A Topological Recursion Algorithm for WKB Solutions

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Eynard–Orantin Topological Recursion

- Framework that recursively computes a set of invariants from a small set of initial data known as a spectral curve.
- First by E-O in 2007, universal structure governing the large N asymptotic expansion of matrix models.
- Correlation functions of the matrix model, when expanded in powers of $1/N^2$, could be computed universally and recursively. The expansion of the correlation function $F = \sum_{g=0}^{\infty} N^{2-2g} F_g$ was interpreted as a sum over the genus of Riemann surfaces.

Eynard–Orantin Topological Recursion

- Input: spectral curve (compact Riemann surface) Σ and $\omega_{0,2}$ and $\omega_{0,1}$
- Output: meromorphic n-forms $\omega_{g,n}$ on Σ^n for $2g-2+n>0$.
- Mirzaknani's recursion formula for Weil-Peterson volume forms.
- Enumerative geomeotry: Gromov-Witten invariants, Hurwitz numbers...

$$\omega_{g,n+1}(z_0, z_I) = \sum_{a \in \mathrm{Ram}(x)} \mathrm{Res}_{z \to a} K(z_0, z) \left(\omega_{g-1,n+2}(z, \sigma(z), z_I) + \sum_{\substack{h+h'=g \\ I \sqcup J = I}}' \omega_{h,|I|+1}(z, z_I)\,\omega_{h',|J|+1}(\sigma(z), z_J) \right) \tag{1}$$

Airy Structures

- An Airy structure (Kontsevich–Soibelman) is a family of at most quadratic operators $\{H_i\}$ satisfying relations

$$[H_i, H_j] = \hbar \sum_k f_{ij}^k H_k.$$

- One seeks a partition function

$$Z = \exp\Big(\sum_{g\geq 0} \hbar^{g-1} F_g\Big)$$

  such that $H_i Z = 0$ for all i.
- Z is uniquely determined and encodes TR invariants of a spectral curve.
- Deformation of spectral curves gives residue constrains, which produce the A, B, C, ε tensors.

Part II: Context and Goal

- In this part, we study WKB solutions of differential operators

$$P(x, \hbar\partial_x; \hbar) \in \mathbb{C}[x][\hbar\partial_x][[\hbar]].$$

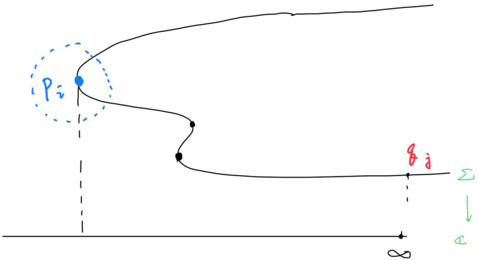
- We assume that the associated spectral curve has only simple ramification.
- Goal: construct a formal WKB solution of $P\psi = 0$ using a topological recursion algorithm.

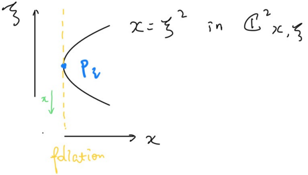
Spectral Curves from Differential Operators

- Write
$$P(x, \hbar\partial_x; \hbar) = \sum_{i\geq 0} \hbar^i \sum_{j,k\geq 0} P_{ijk}\, x^j (\hbar\partial_x)^k. \tag{2}$$
- Phase space: $h\partial_x \sim \xi$. The principal symbol $P_0(x, \xi)$ is defined as the $\hbar$ zeroth-order term.
- The *spectral curve* is
$$\Sigma = \{(x, \xi) \in T^*\mathbb{C} \mid P_0(x, \xi) = 0\}.$$
- Symplectic structure on $T^*\mathbb{C}$ is $dx \wedge d\xi$, and the foliation $x = const$.
- Simple ramification $\partial_\xi P_0 = 0$, $\partial x P_0 \neq 0$ and $\partial_\xi^2 P_0 \neq 0$.

Simple Ramification Assumption

- $\pi : \Sigma \to \mathbb{C}$.
- Locally the spectral curve looks like a fold: $x \sim z^2$.





Local Factorisation to an Airy-Type Operator

- Weierstrass division theorem implies:
  - Microlocally near each simple ramification point, P can be factorised as

$$P = U \circ Q,$$

    where U is microlocally invertible.
  - Q is a second-order operator of the form

$$Q = (\hbar\partial_x)^2 + A(x;\hbar)\,\hbar\partial_x + B(x;\hbar),$$

    further eliminate the first derivative, and after a normalisation, Q can be simplified to *Airy type* $(\hbar\partial_X)^2 - X$.
- Thus, locally near a ramification point, the WKB problem for P reduces to an Airy-type WKB problem.

WKB Ansatz and WKB 1-Forms

- We look for formal WKB solutions of $P\psi = 0$:

$$\psi(x,\hbar) = \exp\left(\hbar^{-1}\left(F_0(x) + \hbar F_1(x) + \hbar^2 F_2(x) + \cdots\right)\right). \qquad (3)$$

- Define WKB 1-forms

$$\alpha_g(x) = dF_g(x), \qquad g \geq 0.$$

- For the Airy equation

$$(\hbar\partial_x)^2 \mathrm{Ai}(x) = x\, \mathrm{Ai}(x),$$

the α_g^{Airy} are explicitly known.

The Tate Symplectic Space H

- In a neighbourhood of a given ramification point with local coordinate z,

$$\alpha^{Airy} = (2z^2 - \frac{\hbar}{2z} - \frac{5\hbar^2}{16z^4} - \frac{15\hbar^3}{32z^7} - \frac{1105\hbar^4}{1024z^{10}} - \frac{1695\hbar^5}{512z^{13}} - \dots)dz \quad (4)$$

- Consider the infinite-dimensional Tate space

$$H = \mathbb{C}[[z]]\, dz \oplus z^{-2}\mathbb{C}[z^{-1}]\, dz; \quad \mathcal{U}^* := \mathbb{C}[[z]]dz, \quad \mathcal{V} := z^{-2}\mathbb{C}[z^{-1}]dz\,.$$

- A meromorphic 1 form with 0 residue can be considered as a point in $\mathcal{H}$.

Lie Algebra $\mathfrak{g}$ and Orbits

- There is a Lie algebra $\mathfrak{g}$ of formal vector fields on $\mathbb{C}^2_{x,y}$, acting (locally) transitively on H.
- The action of $\mathfrak{g}$ preserves the symplectic structure and the foliation.
- $\mathfrak{g} = \mathbb{C}[[x]]\partial_x \ltimes \mathbb{C}[[x]]/\mathbb{C}$.
- define $L_{m-1} := x^m\partial_x$, and $T_n = x^n$. Then the commutation relation:

$$\begin{aligned} [L_{n_1}, L_{n_2}] &= -(n_1 - n_2)L_{n_1+n_2} \\ [L_{n_1}, T_{n_2}] &= n_2 T_{n_1+n_2} \\ [T_{n_1}, T_{n_2}] &= 0. \end{aligned} \tag{5}$$

- For any 1 form $\eta\, dz = \sum_i \eta_i z^i dz$, we have the $\mathfrak{g}$-action

$$\begin{aligned} L_{m-1}\cdot \eta\, dz &= \sum_{i\in\mathbb{Z}} \left(n - 1 + \tfrac{i}{2}\right)\eta_i\, z^{i+2n-2} dz, \qquad n \geq 0, \\ T_n \cdot \eta\, dz &= \eta\, dz + d(x^n), \qquad n \geq 1. \end{aligned} \tag{6}$$

- $\mathfrak{g}$ action preserves the residue of the 1 form.

Lie Algebra $\mathfrak{g}$ and Orbits

- The above $\mathfrak{g}$ action defines a map

$$\mathfrak{g} \to \mathrm{Hom}(U, U) \oplus \mathrm{Hom}(V, V) \oplus (U \otimes V) \oplus (U \otimes V)^* \oplus U^* \oplus V. \quad (7)$$

- The map to the summand U^* is an isomorphism, and the map is 0 to the summand V.
- The morphism therefore can be encoded in the following tensors in terms of U, V:
    1. $\mathfrak{g} \to \mathrm{Hom}(U, U) \subset \mathrm{Hom}(U^*, U^*) \Longleftrightarrow a : U \to U \otimes U$.
    2. $\mathfrak{g} \to (U \otimes V)^* = \mathrm{Hom}(V, U^*) \Longleftrightarrow b : U \otimes V \to U$.
    3. $\mathfrak{g} \to U \otimes V \subset \mathrm{Hom}(U^*, V) \Longleftrightarrow c \in U \otimes U \otimes V$.
    4. $\mathfrak{g} \to \mathrm{Hom}(V, V) \Longleftrightarrow d : V \to U \otimes V$.

Lie Algebra $\mathfrak{g}$ and Orbits

- The formal orbit passing through point $0 \in \mathcal{H}$ gives a graph of the formal germ at 0 of a map from U to V, whose Taylor coefficients are *tensors*:

$$\forall n \geq 2: T_n : \mathrm{Sym}^n(U^*) \to V \iff T_n \in \mathrm{Sym}^n(U) \otimes V, \quad T_2 = c + c^t.$$

- Proposition: These tensors $T_2, T_3, \ldots$ are given as certain sums over trees comprising copies of tensors a, b, c, d.
- Recursion formula:

$$\begin{aligned}\sum_{n\geq 2} T_n\Big(x, \ldots, x, a(g,x) + b(g, \sum_{n\geq 2} T_n(x,\ldots,x)) + g\Big) \\ = c(g,x) + d(g, \sum_{n\geq 2} T_n(x,\ldots,x))\end{aligned} \tag{8}$$

Lie Algebra $\mathfrak{g}$ and Orbits

- α^{Airy} contains a non-zero residue, therefore we need to consider a deformed orbit of a shift $\alpha \mapsto \alpha - \frac{\hbar}{2z}\frac{dz}{z}$.
- We have a similar definition for tensor T_n, but T_n in this case admits a formal $\hbar$ expansion $T_n = \sum_{m\geq 0} \hbar^m T_{n,m}$.
- $T_{0,0} = 0$, $T_{1,0} = 0$ and $T_0 = \alpha^{Airy} - 2z^2 dz + \frac{\hbar}{2z}\frac{dz}{z}$.
- Recursion formula:

$$\sum_{n\geq 0} T_n\Big(x, \dots, x, a(g,x) + b(g, \sum_{n\geq 0} T_n(x,\dots,x)) + g\Big) = c(g,x) + d\big(g, \sum_{n\geq 0} T_n(x,\dots,x)\big) \tag{9}$$

Calculation

- $J_n = \sum_n C_{k,[n]} J_-^{[n]}$
- $J_-^{[d]}$ is homogeneous of total degree d, and $C_{k,[n]}$ the corresponding Taylor coefficient.
- Set $deg(\hbar) = 1$, therefore $deg(x) = \frac{2}{3}$, $deg(y) = deg(z) = \frac{1}{3}$, and accordingly $deg(J_{i-1}) = \frac{2-i}{3}$.

Table

z^i	z^4	z^3	z^2	z^1	z^0	z^{-1}
α	J_{-5}	J_{-4}	$J_{-3}+2$	J_{-2}	J_{-1}	$\frac{-\hbar}{2}$
$\dot{\alpha}$	$\dot{J}_{-5}$	$\dot{J}_{-4}$	$\dot{J}_{-3}$	$\dot{J}_{-2}$	$\dot{t}_0$	0
$L_{-1}\alpha$	$\frac{5}{2}J_{-7}$	$2J_{-6}$	$\frac{3}{2}J_{-5}$	J_{-4}	$\frac{1}{2}(J_{-3}+2)$	0
$L_0\alpha$	$\frac{5}{2}J_{-5}$	$2J_{-4}$	$\frac{3}{2}(J_{-3}+2)$	J_{-2}	$\frac{1}{2}J_{-1}$	0
$L_1\alpha$	$\frac{5}{2}(J_{-3}+2)$	$2J_{-2}$	$\frac{3}{2}J_{-1}$	$\frac{-\hbar}{2}$	$\frac{1}{2}\left(\sum_{m,[n]}\hbar^m C_{1,m,[n]}J_-^n-\beta_1\right)$	0
$L_2\alpha$	$\frac{5}{2}J_{-1}$	$2\frac{-\hbar}{2}$	$\frac{3}{2}\dots$			

TABLE 4.3 – α, its derivative and the L_i action

z^i	z^{-2}	z^{-3}	z^{-4}
α	$\sum_{m,[n]}\hbar^m C_{1,m,[n]}J_-^n-\beta_1$	$\sum_{m,[n]}\hbar^m C_{2,m,[n]}J_-^n-\beta_2$	$\sum_{m,[n]}\hbar^m C_{3,m,[n]}J_-^n-\beta_3$
$\dot{\alpha}$	$\sum_{m,[n]}\hbar^m C_{1,m,[n]}\dot{J}_-^n$	$\sum_{m,[n]}\hbar^m C_{2,m,[n]}\dot{J}_-^n$	$\sum_{m,[n]}\hbar^m C_{3,m,[n]}\dot{J}_-^n$
$L_{-1}\alpha$	$-\frac{1}{2}J_{-1}$	$-\frac{-\hbar}{2}$	$-\frac{3}{2}\left(\sum_{m,[n]}\hbar^m C_{1,m,[n]}J_-^n-\beta_1\right)$
$L_0\alpha$	$-\frac{1}{2}\left(\sum_{m,[n]}\hbar^m C_{1,m,[n]}J_-^n-\beta_1\right)$	$-\left(\sum_{m,[n]}\hbar^m C_{2,m,[n]}J_-^n-\beta_2\right)$	$-\frac{3}{2}\left(\sum_{m,[n]}\hbar^m C_{3,m,[n]}J_-^n-\beta_3\right)$
$L_1\alpha$	$-\frac{1}{2}\left(\sum_{m,[n]}\hbar^m C_{3,m,[n]}J_-^n-\beta_3\right)$	$-\left(\sum_{m,[n]}\hbar^m C_{4,m,[n]}J_-^n-\beta_4\right)$	

TABLE 4.4 – α, its derivative and the L_i action

Construction of the Global Complement V_{global}

- The final step is to intersect the orbit with a shifted affine subspace, which is specified for each differential equation. The unique intersection point is accordingly the solution for the given differential equation.
- Denote the ramification points p_i, and the points at ∞ as q_j.
- Define

$$\mathcal{H}_{global} = \oplus_i \mathcal{H}_i, \quad \mathcal{U}^*_{global} := \oplus_i \mathcal{U}^*_i.$$

- Ω': sheaf of meromorphic 1-forms on the compactified spectral curve $\overline{\Sigma}$, which have *zero* residues at only ramification points $\{p_i\}$ and have no poles at the ∞.

The intersection

- Theorem: There exists $\mathcal{V}_{global} \subset \Omega'(\overline{\Sigma}_0)$, of codimension g, given by finitely many conditions on jets at all $\{q_j\}$, and such $\mathcal{V}_{global}$ is complementary to $\mathcal{U}^*_{global}$.
- Proposition: WKB 1-forms $\alpha_k \in \Omega'(\overline{\Sigma})$ for Schrödinger-like differential equation $P\psi = 0$ belong to the subspace $\mathcal{V}_{global}$ for *sufficiently large* k.
- Therefore the formal WKB solution is obtained by the unique intersection of the $\mathfrak{g}$ orbit passing through Airy points and $\mathcal{V}_{global}$, plus a holomorhic 1 form with finite $\hbar$ expansion.

End

Thank you for your attention!