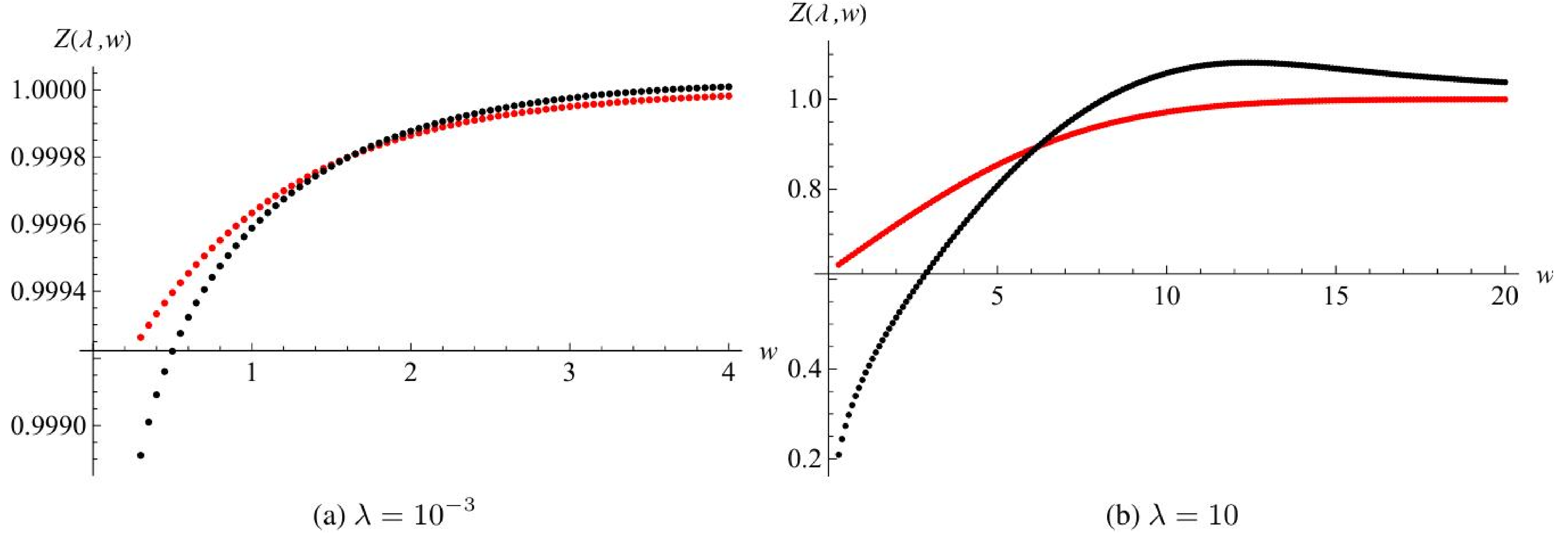
$$I_k(w,t) = \frac{1}{2\pi}\frac{1}{\sqrt{h_k(w)}}\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}\sqrt{1+\frac{4te^{-x}}{C_k(w)}}}$$

The position of the singularity has a branch cut that goes along the positive real line starting at w=0.31.

The multiple solutions of the saddle point equation reflect the fact that the position of our singularity is not an entire function of w !

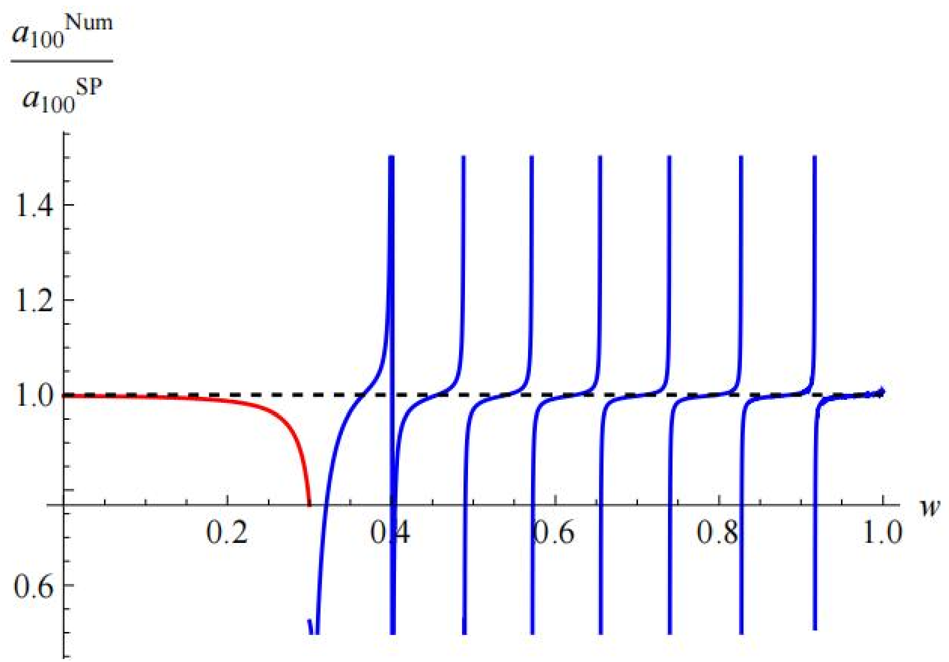
We have to define a lateral resumation:

$$\mathcal{S}_\theta^{\pm} Z(\lambda, w) = \int_0^\infty \frac{\mathrm{d}t}{\lambda}\mathrm{e}^{\frac{-t}{\lambda}} I_0(w \pm i\epsilon), \quad \epsilon \to 0$$

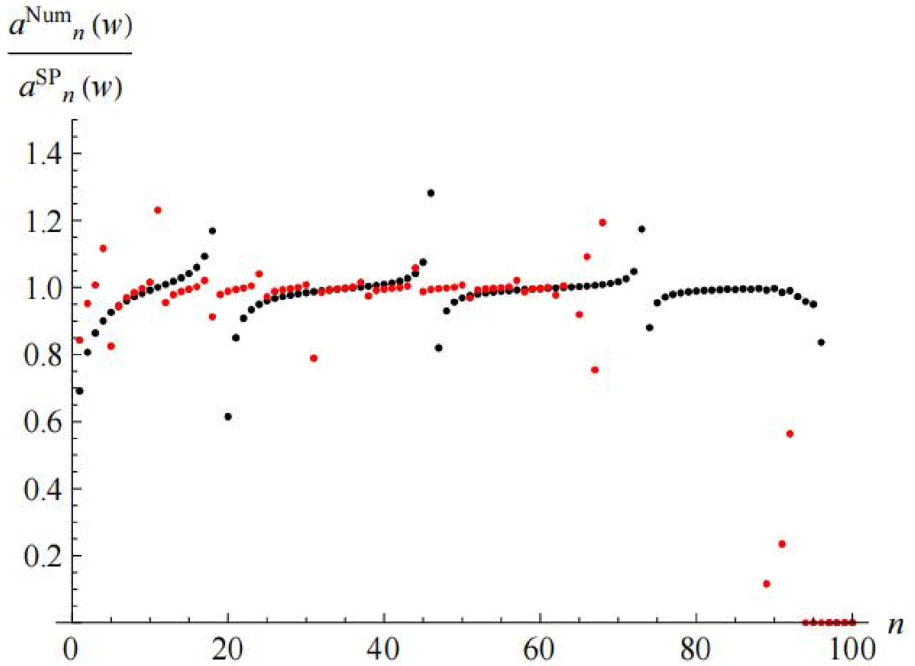


(a) $\lambda = 10^{-3}$ (b) $\lambda = 10$

Black dots correspond to the result of lateral Borel resumation while red ones correspond to the numerical integration of the partition function.



The ratio of the n = 100 coefficient computed numerically to its saddle-point approximation.



The ratio of the coefficient computed numerically to its saddle-point approximation as a function of n for w = 0.5 (black) and w = 1 (red).

$$a_n(w \pm i\epsilon) \;\approx\; \frac{(2n)!}{n!\,\sqrt{4\pi n}}\,\frac{1}{\sqrt{h_0(w)}}\,|C_0(w)|^{-n}\cos\!\left(n\,\arg\left(C_0(w)\right) - \arg\!\left(\frac{1}{\sqrt{h_0(w)}}\right) + \frac{\pi}{256}\right)$$

Possible ansatz for a transseries

To understand better the problem it is instructive to find a differential equation for the partition function.

$$\partial_w Z(w,\lambda) = \lambda\partial_w^2 Z(w,\lambda) - \lambda\partial_\lambda Z(w,\lambda).$$

problems for λ=0 → the boundary layer requires a WKB-like exponential to resolve the singularity, this structure hints that the correct ansatz is a **transseries**

boundary conditions: $Z(w,0) = 1 - \mathrm{e}^{-w}, \quad Z(\infty,\lambda) = 1$

We consider a general transseries ansatz:

$$Z(w,\lambda) = \sum_{k=0}^{1} \sigma(w)^k \mathrm{e}^{-k\frac{A(w)}{\lambda}} \sum_n a_{n,k}(w)\lambda^n$$

↓ equalizing the coefficients in front of λ^{-1}

$$(\partial_w A(w))^2 + \partial_w A(w) = A(w).$$

↓

$$A(w) = \frac{1}{4}W\left(-e^{w-1\pm 2c_1}\right)\left(2 + W\left(-e^{w-1\pm 2c_1}\right)\right)$$

Recusrion relations for a 1-instanton sector coeficients

We can obtain the following recursive differential equation for the perturbative coefficients of the 1-instanton sector:

$$G_n(w)\left(nG_0(w)^2-\frac{G_0'(w)}{G_0(w)}\right)+G_n'(w)+G_0(w)^2\left(-G_{n-1}''(w)\right)=0$$

for $G_n(w)=\sigma(w)a_{1,n}$

We can solve for n=0: $G_0(w)=\dfrac{C_0}{\sqrt{1+W(z)}}$, where $z=-2\,e^{w-2}$ and C_0 is a constant.

$$a_n\sim\frac{\Gamma(n+\alpha)}{A^{n+\alpha}}\left(b_0+\frac{b_1A}{n+\alpha-1}+\frac{b_2A^2}{(n+\alpha-1)(n+\alpha-2)}+\cdots\right),$$

Another assymptotic relation for LOB that realtes perturbative coefficients to the coefficients of the 1-instanton fluctuation series.

The LOB of the coefficient is given by:

$$a_n\approx\frac{(2n)!}{(n!)\sqrt{4\pi n}}C_k(w)^{-n}\frac{1}{\sqrt{h_k(w)}}$$

$$h_k(w)=W_k\left(-2e^{w-2}\right)+1$$

This fixes the constant to 1: $Z_{\text{Instanton}}\approx e^{-S[\phi_1]/\lambda}\cdot\left[\dfrac{1}{\sqrt{\text{Det}\left(\frac{\delta^2S}{\delta\phi^2}|_{\phi_1}\right)}}\right]\cdot(1+\mathcal{O}(\lambda))$

Challenges

- The non-analiticity comes from a branch cut in w not in t
- We do not integrate in w, hence there is no convolution in w.
- Transseries seems to be a correct ansatz
- Resummation for $w < 0.31$ is effected by the resummation for $w > 0.31$

Thank you for your attention!

Martin-Siggia-Rose formalism

$$\partial_t u(x,t) = \mathcal{V}\big(x, u(x,t)\big) + \xi(x,t), \quad \langle \xi(x,t)\xi(x',t')\rangle = D(x,x',t,t')$$

Here $\mathcal{V}\big(x, u(x,t)\big)$ depends on the field and its spatial derivatives at a single moment of time and does not involve time derivatives.

$\xi(x,t)$ is a Gaussian random force with zero mean and a prescribed two-point correlator $D(x,x',t,t')$.

This equation is equivalent to some field theory with doubled amount of fields.

Quenched Edwards–Wilkinson equation

$$\eta\partial_t u(x,t) = c\nabla^2 u(x,t) + F\big(u(x,t),x\big) + m^2\big(w - u(x,t)\big).$$

First, we need to discretize the equation in time:

$$u(x,t+\delta t) = u(x,t) + \frac{\eta}{\delta t}\Big[c\nabla^2 u(x,t) + m^2\big(w - u(x,t)\big) + F\big(u(x,t),x\big)\Big]$$

We want to compute expectations $\langle \mathcal{O}\big(u(x,t+\delta t)\big)\rangle$ for a given realization of the Gaussian random variable $F\big(u(x,t),x\big)$.

Instead of solving it directly, we can implement constraints by introducing a series of δ-functions in the integral.

$$\langle \mathcal{O}(u(x,t+\delta t))\rangle = \frac{\eta}{\delta t}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}\tilde{u}(x,t)}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}u(x,t+\delta t)\,\mathcal{O}(u(x,t+\delta t))$$
$$\times \mathrm{e}^{\tilde{u}(x,t)\left[\frac{\eta}{\delta t}\left(u(x,t+\delta t)-u(x,t)\right)-\nabla^2 u(x,t)-m^2[w-u(x,t)]-F\left(x,u(x,t)\right)\right]}$$

Quenched Edwards–Wilkinson equation

We represented delta-functions via their Fourrier transform by introducing an auxiliary field $\tilde{u}(x,t)$.

We have used the so-called Itô discretization, where the r.h.s. of qEW is evaluated at time *t*. We have used a property of Dirac delta function:

$$\delta\big(f(x)\big)|f'(x)| = \delta(x - x^*), \text{ for any } f(x) \text{ with a single zero at } x^*.$$

In infinite dimensional case, for our equation this yields:

In our scheme, the Jacobian determinant is unity.

$$\prod_x \prod_t \mathrm{d}u(x,t)\delta\big(u(x,t) - u_*(x,t)\big) = \prod_x \prod_t \mathrm{d}u(x,t)\,\delta\big(\mathcal{F}(u(x,t))\big)\left|\mathrm{Det}[\frac{\delta \mathcal{F}}{\delta u}]\right|$$

where $\mathcal{F}(u(x,t)) = \eta\partial_t u(x,t) - \nabla^2 u(x,t) - m^2[w - u(x,t)] - F(x,u(x,t))$.

Quenched Edwards–Wilkinson equation

The strategy forward is now clear: define the path-integral measure:

$$\int \mathcal{D}[\tilde{u}]\mathcal{D}[u] := \prod_x \prod_t \frac{\eta}{\delta t} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tilde{u}(x,t)}{2\pi i} \int_{-\infty}^{\infty} \mathrm{d}u(x, t+\delta t)$$

and action

$$\mathcal{S}[u, \tilde{u}, F] = \int_{x,t} \tilde{u}(x,t)\Big[(\eta\partial_t - \nabla^2 + m^2)(u(x,t) - w) - F(x, u(x,t))\Big]$$

The expectation of an observable $\mathcal{O}$ reads: $\langle \mathcal{O} \rangle = \int \mathcal{D}[\tilde{u}]\mathcal{D}[u]\, \mathrm{e}^{-\mathcal{S}[u,\tilde{u},F]}\, \mathcal{O}.$

The final step is to average over disorder, which by assumption is Gaussian

$$\mathcal{S}\big[u, \tilde{u}\big] = \int_{x,t} \tilde{u}(x,t)\Big[\big(\eta\partial_t - \nabla^2 + m^2\big)\big(u(x,t) - w\big)\Big] - \frac{1}{2}\int_{x,t,t'} \tilde{u}(x,t)\Delta_0\big(u(x,t) - u(x,t')\big)\tilde{u}(x,t')$$