

Guaranteed, locally efficient, and robust a posteriori estimates for parabolic problems

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GdR MaNu, baie de Somme, 18 octobre 2016

Outline

- 1 Introduction
- 2 Equivalence between error and dual norm of the residual
- 3 High-order time discretization & Radau reconstruction
- 4 Missing Galerkin orthogonality
- 5 Guaranteed upper bound
- 6 Local space-time efficiency and robustness
- 7 Conclusions and future directions

Model parabolic problem

The heat equation

$$\partial_t u - \Delta u = f \quad \text{in } \Omega \times (0, T),$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T),$$

$$u(0) = u_0 \quad \text{in } \Omega$$

Spaces

$$X := L^2(0, T; H_0^1(\Omega)),$$

$$\|v\|_X^2 := \int_0^T \|\nabla v\|^2 dt,$$

$$Y := L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)),$$

$$\|v\|_Y^2 := \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 dt + \|v(T)\|^2$$

Weak solution

Find $u \in Y$ with $u(0) = u_0$ such that

$$\int_0^T \langle \partial_t u, v \rangle + (\nabla u, \nabla v) dt = \int_0^T (f, v) dt$$

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An optimal a posteriori estimate for evolutive problems

Guaranteed upper bound

- $\|u - u_{h\tau}\|_{?,\Omega \times (0,T)}^2 \leq \sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
- no undetermined constant: **error control**

Local efficiency

- $\eta_K^n(u_{h\tau}) \leq C_{\text{eff}} \|u - u_{h\tau}\|_{?, \text{neighbors of } K \times (t^{n-1}, t^n)}$
- optimal space-time mesh refinement
- **local** in **time** and in **space** error lower bound

Robustness

- C_{eff} independent of data, domain Ω , **final time** T , meshes, solution u , **polynomial degrees** of $u_{h\tau}$ in space and in time

Asymptotic exactness

- $\sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2 / \|u - u_{h\tau}\|_{?,\Omega \times (0,T)}^2 \searrow 1$
- overestimation factor goes to one with meshes size

Small evaluation cost

- estimators can be evaluated cheaply (locally)

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Previous results – continuous finite elements

- Bieterman and Babuška (1982), introduction
- Picasso / Verfürth (1998), work with the energy norm X :
 - upper bound $\|u - u_{h\tau}\|_X^2 \leq C \sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
 - **constrained lower bound** (h and τ strongly linked)
- Verfürth (2003) (cf. also Bergam, Bernardi, and Mghazli (2005)), work with the Y norm:
 - upper bound $\|u - u_{h\tau}\|_Y^2 \leq C \sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
 - $\sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2 \leq C \|u - u_{h\tau}\|_{Y(l_n)}^2$
 - **robustness** with respect to the **final time**
 - efficiency **local in time** but **global in space**
- Eriksson and Johnson (1991), duality techniques & Makridakis and Nochetto (2003), elliptic reconstruction: $L^2(L^2) / L^\infty(L^2) / L^\infty(L^\infty) /$ higher-order norms
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Equivalence between error and residual (steady case)

Laplace equation

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Residual of $u_h \in H_0^1(\Omega)$

- $\mathcal{R}(u_h) \in H^{-1}(\Omega)$, the **misfit** of u_h in the **weak formulation**:

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v) \quad v \in H_0^1(\Omega)$$

- dual norm of the residual

$$\|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)} := \sup_{v \in H_0^1(\Omega); \|\nabla v\|=1} \langle \mathcal{R}(u_h), v \rangle$$

Energy error is the dual norm of the residual

$$\|\nabla(u - u_h)\| = \sup_{v \in H_0^1(\Omega); \|\nabla v\|=1} (\nabla(u - u_h), \nabla v) = \|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)}$$

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Bounding and localizing dual norms (steady case)

- $V := W_0^{1,p}(\Omega)$, $p > 1$, bounded linear functional $\mathcal{R} \in V'$
- norm $\|\mathcal{R}\|_{V'} := \sup_{v \in V; \|\nabla v\|_p=1} \langle \mathcal{R}, v \rangle_{V',V}$
- localized energy space $V^a := W_0^{1,p}(\omega_a)$ for $\mathbf{a} \in \mathcal{V}_h$
- restriction of \mathcal{R} to $(V^a)'$ (zero extension of $v \in V^a$),
 $\langle \mathcal{R}, v \rangle_{(V^a)',V^a} := \langle \mathcal{R}, v \rangle_{V',V} \quad v \in V^a,$
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Theorem (Bounding and localizing $\|\mathcal{R}\|_{V'}$)

There holds

$$\|\mathcal{R}\|_{V'} \leq (d+1) C_{\text{cont,PF}} \left\{ \frac{1}{(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(V^a)'}^q \right\}^{\frac{1}{q}} \quad \text{if } \underbrace{\langle \mathcal{R}, \psi_{\mathbf{a}} \rangle = 0}_{\text{orthogonality}} \quad \forall \mathbf{a} \in \mathcal{V}_h^{\text{int}}$$

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- 3 High-order time discretization & Radau reconstruction
- 4 Missing Galerkin orthogonality
- 5 Guaranteed upper bound
- 6 Local space-time efficiency and robustness
- 7 Conclusions and future directions

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Error and residual in the unsteady case

Theorem (Parabolic inf-sup identity)

For every $\varphi \in Y$, we have

$$\|\varphi\|_Y^2 = \left[\sup_{v \in X, \|v\|_X=1} \int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v) dt \right]^2 + \|\varphi(0)\|^2.$$

Residual of $u_{h\tau} \in Y$

- $\mathcal{R}(u_{h\tau}) \in X'$, the misfit of $u_{h\tau}$ in the weak formulation:

$$\langle \mathcal{R}(u_{h\tau}), v \rangle := \int_0^T (f, v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v) dt$$

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Y norm error is the dual X norm of the residual + IC error

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Proof of the parabolic inf-sup identity: $\varphi \in Y$

Proof.

- let $w_* \in X$ be defined by, a.e. in $(0, T)$,

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High-order time discretization & Radau reconstruction

CG of degree p in space & DG of degree q in time

Find $u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n}(I_n; V_h^n)$ with $u_{h\tau}(0) = \Pi_h u_0$ such that

$$\int_{I_n} (\partial_t u_{h\tau}, v_{h\tau}) + (\nabla u_{h\tau}, \nabla v_{h\tau}) dt - ((u_{h\tau})|_{n-1}, v_{h\tau}(t_{n-1}^+)) \\ = \int_{I_n} (f, v_{h\tau}) dt \quad \forall v_{h\tau} \in \mathcal{Q}_{q_n}(I_n; V_h^n) \quad \forall 1 \leq n \leq N.$$

- p -degree **continuous** finite elements in **space**

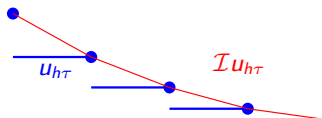
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Radau reconstruction $\mathcal{I}u_{h\tau} \in Y$, $\mathcal{I}u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n+1}(I_n; \widetilde{V}_h^n)$

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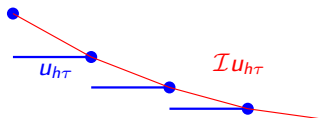
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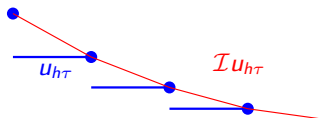
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Missing Galerkin orthogonality

Situation

- $u_{h\tau} \notin Y \Rightarrow$ impossible to estimate $\|u - u_{h\tau}\|_Y$
- $\mathcal{I}u_{h\tau} \in Y \Rightarrow$ **error** $\|u - \mathcal{I}u_{h\tau}\|_Y$
- but $\mathcal{I}u_{h\tau}$ misses the Galerkin orthogonality:

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Remedy

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- augment the norm: $\|v\|_{\mathcal{E}_Y}^2 := \|\mathcal{I}v\|_Y^2 + \|v - \mathcal{I}v\|_X^2$, $v \in Y + V_{h\tau}$
- $\mathcal{I}u = u \Rightarrow$

$$\|u - u_{h\tau}\|_{\mathcal{E}_Y}^2 = \|u - \mathcal{I}u_{h\tau}\|_Y^2 + \|u_{h\tau} - \mathcal{I}u_{h\tau}\|_X^2$$

- we are **adding** to Y norm the **time jumps** in X norm:

$$\int_{I_n} \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2 dt = \frac{\tau_n(q_n+1)}{(2q_n+1)(2q_n+3)} \|\nabla(u_{h\tau})_{n-1}\|^2$$

Equivalence between the Y and \mathcal{E}_Y norms

Global equivalence

$$\|u - \mathcal{I}u_{h\tau}\|_Y \leq \|u - u_{h\tau}\|_{\mathcal{E}_Y} \leq 3\|u - \mathcal{I}u_{h\tau}\|_Y$$

- holds if there is no source term oscillation or no coarsening
- otherwise an additional source term oscillation or coarsening term
- the two norms still may **differ locally**

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Equilibrated flux reconstruction

Definition (Equilibrated flux reconstruction)

For each time-step interval I_n and for each vertex $\mathbf{a} \in \mathcal{V}^n$, let

$$\sigma_{h\tau}^{\mathbf{a},n} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_{h\tau}^{\mathbf{a},n} \\ \nabla \cdot \mathbf{v}_h = \mathbf{g}_{h\tau}^{\mathbf{a},n}}} \int_{I_n} \|\mathbf{v}_h + \boldsymbol{\tau}_{h\tau}^{\mathbf{a},n}\|_{\omega_{\mathbf{a}}}^2 dt,$$

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Then set

$$\sigma_{h\tau} := \sum_{n=1}^N \sum_{\mathbf{a} \in \mathcal{V}^n} \sigma_{h\tau}^{\mathbf{a},n}.$$

Comment

- a priori a local space-time problem, $\mathbf{V}_{h\tau}^{\mathbf{a},n} := \mathcal{Q}_{q_n}(I_n; \mathbf{V}_h^{\mathbf{a},n})$
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where

$$\tau_{h\tau}^{\mathbf{a},n} := \psi_{\mathbf{a}} \nabla \mathbf{u}_{h\tau} |_{\omega_{\mathbf{a}} \times I_n},$$

$$\mathbf{g}_{h\tau}^{\mathbf{a},n} := \psi_{\mathbf{a}} (\Pi_{h\tau}^{\mathbf{a},n} f - \partial_t \mathcal{I} \mathbf{u}_{h\tau}) |_{\omega_{\mathbf{a}} \times I_n} - \nabla \psi_{\mathbf{a}} \cdot \nabla \mathbf{u}_{h\tau} |_{\omega_{\mathbf{a}} \times I_n}.$$

Then set

$$\sigma_{h\tau} := \sum_{n=1}^N \sum_{\mathbf{a} \in \mathcal{V}^n} \sigma_{h\tau}^{\mathbf{a},n}.$$

Comment

- a priori a local space-time problem, $\mathbf{V}_{h\tau}^{\mathbf{a},n} := \mathcal{Q}_{q_n}(I_n; \mathbf{V}_h^{\mathbf{a},n})$
- can be uncoupled to q_n elliptic problems posed in $\mathbf{V}_h^{\mathbf{a},n}$

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Guaranteed upper bound

Theorem (Guaranteed upper bound)

In the absence of data oscillation, there holds

$$\|u - u_{h\tau}\|_{\mathcal{E}_Y}^2 \leq \sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \int_{I_n} \|\sigma_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|_K^2 + \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|_K^2 dt.$$

Outline

- 1 Introduction
- 2 Equivalence between error and dual norm of the residual
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- 6 Local space-time efficiency and robustness**
- 7 Conclusions and future directions

Local space-time efficiency and robustness

Local error contributions

$$\begin{aligned}
 |u - u_{h\tau}|_{\mathcal{E}_Y^{\mathbf{a},n}}^2 &= \int_{I_n} \|\partial_t(u - \mathcal{I}u_{h\tau})\|_{H^{-1}(\omega_{\mathbf{a}})}^2 + \|\nabla(u - \mathcal{I}u_{h\tau})\|_{\omega_{\mathbf{a}}}^2 dt \\
 &\quad + \frac{\tau_n(q_n+1)}{(2q_n+1)(2q_n+3)} \|\nabla(u_{h\tau})_{n-1}\|_{\omega_{\mathbf{a}}}^2
 \end{aligned}$$

Theorem (Local space-time efficiency and robustness)

For each time-step interval I_n and for each element $K \in \mathcal{T}^n$, there holds, in the absence of data oscillation,

$$\int_{I_n} \|\sigma_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|_K^2 + \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|_K^2 dt \leq C_{\text{eff}}^2 \sum_{\mathbf{a} \in \mathcal{V}_K} |u - u_{h\tau}|_{\mathcal{E}_Y^{\mathbf{a},n}}^2.$$

Comments

- **local** in **space** and **time**
- C_{eff} only depends on shape regularity \Rightarrow **robustness**
- **no requirement on coarsening** between \mathcal{T}^{n-1} and \mathcal{T}^n

Local space-time efficiency and robustness

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Conclusions and future directions

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- local **space-time efficiency** is possible (adding the time jumps to the Y -norm error)
- **robustness** with respect to both **spatial** and **temporal degree**
- arbitrarily large **coarsening** allowed

Future directions

- estimates in the X norm
- nonlinear problems

Conclusions and future directions

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