

BPS countings and Resurgence of Topological String

Simon Douaud

Based on arXiv:2412.16140 with **Amir-Kian Kashani-Poor**.



The big picture

Resurgence properties of the **topological string** amplitudes on
(compact one parameter) **Calabi-Yau manifolds** are strongly related
to **BPS states** counting. .

Structure of the talk

- One parameter Calabi-Yau manifolds, periods and BPS states
- Topological string and boundary conditions
- Resurgence
- Numerical results

One parameter Calabi-Yau 3-fold and Mirror symmetry

X is a Calabi-Yau 3-fold : $\exists!$ non vanishing $(3,0)$ closed form Ω .

X can be “deformed” in 2 ways :

- Deformation of the Kähler metric : $h^{1,1}(X)$ moduli parameters.
- Deformation of the complex structure : $h^{2,1}(X)$ moduli parameters.

Mirror symmetry : there exist a Calabi-Yau manifold X^* such that

- $h^{1,1}(X^*) = h^{2,1}(X)$
- $h^{2,1}(X^*) = h^{1,1}(X)$

The two deformations are exchanged.

One parameter Calabi-Yau : $h^{2,1}(X^*) = h^{1,1}(X) = 1$.

Period geometry on the mirror

Let z^i be the complex deformation parameters of X^* .

A central role is played by the periods of $\Omega(z)$,

$$X^I(z) = \int_{A_I} \Omega(z), \quad P_I(z) = \int_{B^I} \Omega(z), \quad (1)$$

defined with regard to a symplectic basis of $H_3(X^*, \mathbb{Z})$,

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J. \quad (2)$$

In the following, we will organize the periods in a period vector

$$\Pi = (P_I, X^I). \quad (3)$$

$$e^{-K} = i \int \Omega \wedge \bar{\Omega}, \quad C_{ijk} = \int \Omega \wedge \partial_i \partial_j \partial_k \Omega, \quad i, j, k = 1, \dots, h^{1,1}(X), \quad (4)$$

Period geometry on the mirror

We specify the study to the so called one parameter hypergeometric Calabi-Yau manifold where the period on X^* satisfy a Pichard-Fuchs differential equation.

$$[\theta^4 - \mu^{-1}z \prod_{k=1}^4 (\theta + a_k)]f(z) = 0, \quad \theta = z \frac{d}{dz}. \quad (5)$$

$$\mathcal{P} \left\{ \begin{array}{ccc} 0 & \mu & \infty \\ 0 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 2 & a_4 \end{array} \right\}. \quad (6)$$

$z = 0$: Large Radius point (X goes to infinite volume)

$z = \mu$: Conifold point, a cycle of X^* goes to zero size.

Period geometry on the mirror

$z = \infty$ classification depending on the exponents

- $(a_1, a_2, a_3, a_4) = (a, b, c, d)$: R-points, X_5 , X_6 , X_8 , X_{10} , $X_{4,3}$ and $X_{6,4}$.
- $(a_1, a_2, a_3, a_4) = (a, b, b, c)$: C-points, $X_{4,2}$, $X_{6,2}$ and $X_{3,2,2}$.
- $(a_1, a_2, a_3, a_4) = (a, a, b, b)$: K-points, $X_{3,3}$, $X_{4,4}$, and $X_{6,6}$.
- $(a_1, a_2, a_3, a_4) = (a, a, a, a)$: M-points, $X_{2,2,2,2}$.

Period geometry on the mirror

Non trivial monodromy M_{pt} around singular points ($pt = 0, \mu, \infty$).

At $z = 0$ we can find a special basis of solution for which the monodromies are integral, i.e $M_{pt} \in Sp(4, \mathbb{Z})$. This is the large radius basis $\mathbf{\Pi} = (P_I, X^I)$. A cycle γ in $H_3(X^*, \mathbb{Z})$ is represented by its charge vectors \mathbf{q} .

$$\int_{\gamma} \Omega = \mathbf{q} \cdot \mathbf{\Pi} = (q_0 P_0 + q_1 P_1 + q_2 X^0 + q_3 X^1) \quad (7)$$

where $\mathbf{q} = (q_0, q_1, q_2, q_3)$ are $D_6 D_4 D_0 D_2$ charges.

Orbits of a charge \mathbf{q} , $\{M_{\infty}^k \mathbf{q}\}$ are important.

For some models, very simple monodromy for example : monodromy group of X_{10} around $z = \infty$ is \mathbb{Z}_{10} i.e $M_{\infty}^{10} = 1$.

Special periods at $z = \infty$

We associate a BPS state to a cycle in H_3 .

Expanding in the Large radius basis $\Pi = (P_I, X^I)$. We have the mass formula

$$M(\mathbf{q}) = e^{K/2} \mathbf{q} \cdot \Pi \quad (8)$$

For certain model can build a 2 dimensional lattice of massless periods $\mathbb{Z}\mathbf{q}_1 + \mathbb{Z}\mathbf{q}_2$ (relates to number theory).

- for R-points : $X_6, X_{4,3}$ and $X_{6,4}$ i.e for $X_{4,3}$ $\mathbf{q}_1 = (1, 0, 0, -2)$, $\mathbf{q}_2 = (1, 1, -3, 1)$.
- All K-points i.e for $X_{3,3}$ $\mathbf{q}_1 = (1, 0, 0, -3)$, $\mathbf{q}_2 = (1, 1, -3, 1)$.
On this subspace M_∞ acts as \mathbb{Z}_n ($n=3$ for $X_{3,3}$).
- All C-points i.e for $X_{6,2}$ $\mathbf{q}_1 = (2, 1, -4, 0)$, $\mathbf{q}_2 = (1, 0, 0, -1)$.

Around $z = 0$, we can compute the BPS invariant associated to the charge \mathbf{q} , $\Omega(\mathbf{q}, z)$.

[2023, *Alexandrov, Pioline, Klemm, Feyzbakhshf, Schimannek*]

Known for \mathbf{q} of the form $(\pm 1, q_2, q_3, q_4)$ (rank 1) and of the form $(0, q_2, q_3, q_4)$.

For $\mathbf{q} = (0, 0, m, d)$, $\Omega(\mathbf{q}, z = 0) = n_0^d$, genus 0 GV invariant.

For $\mathbf{q} = (1, 0, 0, 0)$, $\Omega(\mathbf{q}, z = 0) = 1$.

Topological A and B model

Non-linear sigma model on X (Calabi-Yau metric $g_{i\bar{j}}$ and coordinates ϕ^i)

$$S(\Sigma_g) = \int_{\Sigma_g} g_{i\bar{j}} \bar{\partial} \phi^i \partial \bar{\phi}^{\bar{j}} + \text{fermionic}. \quad (9)$$

Make the theory topological (independent of world-sheet metric) by twisting : lead to two models, A and B.

- A-model : space of marginal deformation of the theory \cong space of Kähler deformation of X .
- B-model : space of marginal deformation of the theory \cong space of complex structure deformation of X .

From mirror symmetry : A model on $X =$ B model on X^* .

Holomorphic anomaly equation

Let z be a deformation parameter of the theory. One deforms the action by adding marginal operator to the action

$$S[z, \bar{z}] = S + z^i O_i + \bar{z}^{\bar{i}} \bar{O}_{\bar{i}}. \quad (10)$$

Topological amplitudes

$$F_g(z, \bar{z}) = \int_{\mathcal{M}_g} \int D[\phi, \psi] O_g^{BRST} e^{iS[z, \bar{z}]}. \quad (11)$$

Satisfies holomorphic anomaly equations

$$\partial_{\bar{k}} F_g = \frac{1}{2} C_{\bar{k}}^{ij} [D_i D_j F_{g-1} + \sum_{r=1}^{g-1} D_i F_r D_j F_{g-r}], \quad D_i F_g = (\partial_i + (2g-2)\partial_i K) F_g. \quad (12)$$

F_g can be computed up to a holomorphic ambiguity f_g .

Associated with boundary conditions : can solve for F_g to high genus (up to 64 for X_5). [2006, Huang, Klemm, Quackenbush]

Frame, Boundary conditions

An holomorphic limit of $F_g(z, \bar{z}) \rightarrow F_g(X_*^I)$ is specified by a choice of A-period (X_*^I) .

At $z = 0, \mu$: universal behavior of F_g in certain frames.

- $z = 0$: Gopakumar-Vafa formula, in the large radius frame specified by the A periods (X^I) .

$$F_g(X^0, X^1) = \sum_{d>0, m} n_0^d \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{1/2}}{dX^1 + mX^0} \right)^{2g-2} + \dots \quad (13)$$

(d, m) are D_2, D_0 charges.

- $z = \mu$: the period P_0 goes to 0 at this point, in the frame (P_0, X^1) we have the singular “gap” behavior

$$F_g(P_0, X^1) = \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{1/2}}{P_0} \right)^{2g-2} + \text{regular}. \quad (14)$$

Resurgence in a nutshell

From Boundary conditions $F_g \sim (2g-2)!$. $F = \sum_g F_g g_s^{2g-2}$ is asymptotic.

Borel transform

$$B[F](\zeta) = \sum_g \frac{F_g}{(2g-2)!} \zeta^{2g-2} \sim -\frac{S_{\mathcal{A}}}{2\pi} \left(\frac{F_{-1}^{\mathcal{A}}}{\zeta - \mathcal{A}} + \text{Log}(\zeta - \mathcal{A}) B[F_{\mathcal{A}}(\zeta - \mathcal{A})] \right) \quad (15)$$

$$F_{\mathcal{A}}(g_s) = \sum_{n \geq 0} F_n^{\mathcal{A}} g_s^n. \quad (16)$$

\mathcal{A} = Borel singularity.

$S_{\mathcal{A}}$ = Stokes Constant.

$(\mathcal{A}, S_{\mathcal{A}})$ = Resurgence data.

$F_n^{\mathcal{A}}$ can be computed in term of F_g using an operator formalism.

[2023, Gu, Kashani-Poor, Klemm, Marino]

Resurgence in a nut-shell

Borel re-summation provides a family of well defined functions (i.e non perturbative) whose expansion around $g_s = 0$ equal to F

$$s_\theta(F) = \frac{1}{g_s} \int_0^{e^{i\theta}\infty} B[F](x) e^{-x/g_s}. \quad (17)$$

Discontinuity when $\theta = \text{Arg}(\mathcal{A}_0)$

$$\begin{aligned} \text{Disc}_{\text{Arg}(\mathcal{A}_0)}(F) &= s_{\text{Arg}(\mathcal{A}_0)^+}(F) - s_{\text{Arg}(\mathcal{A}_0)^-}(F) \\ &= i S_{\mathcal{A}_0} s_{\text{Arg}(\mathcal{A}_0)^-} \left(\sum_{n \geq -1, \ell > 0} F_n^{\ell \mathcal{A}_0} g_s^n e^{-\ell \mathcal{A}_0 / g_s} \right). \end{aligned}$$

Resurgence data of F from boundary conditions

Asymptotics of F_g from resurgence data

$$F_g \sim (2g-2)! \sum_{\mathcal{A}} \frac{S_{\mathcal{A}}}{\mathcal{A}^{2g-1}} \left[F_{-1}^{\mathcal{A}} + \frac{F_0^{\mathcal{A}} \mathcal{A}}{2g-2} + \dots \right]. \quad (18)$$

From boundary conditions one can read off some of the resurgence data at $z=0$ and $z=\mu$, ($\ell \in \mathbb{N}$)

- at $z=0$, $\mathcal{A} = \aleph \ell (dX^1 + mX^0)$, $S_{\ell,d,m} = n_0^d = \Omega((0,0,m,d), z=0)$
- at $z=\mu$, $\ell \mathcal{A} = \aleph P_0$, $S_{\ell} = 1 = \Omega(P_0, z=0)$.

Resurgence data at $z=0, \mu$ is of the form

$$(\aleph \ell \mathbf{q} \cdot \mathbf{\Pi}, \Omega(\mathbf{q}, z=0)). \quad (19)$$

Hypothesis : For any z the resurgence data is of the form

$$(\aleph \ell \mathbf{q} \cdot \mathbf{\Pi}, \Omega(\mathbf{q}, z)). \quad (20)$$

And if $S_{\mathbf{q}} \neq \Omega(\mathbf{q}, z=0)$ it is because of wall-crossing.

- What about $z = \infty$?
- What about the resurgence of

$$F_g^{reg} = F_g - \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{1/2}}{P_0} \right)^{2g-2} \quad (21)$$

at $z = \mu$?

\Rightarrow no analytical control for most models : need to use numerics.

Numerical tools to extract Resurgence Data

Padé approximation : $B[F] \sim \frac{P_g(g_s)}{Q_g(g_s)}$, P_g, Q_g polynomial.
Roots of Q_g accumulate around line starting at \mathcal{A} .

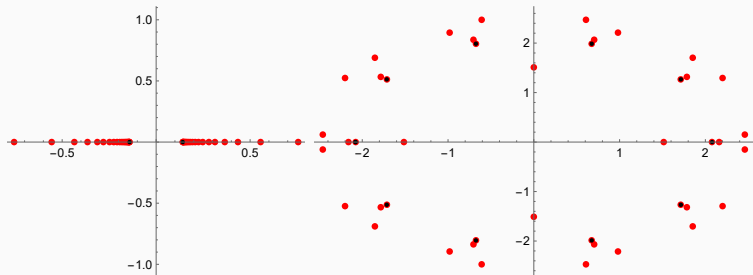


Figure 1: Example of Borel plane for X_5 around $z = \mu$ (left) and around $z = \infty$ (right).

Numerical tools to extract resurgence data

From asymptotic

$$S_{\mathcal{A}} \sim \frac{2\pi}{2} \left(\frac{(2g-2)!F_{-1}^{\mathcal{A}}}{\mathcal{A}^{2g-1}} + \frac{(2g-3)!F_0^{\mathcal{A}}}{\mathcal{A}^{2g-2}} + \dots \right)^{-1} F_g, \quad (22)$$

when \mathcal{A} is leading (i.e smallest singularity).

$$S_{\mathcal{A}} \sim -ie^{\mathcal{A}/g_s} \frac{\text{Disc}_{\arg(\mathcal{A})}(F)}{\sum_{n \geq -1} F_n^{\mathcal{A}} g_s^n}. \quad (23)$$

Disc_{θ} is computed numerically from the residue of the Padé approximation. Removing contribution of singularity

$$F_g - (2g-2)! \frac{S_{\mathcal{A}}}{\mathcal{A}^{2g-1}} \left[F_{-1}^{\mathcal{A}} + \frac{F_0^{\mathcal{A}} \mathcal{A}}{2g-2} + \dots \right]. \quad (24)$$

Numerical results : X_6

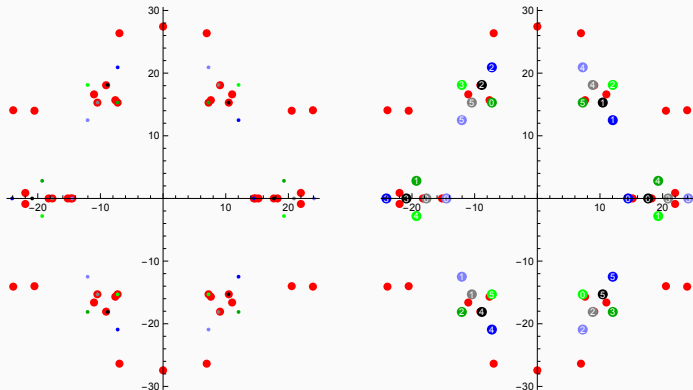


Figure 2: Borel plane of X_6 at $z = 10^4 \mu$ in the frame $(\mathbf{q}_1, (0, 1, -1, 0))$. The (light) blue points are orbits of $(-)\backslash P_0$ under M_∞ , the (light) green points of $(-)\backslash X^0$, the (gray) black points of $(-)\backslash X^1$.

Numerical results : X_6

- The massless periods are not Borel singularities i.e $S_{\mathcal{A}} = 0$ but at $z = 0$, $\Omega(\mathbf{q}, z = 0) = 0$ so not stable at $z = 0$ and not stable at $z = \infty$ as well.
- Monodromy invariance : if $\mathbf{q} \cdot \mathbf{\Pi}$ is a singularity then $M_{\infty}^k(\mathbf{q}) \cdot \mathbf{\Pi}$ is.
- Stokes constant of orbits of X^0 , P_0 and X^1 are equal to $\Omega(X^0/P_0/X^1, z = 0)$.

Numerical results : X_{10}

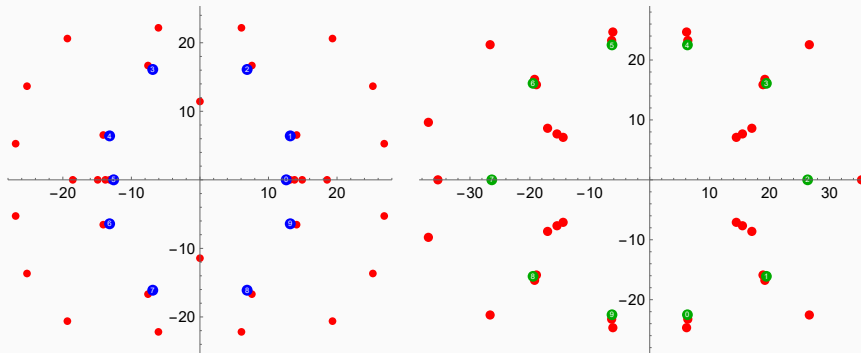


Figure 3: Borel plane of X_{10} in the frame (P_0, X^1) at $z = 10^4 \mu$. On the right, the leading singularities are subtracted. Blue dots indicate orbit elements of $\mathbb{N}P_0$ under M_∞ , green dots orbit elements of $\mathbb{N}X^0$.

Stokes constant of orbits of X^0 , P_0 and X^1 are equal to $\Omega(P_0/X^0, z = 0)$.

Numerical results : $X_{4,3}$

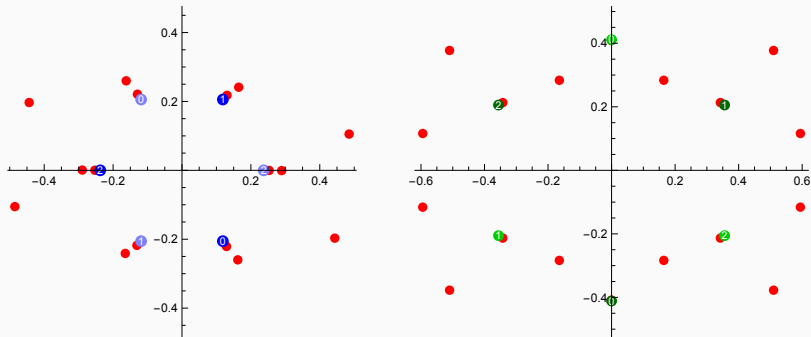


Figure 4: Borel plane of $X_{4,3}$ at $z = 10^8 \mu$ in the frame $(\mathbf{q}_1, (0, 1, -2, 0))$. On the right, the leading singularities are subtracted. The (light) blue dots indicate the orbit elements of $\aleph \mathbf{q}_1 \cdot \Pi$ ($-\aleph \mathbf{q}_1 \cdot \Pi$) under the monodromy M_∞ , the (light) green dots the orbit elements of $\aleph(\mathbf{q}_1 - \mathbf{q}_2) \cdot \Pi$ ($\aleph(\mathbf{q}_2 - \mathbf{q}_1) \cdot \Pi$).

$$S_{M_\infty^k \mathbf{q}_1} = 27 = \Omega(\mathbf{q}_1, z = 0).$$

Numerical results : $X_{4,4}$

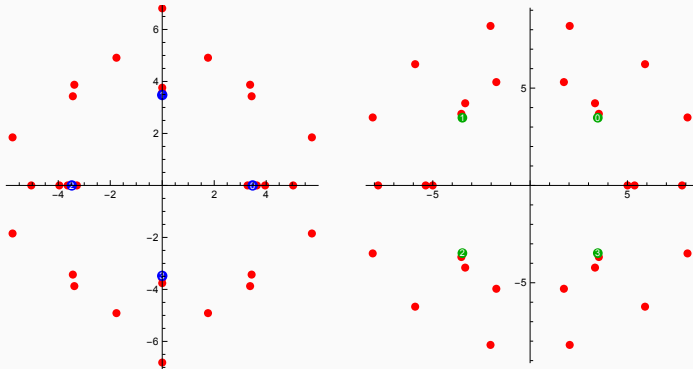


Figure 5: Borel plane of $X_{4,4}$ at $z = 10^6 \mu$ in the frame $(\mathbf{q}_1 \cdot \Pi, P_0)$. On the right, the leading singularities are subtracted. The blue dots indicate the orbit elements of $\aleph \mathbf{q}_1 \cdot \Pi$ ($-\mathbf{q}_1 = \mathbf{q}_1 \cdot M_\infty^2$) under the monodromy M_∞ , the green dots the orbit elements of $\aleph(\mathbf{q}_1 - \mathbf{q}_2) \cdot \Pi$.

$S_{M_\infty^k \mathbf{q}_1} = 1408 = \Omega(\mathbf{q}_1, z = 0)$. $S_{M_\infty^k(\mathbf{q}_1 + \mathbf{q}_2)} = 9984 \neq \Omega(\mathbf{q}_1 + \mathbf{q}_2, z = 0)$.
Wall crossing ?

C-point : analytical behaviour

For those models : lattice of vanishing periods $\mathbb{Z}\mathbf{q}_1 + \mathbb{Z}\mathbf{q}_2$.

In a A-frame containing $\mathbf{q}_1 \cdot \Pi_{LR}$. we have the following behaviour of F_g for X_{42} and X_{62}

$$F_g = S \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{1/2}}{\mathbf{q}_1 \cdot \Pi_{LR}} \right)^{2g-2} + \text{regular}, \quad (25)$$

with $S = 2$ for $X_{4,2}$ and $S = 1$ for $X_{6,2}$. $\Rightarrow \mathcal{A} = \mathbb{N}\ell \mathbf{q}_1 \cdot \Pi_{LR}, S_\ell = S$.

No more singular terms $\Rightarrow S_{\mathbf{q}_2} = 0$, coherent with the fact that for the two models $\Omega(\mathbf{q}_2, z = 0) = 0$, no wall crossing. Also verified numerically.

Wall-crossing around conifold point

$$F_g^{red} = F_g - \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{1/2}}{P_0} \right)^{2g-2} \quad (26)$$

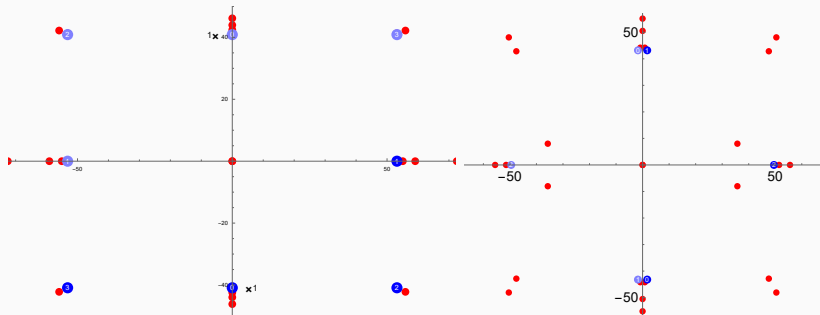


Figure 6: Borel plane of X_5 in the frame (P_0, X^1) upon removing the leading gap singularity, at $z = 1.3\mu$ on the right, $z = 0.7\mu$ on the left. (Light) blue dots : $(-)\aleph\{X^0, P_0 - X^0, X^1, X^1 + X^0, X^1 - X^0\}$. At $z = 0.7\mu$ there is no red points associated to $\pm(P_0 - X^0)$ (number 1). At $z = 1.3\mu$ we have enough precision to identify $\aleph(X^1 + X^0)$ and $\aleph(X^1 - X^0)$.

Wall-crossing around conifold point

Decay of $X^0 - P_0 = \gamma_1 + \gamma_2$, $\gamma_1 = X^0$, $\gamma_2 = -P_0$.

$$\Omega(\gamma_1 + \gamma_2, \mu^-) = 0.$$

$$\Omega(\gamma_1 + \gamma_2, \mu^+) - \Omega(\gamma_1 + \gamma_2, \mu^-) = (-1)^{\langle \gamma_1, \gamma_2 \rangle - 1} |\langle \gamma_1, \gamma_2 \rangle| \Omega(\gamma_1, \mu) \Omega(\gamma_2, \mu) \quad (27)$$

$$\Rightarrow \Omega(X^0 - P_0, \mu^+) = \Omega(X^0, \mu), \quad (28)$$

verified numerically for almost all model.

Conclusion and questions

- Numerical evidence for our hypothesis :
For any z the resurgence data is of the form

$$(\aleph \ell \mathbf{q} \cdot \Pi_{LR}, \Omega(\mathbf{q}, z)). \quad (29)$$

And if $\Omega(\mathbf{q}, z) = S_{\mathbf{q}} \neq \Omega(\mathbf{q}, z = 0)$ it is because of wall-crossing.

- Can we get those results analytically ?
- Are all BPS states present in the resurgence data ? Or only a “topological ” sub-sector ?
- Importance of choice of frame