DDFV and HHO schemes for incompressible Navier-Stokes equations

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GdR MaNu,

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1. Nonhomogeneous incompressible Navier-Stokes problem with DDFV
2. Steady incompressible Navier-Stokes problem with HHO

1. Nonhomogeneous incompressible Navier-Stokes problem with DDFV
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Unknowns

- ρ : the density
- $\mathbf{u}$: the velocity field
- p : the pressure

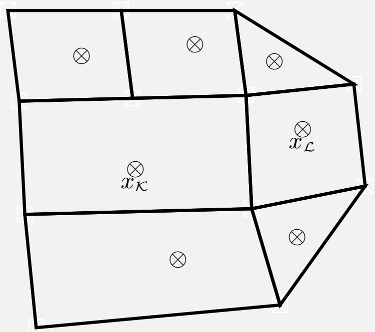
Equations

$$
(\mathrm{NS}) \quad \begin{cases}
\partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \mathrm{div}\left(-2\eta\mathrm{D}\mathbf{u} + p\mathrm{Id}\right) = \mathbf{f}, \text{ in }]0,T[\times\Omega, \\
\mathrm{div}(\mathbf{u}) = 0, \text{ in }]0,T[\times\Omega, \\
\partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0, \text{ in }]0,T[\times\Omega., \\
\mathbf{u} = 0, \text{ on }]0,T[\times\partial\Omega, \\
\mathbf{u}(0,.) = \mathbf{u}_{\mathrm{init}} \text{ in } \Omega, \\
\rho(0,.) = \rho_{\mathrm{init}} \text{ in } \Omega, \\
\int_\Omega p(.,x)\mathrm{d}x = 0,
\end{cases}
$$

with $\mathrm{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$

$$
\text{(NS)} \quad \begin{cases} \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \operatorname{div}\left(-2\eta \mathrm{D}\mathbf{u} + p\mathrm{Id}\right) = \mathbf{f}, \text{ in }]0,T[\times\Omega, \\ \operatorname{div}(\mathbf{u}) = 0, \text{ in }]0,T[\times\Omega, \\ \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \text{ in }]0,T[\times\Omega. \end{cases}
$$

- Non-linear term
- Variable density
- Coupling between the equations
- Time evolutive model



$\mathcal{K}$ Primal cells — $\mathcal{K}^*$ Dual cells — $\mathcal{D}$ Diamond cells

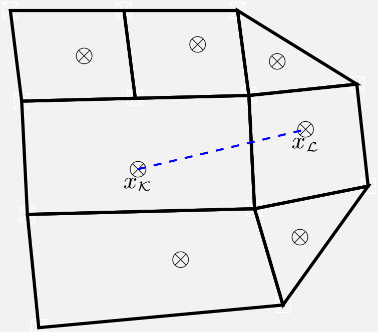
$\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}$ $\quad \rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}$ $\quad \rightsquigarrow p_{\mathfrak{D}} = (p_{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}}$

$\rightsquigarrow \mathbf{u}_{\mathcal{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*}),$ $\quad \rightsquigarrow \rho_{\mathfrak{D}} = (\rho_{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}}$

$\rightsquigarrow$ Discrete operators : $\nabla^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}$ and $\mathbf{div}^{\mathcal{T}}(\xi^{\mathfrak{D}})$.

Hermeline 2000, Domelevo & Omnès 2005

Andreianov & Boyer & Hubert 2007



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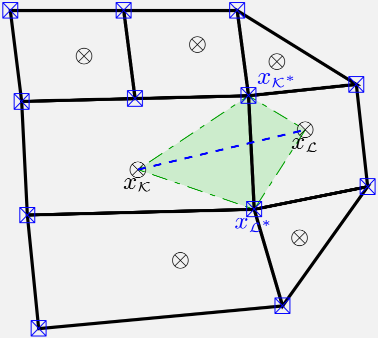

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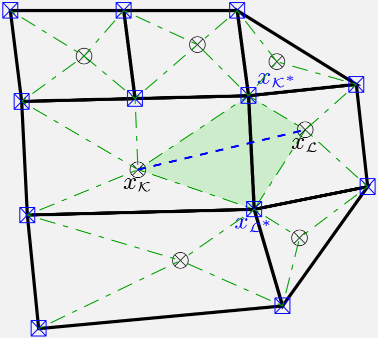

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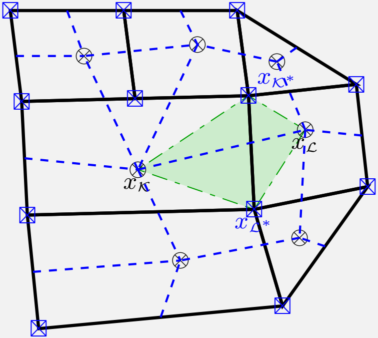

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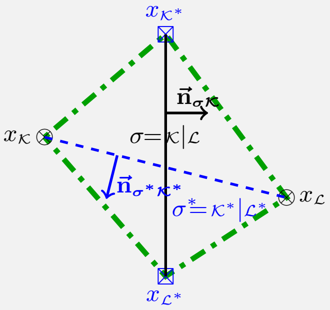
$$\nabla^{\mathfrak{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$$

$$\text{where} \begin{cases} \nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}}(x_{\mathcal{L}} - x_{\mathcal{K}}) = \mathbf{u}_{\boldsymbol{\mathcal{L}}} - \mathbf{u}_{\boldsymbol{\mathcal{K}}}, \\ \nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}}(x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = \mathbf{u}_{\boldsymbol{\mathcal{L}^*}} - \mathbf{u}_{\boldsymbol{\mathcal{K}^*}}. \end{cases}$$



Diamond

$$\nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}} = \frac{1}{2m_{\mathcal{D}}} \Big[m_\sigma (\mathbf{u}_{\boldsymbol{\mathcal{L}}} - \mathbf{u}_{\boldsymbol{\mathcal{K}}}) \otimes \vec{\mathbf{n}}_{\sigma\mathcal{K}} + m_{\sigma^*} (\mathbf{u}_{\boldsymbol{\mathcal{L}^*}} - \mathbf{u}_{\boldsymbol{\mathcal{K}^*}}) \otimes \vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*} \Big].$$

$$\rightsquigarrow \mathrm{D}^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}} = \frac{1}{2} \left(\nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}} + {}^t(\nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}}) \right).$$

$$\rightsquigarrow \mathrm{div}^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}} = \mathrm{Tr} \nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}}.$$

Discrete divergence $\mathbf{div}^{\mathcal{T}} : (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \longrightarrow (\mathbb{R}^2)^{\mathcal{T}}$

$$\begin{aligned}
\mathcal{K} \in \mathfrak{M}, \quad \frac{1}{m_{\mathcal{K}}} \int_{\mathcal{K}} \mathrm{div}(\xi) &= \frac{1}{m_{\mathcal{K}}} \sum_{\sigma \subset \partial\mathcal{K}} \int_{\sigma} \xi \, \vec{\mathbf{n}}_{\sigma\mathcal{K}}. \\
\mathbf{div}^{\mathcal{K}} \xi^{\mathfrak{D}} &= \frac{1}{m_{\mathcal{K}}} \sum_{\sigma \subset \partial\mathcal{K}} m_{\sigma} \xi^{\mathcal{D}} \vec{\mathbf{n}}_{\sigma\mathcal{K}}. \\
\mathcal{K}^* \in \mathfrak{M}^*, \quad \mathbf{div}^{\mathcal{K}^*} \xi^{\mathfrak{D}} &= \frac{1}{m_{\mathcal{K}^*}} \sum_{\sigma^* \subset \partial\mathcal{K}^*} m_{\sigma^*} \xi^{\mathcal{D}} \vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}.
\end{aligned}$$

Discrete duality property

- On the continuous level : the Stokes formula

$$\int_{\Omega} \mathrm{div}\xi \cdot \mathbf{u} = -\int_{\Omega} \xi : \nabla \mathbf{u}$$

- On the discrete level

$$[\![\mathbf{div}^{\mathcal{T}} \xi_{\mathfrak{D}}, \mathbf{u}_{\mathcal{T}}]\!]_{\mathcal{T}} = -(\xi_{\mathfrak{D}} : \nabla^{\mathfrak{D}} \mathbf{u}_{\mathcal{T}})_{\mathfrak{D}}$$

On the continuous level

$$\int_{\mathcal{D}} \operatorname{div}(\rho \mathbf{u}) = \sum_{\mathfrak{s}\in\partial\mathcal{D}} \int_{\mathfrak{s}} \rho \mathbf{u} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathcal{D}}$$

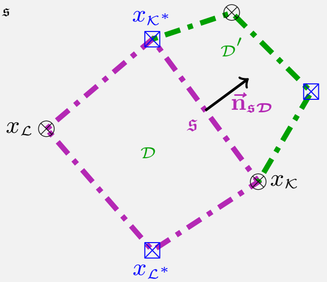
On the discrete level, with upwind fluxes

$$\operatorname{divc}^{\mathfrak{D}} : \mathbb{R}^{\mathfrak{D}} \times \left(\mathbb{R}^2\right)^{\mathcal{T}} \mapsto \mathbb{R}^{\mathfrak{D}} : \quad m_{\mathcal{D}} \operatorname{divc}^{\mathcal{D}}(\rho_{\mathfrak{D}}, \mathbf{u}_{\mathcal{T}}) = \sum_{\mathfrak{s}\in\partial\mathcal{D}} F_{\mathfrak{s},\mathcal{D}}$$

$$F_{\mathfrak{s},\mathcal{D}} = m_{\mathfrak{s}} \left(\left(u_{\mathfrak{s},\mathcal{D}}\right)^+ \rho_{\mathcal{D}} - \left(u_{\mathfrak{s},\mathcal{D}}\right)^- \rho_{\mathcal{D}'} \right) \sim \int_{\mathfrak{s}} \rho \mathbf{u} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathcal{D}}$$

where

- $u_{\mathfrak{s},\mathcal{D}} = \frac{\mathbf{u}_{\mathcal{K}} + \mathbf{u}_{\mathcal{K}^*}}{2} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathcal{D}}$ for $\mathfrak{s} = [x_{\mathcal{K}}, x_{\mathcal{K}^*}] \in \partial\mathcal{D}$,
- $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$
- $\operatorname{div}^{\mathcal{D}} \mathbf{u}_{\mathcal{T}} = \frac{1}{m_{\mathcal{D}}} \sum_{\mathfrak{s}\in\partial\mathcal{D}} m_{\mathfrak{s}} u_{\mathfrak{s},\mathcal{D}}$
- $\operatorname{div}^{\mathcal{D}} \mathbf{u}_{\mathcal{T}} = \operatorname{divc}^{\mathcal{D}}(\mathbf{1}_{\mathfrak{D}}, \mathbf{u}_{\mathcal{T}})$



SCHEME

$$\rightsquigarrow \frac{\rho_{\mathfrak{D}}^{n+1} - \rho_{\mathfrak{D}}^{n}}{\delta t} + \mathrm{divc}^{\mathfrak{D}}(\rho_{\mathfrak{D}}^{n}, \mathbf{u}_{\mathcal{T}}^{n}) = 0.$$

MAXIMUM PRINCIPLE

- $\rho_{\mathfrak{D}}^{n} \geq 0$
- $\delta t \leq \left(\|\mathbf{u}_{\mathcal{T}}\|_{\infty} \frac{1}{m_{\mathcal{D}}} \sum_{\mathfrak{s} \in \partial \mathcal{D}} m_{\mathfrak{s}} \right)^{-1}$

▶ $\rho_{\mathfrak{D}}^{n+1} \geq 0$

HOMOGENEOUS STATES ARE PRESERVED

- $\rho_{\mathfrak{D}}^{n} \equiv 1$
- $\mathrm{div}^{\mathcal{D}} \mathbf{u}_{\mathcal{T}}^{n} = 0$

▶ $\rho_{\mathfrak{D}}^{n+1} \equiv 1$

PROJECTION

- $\rho_{\mathcal{K}}^{n+1} = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma \in \partial \mathcal{K}} m(\mathcal{D} \cap \mathcal{K}) \rho_{\mathcal{D}}^{n+1}, \forall \mathcal{K} \in \mathfrak{M},$
- $\rho_{\mathcal{K}^*}^{n+1} = \frac{1}{m_{\mathcal{K}^*}} \sum_{\sigma^* \in \partial \mathcal{K}^*} m(\mathcal{D} \cap \mathcal{K}^*) \rho_{\mathcal{D}}^{n+1}, \forall \mathcal{K}^* \in \mathfrak{M}^*.$

ON THE CONTINUOUS LEVEL

$$\int_{\mathcal{K}} \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \sum_{\sigma \in \partial \mathcal{K}} \int_\sigma (\rho \mathbf{u} \cdot \vec{\mathbf{n}}) \mathbf{u}$$

ON THE DISCRETE LEVEL, WITH UPWIND FLUXES

$b^{\mathcal{T}} : \mathbb{R}^{\mathfrak{D}} \times \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \left(\mathbb{R}^2\right)^{\mathcal{T}} \mapsto \left(\mathbb{R}^2\right)^{\mathcal{T}}$

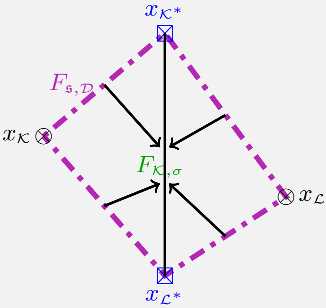
$$b_{\mathcal{K}}(\rho_{\mathfrak{D}}, \mathbf{v}_{\boldsymbol{\mathcal{T}}}, \mathbf{u}_{\boldsymbol{\mathcal{T}}}) = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma \in \partial \mathcal{K}} (F_{\mathcal{K},\sigma}(\rho_{\mathfrak{D}}, \mathbf{v}_{\boldsymbol{\mathcal{T}}}))^+ \mathbf{u}_{\boldsymbol{\mathcal{K}}} - (F_{\mathcal{K},\sigma}(\rho_{\mathfrak{D}}, \mathbf{v}_{\boldsymbol{\mathcal{T}}}))^- \mathbf{u}_{\boldsymbol{\mathcal{L}}}$$

How to define $F_{\mathcal{K},\sigma}(\rho_{\mathfrak{D}}, \mathbf{u}_{\boldsymbol{\mathcal{T}}}) \sim \int_\sigma \rho \mathbf{u} \cdot \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}$? using $F_{\mathfrak{s},\mathcal{D}} \sim \int_{\mathfrak{s}} \rho \mathbf{u} \cdot \vec{\mathbf{n}}_{\boldsymbol{\mathfrak{s}\mathcal{D}}}$.

$$F_{\mathcal{K},\sigma} = -\frac{m(\mathcal{D} \cap \mathcal{L})}{m_{\mathcal{D}}} \sum_{\mathfrak{s} \in \partial \mathcal{D}, \mathfrak{s} \subset \mathcal{K}} F_{\mathfrak{s},\mathcal{D}} + \frac{m(\mathcal{D} \cap \mathcal{K})}{m_{\mathcal{D}}} \sum_{\mathfrak{s} \in \partial \mathcal{D}, \mathfrak{s} \subset \mathcal{L}} F_{\mathfrak{s},\mathcal{D}}$$

such that

$$m_{\mathcal{K}} \frac{\rho_{\mathcal{K}}^{n+1} - \rho_{\mathcal{K}}^n}{\delta t} + \sum_{\sigma \in \partial \mathcal{K}} F_{\mathcal{K},\sigma}^n = 0.$$



DDFV SCHEME : Let $(\rho_{\mathfrak{D}}^n, \mathbf{u}_{\mathcal{T}}^n) \in \mathbb{R}^{\mathfrak{D}} \times \mathbb{E}_0$.
Find $\rho_{\mathfrak{D}}^{n+1} \in \mathbb{R}^{\mathfrak{D}}$ such that,

$$\frac{\rho_{\mathfrak{D}}^{n+1} - \rho_{\mathfrak{D}}^n}{\delta t} + \mathrm{divc}^{\mathfrak{D}}(\rho_{\mathfrak{D}}^n, \mathbf{u}_{\mathcal{T}}^n) = 0.$$

Find $\mathbf{u}_{\mathcal{T}}^{n+1} \in \mathbb{E}_0$ and $p_{\mathfrak{D}}^{n+1} \in \mathbb{R}^{\mathfrak{D}}$ such that,

$$\begin{cases} \dfrac{\rho_{\mathcal{T}}^{n+1}\mathbf{u}_{\mathcal{T}}^{n+1} - \rho_{\mathcal{T}}^n\mathbf{u}_{\mathcal{T}}^n}{\delta t} + b^{\mathcal{T}}(\rho_{\mathfrak{D}}^n, \mathbf{u}_{\mathcal{T}}^n, \mathbf{u}_{\mathcal{T}}^{n+1}) + \mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1} + p_{\mathfrak{D}}^{n+1}\mathrm{Id}) = \mathbf{f}_{\mathcal{T}}^{n+1}, \\ \mathrm{div}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1} = 0, \qquad \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p_{\mathcal{D}}^{n+1} = 0. \end{cases}$$

ENERGY STABILITY

$$\frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n+1}}\mathbf{u}_{\mathcal{T}}^{n+1}\|_{\mathcal{T}}^2 - \frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n}}\mathbf{u}_{\mathcal{T}}^{n}\|_{\mathcal{T}}^2 + \frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n}}(\mathbf{u}_{\mathcal{T}}^{n+1} - \mathbf{u}_{\mathcal{T}}^n)\|_{\mathcal{T}}^2$$
$$+\underline{C}_\eta\|\nabla^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}\|_2^2 \leq [\![\mathbf{f}_{\mathcal{T}}^{n+1}, \mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}.$$

▶ Existence and uniqueness

- Multiply momentum conservation equation by $\mathbf{u}_{\mathcal{T}}^{n+1}$:

$$\frac{1}{\delta t}[\![\rho_{\mathcal{T}}^{n+1}\mathbf{u}_{\mathcal{T}}^{n+1},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}-\frac{1}{\delta t}[\![\rho_{\mathcal{T}}^{n}\mathbf{u}_{\mathcal{T}}^{n},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}+[\![b^{\mathcal{T}}(\rho_{\mathfrak{D}}^{n},\mathbf{u}_{\mathcal{T}}^{n},\mathbf{u}_{\mathcal{T}}^{n+1}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}$$
$$+[\![\mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}+p_{\mathfrak{D}}^{n+1}\mathrm{Id}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}=[\![\mathbf{f}_{\mathcal{T}}^{n+1},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}.$$

$$[\![\mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}+p_{\mathfrak{D}}^{n+1}\mathrm{Id}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}.$$

- Stokes formula
  - $[\![\mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}=(2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}:\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1})_{\mathfrak{D}}$
  - $[\![\mathbf{div}^{\mathcal{T}}(p_{\mathfrak{D}}^{n+1}\mathrm{Id}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}=-(p_{\mathfrak{D}}^{n+1}\mathrm{Id}:\nabla^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1})_{\mathfrak{D}}.$

- Constraint equation
  - $[\![\mathbf{div}^{\mathcal{T}}(p_{\mathfrak{D}}^{n+1}\mathrm{Id}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}=-(p_{\mathfrak{D}}^{n+1},\mathrm{div}^{\mathfrak{D}}(\mathbf{u}_{\mathcal{T}}^{n+1}))_{\mathfrak{D}}=0.$
- Korn inequality
  - $[\![\mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}+p_{\mathfrak{D}}^{n+1}\mathrm{Id}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}\geq\underline{C}_{\eta}\|\nabla^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}\|_2^2$

Non-linear term $2[\![b^{\mathcal{T}}(\rho^n_{\mathfrak{D}}, \mathbf{u}^n_{\mathcal{T}}, \mathbf{u}^{n+1}_{\mathcal{T}}), \mathbf{u}^{n+1}_{\mathcal{T}}]\!]_{\mathcal{T}} = P + D.$

- Definition of P

$$P := \sum_{\mathcal{K}\in\mathfrak{M}} \mathbf{u}^{n+1}_{\mathcal{K}} \cdot \sum_{\sigma\in\partial\mathcal{K}} F^+_{\mathcal{K},\sigma}\mathbf{u}_{\mathcal{K}} - F^-_{\mathcal{K},\sigma}\mathbf{u}_{\mathcal{L}}$$

- Integration by parts

$$P = \sum_{\sigma\in\mathfrak{D}} \left(F^+_{\mathcal{K},\sigma}\mathbf{u}_{\mathcal{K}} - F^-_{\mathcal{K},\sigma}\mathbf{u}_{\mathcal{L}}\right) \cdot (\mathbf{u}^{n+1}_{\mathcal{K}} - \mathbf{u}^{n+1}_{\mathcal{L}})$$

- $|x| = x^+ + x^-$

$$P = \sum_{\sigma\in\mathfrak{D}} |F_{\mathcal{K},\sigma}|\mathbf{u}^{n+1}_{\mathcal{K}} \cdot (\mathbf{u}^{n+1}_{\mathcal{K}} - \mathbf{u}^{n+1}_{\mathcal{L}}) - F^-_{\mathcal{K},\sigma}\left((\mathbf{u}^{n+1}_{\mathcal{K}})^2 - (\mathbf{u}^{n+1}_{\mathcal{L}})^2\right)$$

- Develop $(a-b)a = \frac{1}{2}((a-b)^2 + a^2 - b^2)$ and $x = x^+ - x^-$

$$P = \frac{1}{2}\sum_{\sigma\in\mathfrak{D}} \underbrace{|F_{\mathcal{K},\sigma}|(\mathbf{u}^{n+1}_{\mathcal{K}} - \mathbf{u}^{n+1}_{\mathcal{L}})^2}_{\geq 0} + \frac{1}{2}\sum_{\sigma\in\mathfrak{D}} F_{\mathcal{K},\sigma}\left((\mathbf{u}^{n+1}_{\mathcal{K}})^2 - (\mathbf{u}^{n+1}_{\mathcal{L}})^2\right)$$

- Integration by parts

$$P \geq \frac{1}{2} \sum_{\sigma \in \mathfrak{D}} F_{\mathcal{K},\sigma} \left((\mathbf{u}_{\mathcal{K}}^{n+1})^2 - (\mathbf{u}_{\mathcal{L}}^{n+1})^2 \right) \geq \frac{1}{2} \sum_{\mathcal{K} \in \mathfrak{M}} (\mathbf{u}_{\mathcal{K}}^{n+1})^2 \sum_{\sigma \in \partial\mathcal{K}} F_{\mathcal{K},\sigma}$$

- Mass conservation equation

$$P \geq \frac{1}{2} \sum_{\mathcal{K} \in \mathfrak{M}} (\mathbf{u}_{\mathcal{K}}^{n+1})^2 m_{\mathcal{K}} \frac{\rho_{\mathcal{K}}^n - \rho_{\mathcal{K}}^{n+1}}{\delta t}$$

- Finally

$$[\![b^{\mathcal{T}}(\rho_{\mathfrak{D}}^n, \mathbf{u}_{\mathcal{T}}^n, \mathbf{u}_{\mathcal{T}}^{n+1}), \mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}} \geq \frac{1}{2\delta t} [\![\rho_{\mathcal{T}}^n - \rho_{\mathcal{T}}^{n+1}, (\mathbf{u}_{\mathcal{T}}^{n+1})^2]\!]_{\mathcal{T}}.$$

- The weak formulation

$$\frac{1}{\delta t}[\![\rho_{\mathcal{T}}^{n+1}\mathbf{u}_{\mathcal{T}}^{n+1},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}} - \frac{1}{\delta t}[\![\rho_{\mathcal{T}}^{n}\mathbf{u}_{\mathcal{T}}^{n},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}} + [\![b^{\mathcal{T}}(\rho_{\mathfrak{D}}^{n},\mathbf{u}_{\mathcal{T}}^{n},\mathbf{u}_{\mathcal{T}}^{n+1}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}$$
$$+ [\![\mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1} + p_{\mathfrak{D}}^{n+1}\mathrm{Id}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}} = [\![\mathbf{f}_{\mathcal{T}}^{n+1},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}.$$

becomes

$$\frac{1}{2\delta t}[\![\rho_{\mathcal{T}}^{n+1}\mathbf{u}_{\mathcal{T}}^{n+1},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}} - \frac{1}{\delta t}[\![\rho_{\mathcal{T}}^{n}\mathbf{u}_{\mathcal{T}}^{n},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}} + \frac{1}{2\delta t}[\![\rho_{\mathcal{T}}^{n}\mathbf{u}_{\mathcal{T}}^{n+1},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}$$
$$+ \underline{\mathrm{C}}_{\eta}\|\nabla^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}\|_2^2 \leq [\![\mathbf{f}_{\mathcal{T}}^{n+1},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}.$$

- Evolution term

  - $$\frac{1}{2\delta t}[\![\rho_{\mathcal{T}}^{n}\mathbf{u}_{\mathcal{T}}^{n+1},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}} - \frac{1}{\delta t}[\![\rho_{\mathcal{T}}^{n}\mathbf{u}_{\mathcal{T}}^{n},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}$$
$$= \frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n}}(\mathbf{u}_{\mathcal{T}}^{n+1} - \mathbf{u}_{\mathcal{T}}^{n})\|_{\mathcal{T}}^2 - \frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n}}\mathbf{u}_{\mathcal{T}}^{n}\|_{\mathcal{T}}^2.$$

- Sum up

$$\frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n+1}}\mathbf{u}_{\mathcal{T}}^{n+1}\|_{\mathcal{T}}^2 - \frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n}}\mathbf{u}_{\mathcal{T}}^{n}\|_{\mathcal{T}}^2$$
$$+\frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n}}(\mathbf{u}_{\mathcal{T}}^{n+1} - \mathbf{u}_{\mathcal{T}}^{n})\|_{\mathcal{T}}^2 + \underline{\mathrm{C}}_{\eta}\|\nabla^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}\|_2^2 \leq [\![\mathbf{f}_{\mathcal{T}}^{n+1},\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}.$$

Exact solution

$$\mathbf{u} = \begin{pmatrix} -\cos(2\pi x)\sin(2\pi y)e^{-2t\eta} \\ \sin(2\pi x)\cos(2\pi y)e^{-2t\eta} \end{pmatrix}$$

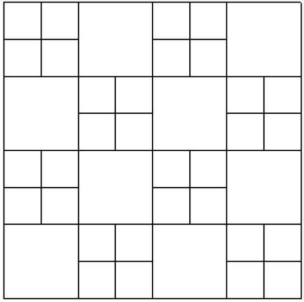
$$p = -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y))e^{-4t\eta}$$

$$\rho = 1$$

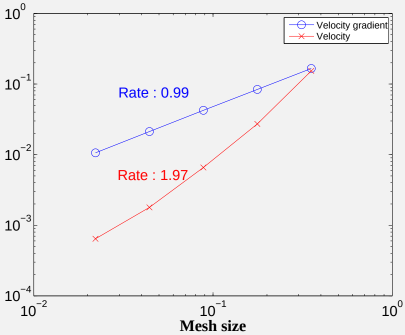
$$\eta = 1$$

$T = 1$ and $\delta t = 5.10^{-3}$

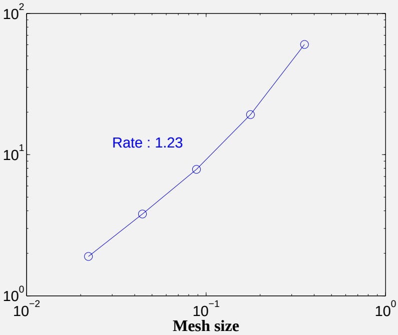
Primal mesh



Velocity gradient & Velocity



Pressure



- $\rho_{\mathfrak{D}}^n \equiv 1$

Exact solution

$$\mathbf{u} = \begin{pmatrix} -y\cos(t) \\ x\cos(t) \end{pmatrix}$$

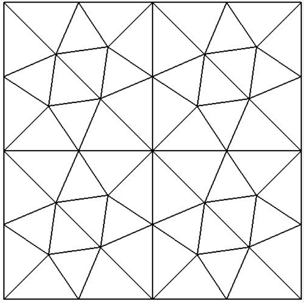
$$p = \sin(x)\sin(y)\sin(t)$$

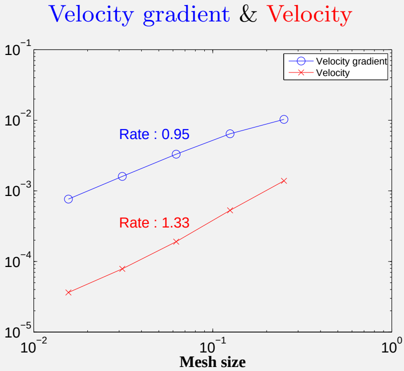
$$\rho(r,\theta,t) = 2 + r\cos(\theta - \sin(t))$$

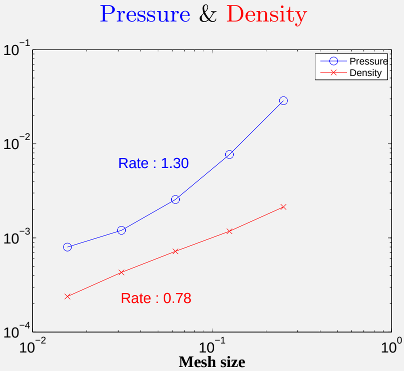
$$\eta = 1$$

$T = 3.10^{-2}$ and $\delta t = 7,5.10^{-5}$

Primal mesh




Velocity gradient & Velocity
Velocity gradient
Velocity
Rate : 0.95
Rate : 1.33
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-2}
10^{-1}
10^{0}
Mesh size


Pressure & Density
Pressure
Density
Rate : 1.30
Rate : 0.78
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
10^{0}
Mesh size

Parameter

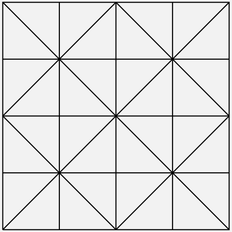
- $\Omega =]-0.5, 0.5[\times]-2, 2[.$
- $\rho_{\text{init}}(x, y) = 2 + \tanh\left(\frac{y + 0.1\cos(2\pi x)}{0.01}\right),$
- $\mathbf{u}_{\text{Init}} \equiv 0,$
- $\eta = \frac{1}{1000},$
- $\mathbf{f} = (0, -\rho)$

Primal mesh

$\mathbf{u} = 0$

$\mathbf{u} \cdot \vec{\mathbf{n}} = 0$

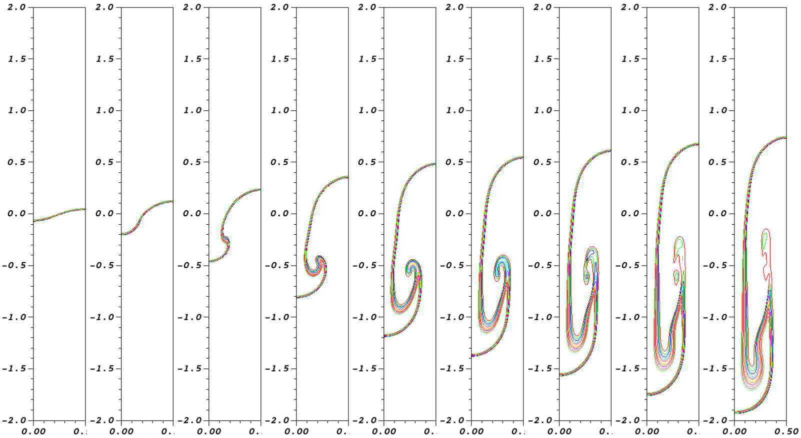
$2\eta \mathrm{D}\mathbf{u}\vec{\mathbf{n}} \cdot \vec{\boldsymbol{\tau}} = 0$

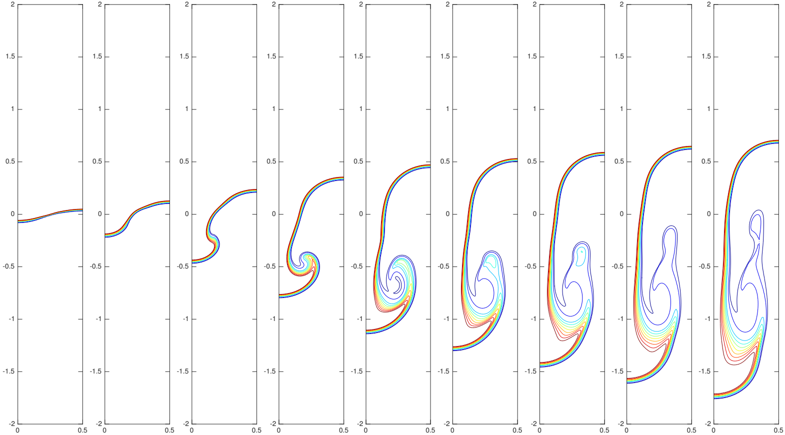


$\mathbf{u} \cdot \vec{\mathbf{n}} = 0$

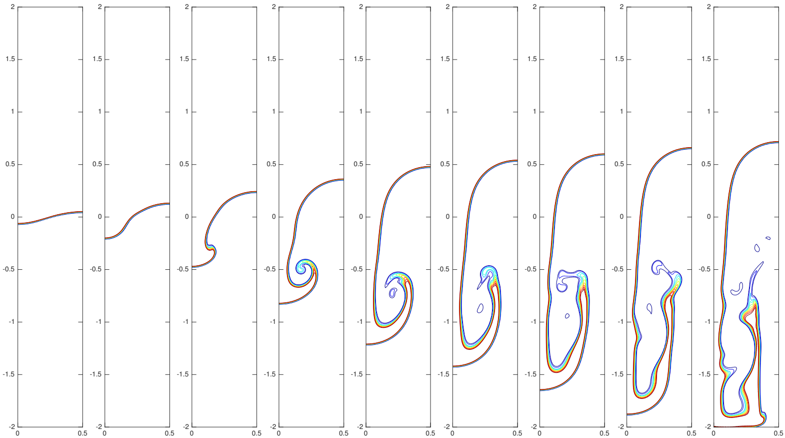
$2\eta \mathrm{D}\mathbf{u}\vec{\mathbf{n}} \cdot \vec{\boldsymbol{\tau}} = 0$

$\mathbf{u} = 0$


2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
0.00
0.50



Calgaro & Creusé & Goudon 2008.



Calgaro & Creusé & Goudon 2008.

1. NONHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES PROBLEM WITH DDFV

2. STEADY INCOMPRESSIBLE NAVIER-STOKES PROBLEM WITH HHO

Equations

$$\text{(sNS)} \quad \begin{cases} -\nu \triangle \mathbf{u} + \nabla \mathbf{u}\, \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \int_\Omega p = 0, & \end{cases}$$

Weak formulation Find $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that

$$\begin{aligned} \nu a(\mathbf{u}, \mathbf{v}) + t(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} && \forall \mathbf{v} \in H_0^1(\Omega)^2, \\ -b(\mathbf{u}, q) &= 0 && \forall q \in L_0^2(\Omega), \end{aligned}$$

with

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}, \qquad b(\mathbf{v}, q) := -\int_\Omega (\operatorname{div} \mathbf{v}) q,$$

$$t(\mathbf{w}, \mathbf{u}, \mathbf{v}) := \frac{1}{2} \int_\Omega \mathbf{v}^t \nabla \mathbf{u}\, \mathbf{w} - \frac{1}{2} \int_\Omega \mathbf{u}^t \nabla \mathbf{v}\, \mathbf{w}.$$

MESH REGULARITY

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of polyhedral meshes

- $\mathcal{T}_h = \{T\}$ a finite collection of nonempty disjoint open polyhedral elements $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$
- for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is
  - shape-regular in the usual sense of Ciarlet ;
  - contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \sim h_T$.

MAIN CONSEQUENCES :

- Trace and inverse inequalities
- Optimal approximation for broken polynomial spaces

Di Pietro & Ern 2012 and **Di Pietro & Droniou 2016**

Hybrid space : element-based and face-based velocity DOFs

$k \geq 0$ a polynomial degree.

$$\underline{\mathbf{U}}_h^k := \left(\bigtimes_{T\in\mathcal{T}_h} \mathbb{P}^k(T)^2 \right) \times \left(\bigtimes_{F\in\mathcal{F}_h} \mathbb{P}^k(F)^2 \right).$$

$$\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T\in\mathcal{T}_h}, (\mathbf{v}_F)_{F\in\mathcal{F}_h}) \in \underline{\mathbf{U}}_h^k$$

Seminorm

$$\|\underline{\mathbf{v}}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{\mathbf{v}}_T\|_{1,T}^2,$$

where, for all $T \in \mathcal{T}_h$,

$$\|\underline{\mathbf{v}}_T\|_{1,T}^2 := \|\nabla \mathbf{v}_T\|_T^2 + |\underline{\mathbf{v}}_T|_{1,\partial T}^2, \qquad |\underline{\mathbf{v}}_T|_{1,\partial T}^2 := \sum_{F\in\mathcal{F}_T} h_F^{-1} \|\mathbf{v}_F - \mathbf{v}_T\|_F^2.$$

Gradient reconstruction

$$(\mathbf{G}_h^l \underline{\mathbf{v}}_h)_{|T} := \mathbf{G}_T^l \underline{\mathbf{v}}_T.$$

$\mathbf{G}_T^l : \underline{\mathbf{U}}_T^k \to \mathbb{P}^l(T)^{2\times 2}$ such that, for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ and all $\boldsymbol{\tau} \in \mathbb{P}^l(T)^{2\times 2}$,

$$\int_T \mathbf{G}_T^l \underline{\mathbf{v}}_T : \boldsymbol{\tau} = -\int_T \mathbf{v}_T \cdot (\operatorname{div} \boldsymbol{\tau}) + \sum_{F \in \mathcal{F}_T} \int_F \mathbf{v}_F \cdot (\boldsymbol{\tau}\, \mathbf{n}_{TF})$$

- $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$
- $\mathbf{n}_{TF}$ the normal to F pointing out of T

Integration by parts :

$$\int_T \mathbf{G}_T^l \underline{\mathbf{v}}_T : \boldsymbol{\tau} = \int_T \nabla \mathbf{v}_T : \boldsymbol{\tau} + \sum_{F \in \mathcal{F}_T} \int_F (\mathbf{v}_F - \mathbf{v}_T) \cdot (\boldsymbol{\tau}\, \mathbf{n}_{TF}),$$

Divergence operator

$$D_T^k = \operatorname{tr}(\mathbf{G}_T^k).$$

Discretisation of $a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u}:\nabla\mathbf{v}$.

First choice

$$a_h(\underline{\mathbf{u}}_h,\underline{\mathbf{v}}_h) := \int_\Omega \mathbf{G}_h^k\underline{\mathbf{u}}_h:\mathbf{G}_h^k\underline{\mathbf{v}}_h$$

- Bilinear form is in general not stable.

Add a local stabilization term :

$$a_h(\underline{\mathbf{u}}_h,\underline{\mathbf{v}}_h) := \int_\Omega \mathbf{G}_h^k\underline{\mathbf{u}}_h:\mathbf{G}_h^k\underline{\mathbf{v}}_h + s_h(\underline{\mathbf{u}}_h,\underline{\mathbf{v}}_h).$$

- Coercivity w.r.t. to the local seminorm $\|\cdot\|_{1,h}$

$$C_a^{-1}\|\underline{\mathbf{v}}_h\|_{1,h}^2 \le a_h(\underline{\mathbf{v}}_h,\underline{\mathbf{v}}_h) \le C_a\|\underline{\mathbf{v}}_h\|_{1,h}^2,$$

- Consistency property : For all $\mathbf{v}\in H^{k+2}(\Omega)^d$,

$$s_h(\underline{\mathbf{I}}_h^k\mathbf{v},\underline{\mathbf{I}}_h^k\mathbf{v})^{\frac{1}{2}} \lesssim h^{k+2}\|\mathbf{v}\|_{H^{k+2}(\Omega)}.$$

Discretisation of $t(\mathbf{w},\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_\Omega \mathbf{v}^t \nabla \mathbf{u}\,\mathbf{w} - \frac{1}{2}\int_\Omega \mathbf{u}^t \nabla \mathbf{v}\,\mathbf{w}$.

TRILINEAR FORM

$$t_T(\underline{\mathbf{w}}_T,\underline{\mathbf{u}}_T,\underline{\mathbf{v}}_T) := \frac{1}{2}\int_\Omega \mathbf{v}_T^t \mathbf{G}_T^{2k}\underline{\mathbf{u}}_T\,\mathbf{w}_T - \frac{1}{2}\int_\Omega \mathbf{u}_T^t \mathbf{G}_T^{2k}\underline{\mathbf{v}}_T\,\mathbf{w}_T.$$

Otherwise

$$\begin{aligned} t_T(\underline{\mathbf{w}}_T,\underline{\mathbf{u}}_T,\underline{\mathbf{v}}_T) =& \frac{1}{2}\int_T \mathbf{v}_T^t \nabla \mathbf{u}_T\,\mathbf{w}_T - \frac{1}{2}\int_T \mathbf{u}_T^t \nabla \mathbf{v}_T\,\mathbf{w}_T \\ &+ \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F (\mathbf{u}_F\cdot\mathbf{v}_T - \mathbf{v}_F\cdot\mathbf{u}_T)(\mathbf{w}_T\cdot\mathbf{n}_{TF}). \end{aligned}$$

SKEW-SYMMETRY. For all $\underline{\mathbf{v}}_h,\underline{\mathbf{w}}_h \in \underline{\mathbf{U}}_{h,0}^k$, it holds $t_h(\underline{\mathbf{w}}_h,\underline{\mathbf{v}}_h,\underline{\mathbf{v}}_h) = 0$.

BOUNDEDNESS. For all $\underline{\mathbf{u}}_h,\underline{\mathbf{v}}_h,\underline{\mathbf{w}}_h \in \underline{\mathbf{U}}_{h,0}^k$, it holds

$$|t_h(\underline{\mathbf{w}}_h,\underline{\mathbf{u}}_h,\underline{\mathbf{v}}_h)| \le C_t \|\underline{\mathbf{w}}_h\|_{1,h}\|\underline{\mathbf{u}}_h\|_{1,h}\|\underline{\mathbf{v}}_h\|_{1,h}.$$

Discretisation of $b(\mathbf{v}, q) = -\int_\Omega (\operatorname{div} \mathbf{v})q$.

BILINEAR FORM

$$b_h(\underline{\mathbf{v}}_h, q_h) := -\int_\Omega D_h^k \underline{\mathbf{v}}_h q_h.$$

INF-SUP STABILITY. For all $q_h \in P_h^k$, it holds

$$\|q_h\|_{L^2(\Omega)} \lesssim \sup_{\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k, \|\underline{\mathbf{v}}_h\|_{1,h}=1} b_h(\underline{\mathbf{v}}_h, q_h).$$

Non-linear scheme Find $(\underline{\mathbf{u}}_h, p_h) \in \underline{\mathbf{U}}_{h,0}^k \times P_h^k$ such that

$$(\text{sNS}_h) \quad \begin{cases} \nu a_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) + t_h(\underline{\mathbf{u}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) + b_h(\underline{\mathbf{v}}_h, p_h) = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h, \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k, \\ -b_h(\underline{\mathbf{u}}_h, q_h) = 0, \forall q_h \in P_h^k. \end{cases}$$

Existence and a priori bounds

There exists a solution $(\underline{\mathbf{u}}_h, p_h) \in \underline{\mathbf{U}}_{h,0}^k \times P_h^k$ to (sNS_h), which

$$\|\underline{\mathbf{u}}_h\|_{1,h} \leq C\nu^{-1}\|\mathbf{f}\|_{L^2(\Omega)^2}, \quad \|p_h\|_{L^2(\Omega)} \leq C\left(\|\mathbf{f}\|_{L^2(\Omega)^2} + \nu^{-2}\|\mathbf{f}\|_{L^2(\Omega)^2}^2\right),$$

with $C > 0$ real number independent of both h and ν.

Uniqueness of the discrete solution

Under a data smallness condition the solution $(\underline{\mathbf{u}}_h, p_h)$ of (sNS_h) is unique.

Convergence to minimal regularity solutions

- Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ admissible mesh,
- $(\underline{\mathbf{u}}_h, p_h) \in \underline{\mathbf{U}}_{h,0}^k \times P_h^k$ solves (sNS$_h$),
- $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times P$ solves (sNS).

Then, it holds up to a subsequence,

- $\mathbf{u}_h \to \mathbf{u}$ strongly in $L^p(\Omega)^2$ for all $p \in [1, +\infty)$;
- $\mathbf{G}_h^k \underline{\mathbf{u}}_h \to \nabla \mathbf{u}$ strongly in $L^2(\Omega)^{2\times 2}$;
- $s_h(\underline{\mathbf{u}}_h, \underline{\mathbf{u}}_h) \to 0$;
- $p_h \to p$ strongly in $L^2(\Omega)$.

If the continuous solution is unique, convergence to the whole sequence.

CONVERGENCE RATES FOR SMALL DATA

- Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ admissible mesh,
- $(\underline{\mathbf{u}}_h, p_h) \in \underline{\mathbf{U}}_{h,0}^k \times P_h^k$ solves (sNS$_h$),
- $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times P$ solves (sNS),
- assume uniqueness (both smallness conditions),
- $(\mathbf{u}, p) \in H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)$.

Then, there is $C > 0$ independent of both h and ν such that

$$\begin{aligned}&\|\underline{\mathbf{u}}_h - \underline{\mathbf{I}}_h^k \mathbf{u}\|_{1,h} + \nu^{-1}\|p_h - \pi_h^k p\|_{L^2(\Omega)}\\&\qquad \le C h^{k+1}\Big(\big(1+\nu^{-1}\|\mathbf{u}\|_{H^2(\Omega)^d}\big)\,\|\mathbf{u}\|_{H^{k+2}(\Omega)^d} + \nu^{-1}\|p\|_{H^{k+1}(\Omega)}\Big)\end{aligned}$$

and

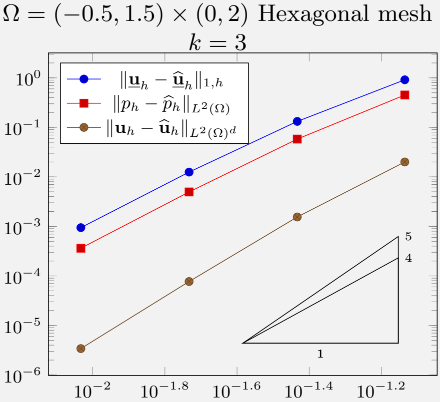
$$\begin{aligned}&\|\mathbf{G}_h^k \underline{\mathbf{u}}_h - \nabla\mathbf{u}\|_{L^2(\Omega)^{d\times d}} + s_h(\underline{\mathbf{u}}_h, \underline{\mathbf{u}}_h)^{\frac{1}{2}} + \nu^{-1}\|p_h - p\|_{L^2(\Omega)}\\&\qquad \le C h^{k+1}\Big(\big(1+\nu^{-1}\|\mathbf{u}\|_{H^2(\Omega)^d}\big)\,\|\mathbf{u}\|_{H^{k+2}(\Omega)^d} + \nu^{-1}\|p\|_{H^{k+1}(\Omega)}\Big).\end{aligned}$$

Exact solution

$$\mathbf{u} = \begin{pmatrix} 1 - \exp(\lambda x)\cos(2\pi y) \\ \frac{\lambda}{2\pi}\exp(\lambda x)\sin(2\pi y) \end{pmatrix}$$

$$p = -\frac{1}{2}\exp(2\lambda x) + \frac{\lambda}{2}\left(\exp(4\lambda) - 1\right)$$

$$\nu = 1$$



Thanks you !

$$s_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) := \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} h_F^{-1} \int_F \boldsymbol{\delta}_{TF}^k \underline{\mathbf{u}}_T \cdot \boldsymbol{\delta}_{TF}^k \underline{\mathbf{v}}_T,$$

where

$$\boldsymbol{\delta}_{TF}^k \underline{\mathbf{v}}_T := \boldsymbol{\pi}_F^k \left(\mathbf{v}_F - \mathbf{r}_T^{k+1} \underline{\mathbf{v}}_T - \boldsymbol{\pi}_T^k (\mathbf{v}_T - \mathbf{r}_T^{k+1} \underline{\mathbf{v}}_T) \right).$$

Velocity reconstruction operator

$$(\mathbf{r}_h^{k+1} \underline{\mathbf{v}}_h)_{|T} := \mathbf{r}_T^{k+1} \underline{\mathbf{v}}_T.$$

$\mathbf{r}_T^{k+1} : \underline{\mathbf{U}}_T^k \to \mathbb{P}^{k+1}(T)^2$ such that, for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$,

$$\int_T \nabla \mathbf{r}_T^{k+1} \underline{\mathbf{v}}_T : \nabla \mathbf{w} = -\int_T \mathbf{v}_T \cdot \triangle \mathbf{w} + \sum_{F\in\mathcal{F}_T} \int_F \mathbf{v}_F \cdot (\nabla \mathbf{w}\, \mathbf{n}_{TF}) \qquad \forall \mathbf{w} \in \mathbb{P}^{k+1}(T)^2,$$

with closure condition $\int_T (\mathbf{r}_T^{k+1} \underline{\mathbf{v}}_T - \mathbf{v}_T) = O$.

L^2-orthogonal projector

$\boldsymbol{\pi}_X^k : L^1(X) \to \mathbb{P}^k(X)$ the L^2-orthogonal projector s.t., for all $v \in L^1(X)$,

$$\int_X (v - \boldsymbol{\pi}_X^k v) w = 0 \qquad \forall w \in \mathbb{P}^k(X).$$

$$\underline{\mathbf{I}}_h^k \mathbf{v} := \left((\boldsymbol{\pi}_T^k \mathbf{v})_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}_F^k \mathbf{v})_{F\in\mathcal{F}_h} \right).$$