The Hybrid High-Order method for the polytopal discretization of diffusion problems featuring variable coefficients

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Why polygons/polyhedra?

1 - In many situations, the mesh cannot be adapted to the numerical method but must be taken as a datum $\rightsquigarrow$ ex. in subsurface modeling

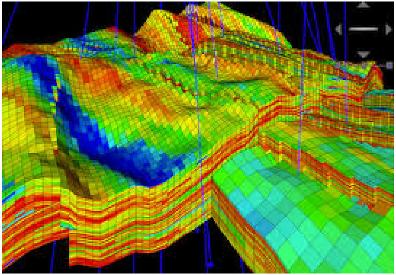

Figure : Courtesy of IFPEn

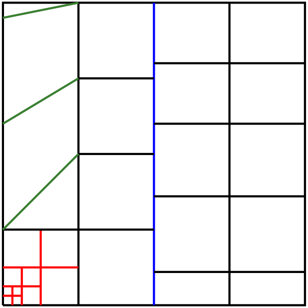

Figure : Simplified 2D example of CPG mesh with LGR, erosions and faults

Why polygons/polyhedra?

2 - They provide geometric flexibility in the modeling... $\rightsquigarrow$ ex. of filled elastomers

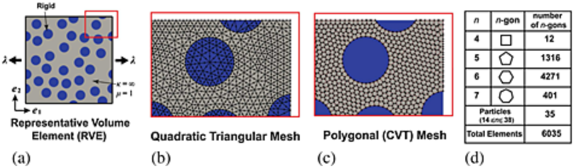

Figure : In [Chi, Talischi, Lopez-Pamies and Paulino, 15]

Why polygons/polyhedra?

3 - . . . and can even provide more accurate solutions!

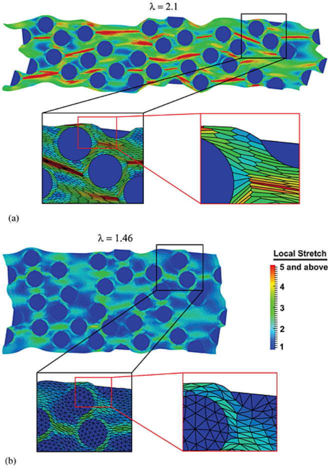

Figure : In [Chi, Talischi, Lopez-Pamies and Paulino, 15]

Why new-generation methods?

$\rightsquigarrow$ We are interested in the arbitrary-order approximation of elliptic problems on polytopal meshes

Classical approaches

- (Polytopal) Finite Element (FE) methods [Wachspress, 75 + Tabarraei and Sukumar, 04 + Gillette, Rand and Bajaj, 16 + Christiansen and Gillette, 16]
- Discontinuous Galerkin (DG) methods [Arnold, Brezzi, Cockburn and Marini, 02 + Di Pietro and Ern, 12 + Bassi, Botti, Colombo, Di Pietro and Tesini, 12 + Antonietti, Giani and Houston, 13 + Cangiani, Georgoulis and Houston, 14]

Limitations

- for FE: basis functions are hard to construct (due to continuity requirements) and non-polynomial, local conservativity is not ensured, stencils are large (vertex-based)
- for DG: the number of DoFs rapidly explodes with the order of approximation ($\sim k^3$ in 3D), mesh-dependent parameters must be tuned by the end-user to ensure stability

New-generation polytopal discretization methods

Hybridizable DG (HDG) methods [Cockburn, Gopalakrishnan and Lazarov, 09]

- skeletal methods: number of globally coupled DoFs $\sim k^2$ in 3D (after local elimination of cell unknowns)
- compact (face-based) stencil
- conservativity at the discrete level
- polynomial basis functions
- BUT non-optimally convergent on general element shapes for equal-order trace and potential unknowns

Virtual Element (VE) methods

[Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini and Russo, 13]

Weak Galerkin (WG) methods [Wang and Ye, 13]

- proved equivalent to HDG methods [Cockburn, 16]

Hybrid High-Order (HHO) methods [Di Pietro, Ern and SL, 14]

- bridged to HDG methods [Cockburn, Di Pietro and Ern, 16]
- HHO = HDG on steroids!

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References

⇝ Di Pietro, D. A. and Ern, A. (2015) Hybrid High-Order methods for variable-diffusion problems on general meshes. C. R. Acad. Sci. Paris, Ser. I, 353:31–34.

⇝ Di Pietro, D. A., Ern, A., and SL (2016) A review of Hybrid High-Order methods: formulations, computational aspects, comparison with other methods. In *Building Bridges: Connections and Challenges in Modern Approaches to Numerical PDEs*, nbr 114 in Lecture Notes in Computational Science and Engineering. Springer.

Model problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be an open, connected, bounded polytopal domain.

Problem: Let $f \in L^2(\Omega)$. Find a potential $u : \Omega \to \mathbb{R}$ such that

$$\boxed{\begin{aligned} -\mathrm{div}(\mathbb{A}\boldsymbol{\nabla} u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}}$$

Assumption: $\mathbb{A}$ is a symmetric matrix field on Ω s.t. $\mathbb{A} \in L^\infty(\Omega, \mathcal{S}_\alpha^\beta)$, i.e. for a.e. $\boldsymbol{x} \in \Omega$ and for all $\boldsymbol{\xi} \in \mathbb{R}^d$ s.t. $|\boldsymbol{\xi}| = 1$,

$$0 < \alpha \leqslant \mathbb{A}(\boldsymbol{x})\boldsymbol{\xi}{\cdot}\boldsymbol{\xi} \leqslant \beta < +\infty.$$

We let $\rho := \beta/\alpha$ denote the (global) heterogeneity/anisotropy ratio of $\mathbb{A}$.

Admissible meshes

Definition

The mesh $\mathcal{T}_h$ is admissible if (i) $\mathcal{T}_h$ is a finite collection of disjoint polygons/polyhedra T s.t. $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T}$, (ii) $\mathcal{T}_h$ admits a matching simplicial submesh $\mathfrak{T}_h$ that is

- shape-regular in the usual sense of Ciarlet;
- contact-regular: every simplex $S \subseteq T$ of $\mathfrak{T}_h$ is s.t. $h_S \approx h_T$.

On any $T \in \mathcal{T}_h$, we let $\rho_T := \beta_T / \alpha_T$ denote the (local) heterogeneity/anisotropy ratio of $\mathbb{A}_{|T}$.

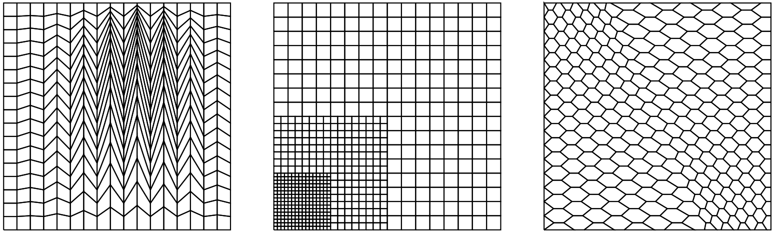

Figure : Admissible meshes in 2D

Discrete unknowns

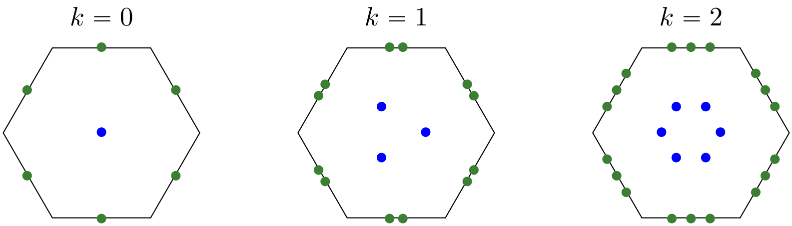

Figure : DoFs associated with local unknowns, $d = 2$

Local hybrid set of unknowns

$$\boxed{\underline{\mathrm{U}}_T^k := \mathbb{P}_d^k(T) \times \mathbb{P}_{d-1}^k(\mathcal{F}_T)}$$

Local reduction operator

$\underline{\mathrm{I}}_T^k : H^1(T) \to \underline{\mathrm{U}}_T^k$ s.t., for any $v \in H^1(T)$, $\underline{\mathrm{I}}_T^k v := \left(\Pi_T^k v, \Pi_{\partial T}^k v\right)$, with $(\Pi_{\partial T}^k v)_{|F} := \Pi_F^k v$ for all $F \in \mathcal{F}_T$

Reconstruction operator

Local reconstruction operator $\mathrm{p}_T^{k+1} : \underline{\mathrm{U}}_T^k \to \mathbb{P}_d^{k+1}(T)$

For $\underline{\mathrm{v}}_T = (\mathrm{v}_T, \mathrm{v}_{\partial T}) \in \underline{\mathrm{U}}_T^k$, $\mathrm{p}_T^{k+1}\underline{\mathrm{v}}_T \in \mathbb{P}_d^{k+1}(T)$ s.t. $\int_T \mathrm{p}_T^{k+1}\underline{\mathrm{v}}_T = \int_T \mathrm{v}_T$ solves, for all $\mathrm{w} \in \mathbb{P}_d^{k+1}(T)$,

$$\boxed{\left(\mathbb{A}\boldsymbol{\nabla}\mathrm{p}_T^{k+1}\underline{\mathrm{v}}_T, \boldsymbol{\nabla}\mathrm{w}\right)_T = -(\mathrm{v}_T, \mathrm{div}(\mathbb{A}\boldsymbol{\nabla}\mathrm{w}))_T + \left(\mathrm{v}_{\partial T}, \mathbb{A}\boldsymbol{\nabla}\mathrm{w}\cdot\boldsymbol{n}_{\partial T}\right)_{\partial T}}$$

$\rightsquigarrow$ heterogeneity/anisotropy included in the reconstruction procedure

Computation

Requires to invert a SPD matrix of size N_d^{k+1}, with $\mathrm{N}_l^q := \dim(\mathbb{P}_l^q)$

Approximation properties of $\mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k$

- $\mathbb{A}_{|T} \in \left[\mathbb{P}_d^0(T)\right]^{d\times d}$: $\|\mathbb{A}^{1/2}\boldsymbol{\nabla}(v - \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v)\|_{0,T} = \inf_{\mathrm{w}\in\mathbb{P}_d^{k+1}(T)} \|\mathbb{A}^{1/2}\boldsymbol{\nabla}(v-\mathrm{w})\|_{0,T}$
- $\mathbb{A}_{|T} \in [\mathrm{Lips}(T)]^{d\times d}$: with $\gamma = 1/2$ if $\mathbb{A}_{|T}$ is constant, $\gamma = 1$ otherwise,
  $\|v - \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v\|_{0,T} + h_T\|\boldsymbol{\nabla}(v - \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v)\|_{0,T} \lesssim \rho_T^{\gamma} h_T^{k+2}\|v\|_{k+2,T}$

Stabilization

$$\boxed{a_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) := \left(\mathbb{A}\boldsymbol{\nabla}\mathrm{p}_T^{k+1}\underline{\mathrm{u}}_T, \boldsymbol{\nabla}\mathrm{p}_T^{k+1}\underline{\mathrm{v}}_T\right)_T + j_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T)}$$

Local stabilization $j_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}$

For any $\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$, and with $\kappa_{T,F} := \|\mathbb{A}_{|T}\boldsymbol{n}_{T,F}\cdot\boldsymbol{n}_{T,F}\|_{\infty,F}$,

$$j_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) := \sum_{F\in\mathcal{F}_T} \frac{\kappa_{T,F}}{h_F}\left(\Pi_F^k\left(\mathrm{q}_T^{k+1}\underline{\mathrm{u}}_T - \mathrm{u}_F\right), \Pi_F^k\left(\mathrm{q}_T^{k+1}\underline{\mathrm{v}}_T - \mathrm{v}_F\right)\right)_F,$$

where $\mathrm{q}_T^{k+1}\underline{\mathrm{w}}_T := \mathrm{w}_T + \left(\mathrm{p}_T^{k+1}\underline{\mathrm{w}}_T - \Pi_T^k\mathrm{p}_T^{k+1}\underline{\mathrm{w}}_T\right)$

$\rightsquigarrow$ the use of Π_F^k is reminiscent of Lehrenfeld-Schöberl stabilization for HDG with cell unknowns of order $(k+1)$ [Lehrenfeld, 10]

$\rightsquigarrow$ the operator q_T^{k+1} is new and opens the door to cell unknowns of order $\{(k-1), k\}$

Approximation properties

$$j_T(\underline{\mathrm{I}}_T^k v, \underline{\mathrm{I}}_T^k v)^{1/2} \lesssim \beta_T^{1/2}\rho_T^{\gamma}h_T^{k+1}\|v\|_{k+2,T}$$

Discrete problem

Global hybrid set of unknowns

$$\boxed{\underline{\mathrm{U}}_h^k := \mathbb{P}_d^k(\mathcal{T}_h) \times \mathbb{P}_{d-1}^k(\mathcal{F}_h)}$$

Discrete problem

Find $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_{h,0}^k$ s.t.

$$a_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) = (f, \mathrm{v}_{\Omega_h})_\Omega \qquad \text{for all } \underline{\mathrm{v}}_h = (\mathrm{v}_{\Omega_h}, \mathrm{v}_{\partial\mathcal{T}_h}) \in \underline{\mathrm{U}}_{h,0}^k$$

with $a_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) := \sum_{T\in\mathcal{T}_h} a_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T)$

Stability

$$\rho_T^{-1}\left(\|\mathbb{A}^{1/2}\nabla \mathrm{v}_T\|_{0,T}^2 + \sum_{F\in\mathcal{F}_T} \frac{\kappa_{T,F}}{h_F}\|\Pi_F^k(\mathrm{v}_T - \mathrm{v}_F)\|_{0,F}^2\right) \lesssim a_T(\underline{\mathrm{v}}_T, \underline{\mathrm{v}}_T)$$

Guaranteed error estimates

Theorem (Energy-norm error estimate)

Assume $\mathbb{A} \in \left[\mathbb{P}^l_d(\mathcal{T}_h)\right]^{d\times d}$ *for some* $l \in \mathbb{N}$, *and* $u \in H^1_0(\Omega) \cap H^{k+2}(\mathcal{T}_h)$.

$$\|\mathbb{A}^{1/2}(\nabla u - \nabla_h \mathrm{p}^{k+1}_{\Omega_h}\underline{\mathrm{u}}_h)\|_{0,\Omega} \lesssim \left\{ \sum_{T\in\mathcal{T}_h} \beta_T \rho_T^{2\gamma} h_T^{2(k+1)} \|u\|^2_{k+2,T} \right\}^{1/2}$$

Theorem (L^2-norm error estimate)

Assume $\mathbb{A} \in \left[\mathbb{P}^0_d(\mathcal{T}_h)\right]^{d\times d}$, *elliptic regularity* $\|z(g)\|_{2,\mathcal{T}_h} \lesssim \alpha^{-1}\|g\|_{0,\Omega}$, *and* $f \in H^{k+\delta}(\Omega)$ *with* $\delta = 1$ *if* $k = 0$ *and* $\delta = 0$ *if* $k \geqslant 1$.

$$\alpha\|u - \mathrm{p}^{k+1}_{\Omega_h}\underline{\mathrm{u}}_h\|_{0,\Omega} \lesssim \beta^{1/2}\rho^{1/2}\, h \left\{ \sum_{T\in\mathcal{T}_h} \beta_T \rho_T h_T^{2(k+1)} \|u\|^2_{k+2,T} \right\}^{1/2} + h^{k+2}\|f\|_{k+\delta,\Omega}$$

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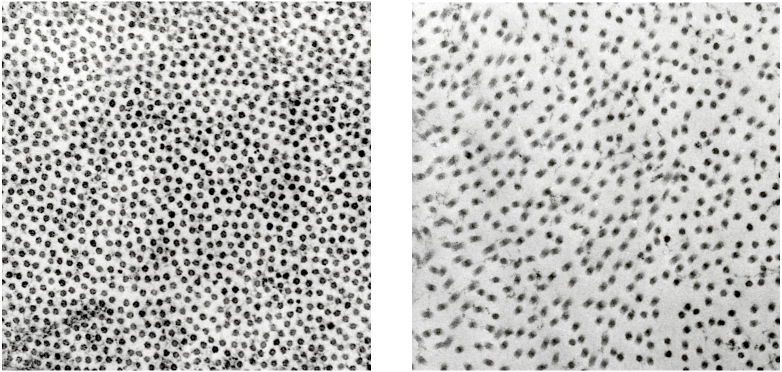
Reference

⇝ Cicuttin, M., Ern, A., and SL (In preparation) A Multi-scale Hybrid High-Order method.

A multi-scale medium: the human cornea



(a) Healthy cornea　　(b) Oedematous cornea

Figure : Cross-section of the corneal stroma (courtesy of K. Plamann, Laboratoire d'Optique Appliquée, Palaiseau)

Model oscillatory problem

Problem: Let $f \in L^2(\Omega)$. Find a potential $u_\varepsilon : \Omega \to \mathbb{R}$ such that

$$\boxed{\begin{aligned} -\mathrm{div}(\mathbb{A}_\varepsilon \boldsymbol{\nabla} u_\varepsilon) &= f \quad \text{in } \Omega \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega \end{aligned}}$$

Assumption: $\mathbb{A}_\varepsilon(\cdot) = \mathbb{A}(\cdot/\varepsilon)$ in Ω, for $\mathbb{A}$ a symmetric $\mathbb{Z}^d$-periodic matrix field on $\mathbb{R}^d$ such that $\mathbb{A} \in L^\infty(\mathbb{R}^d; \mathcal{S}_\alpha^\beta)$

The theory of homogenization ensures the existence of $\mathbb{A}_0 \in \mathcal{S}_\alpha^\beta$ s.t. the (whole) family $(\mathbb{A}_\varepsilon)_{\varepsilon>0}$ G-converges to $\mathbb{A}_0$. The homogenized problem is

$$\begin{cases} -\mathrm{div}(\mathbb{A}_0 \boldsymbol{\nabla} u_0) = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

Background on multi-scale numerical methods (1/2)

- The H^{k+2}-norm of u_ε scales as $\varepsilon^{-(k+1)}$.
- Mono-scale methods hence provide an energy-norm decay of the error of order $\left(\frac{h}{\varepsilon}\right)^{k+1}$.
- To be accurate, mono-scale methods must rely on a mesh resolving the fine scale, i.e. with $h \ll \varepsilon$.
- Since ε is supposedly much smaller than the size of Ω (that is assumed to be of order one), an accurate approximation implies an overwhelming number of DoFs.
- In a multi-query context (the solution is needed for a large number of rhs), a mono-scale solve is unaffordable!
- Multi-scale methods aim at resolving the fine scale in an offline stage.

Background on multi-scale numerical methods (2/2)

Let $\mathcal{T}_H$ be a coarse matching simplicial mesh of Ω, and let $k = 0$.

The Multi-scale Finite Element Method (MsFEM) [Hou and Wu, 97]

- Offline stage: for each node $\boldsymbol{x}^n$ of $\mathcal{T}_H$, solve, for each cell T s.t. $\boldsymbol{x}^n \in T$,

$$-\mathrm{div}(\mathbb{A}_\varepsilon \boldsymbol{\nabla} \varphi^n_{\varepsilon|T}) = 0 \quad \text{in } T, \qquad \varphi^n_{\varepsilon|T} = \Phi^n_{|T} \quad \text{on } \partial T$$

  $\rightsquigarrow$ this stage is independent of the rhs and fully parallelizable

- Online stage: for any rhs $f \in L^2(\Omega)$, solve

$$(\mathbb{A}_\varepsilon \boldsymbol{\nabla} u_{\varepsilon,H}, \boldsymbol{\nabla} v_{\varepsilon,H})_\Omega = (f, v_{\varepsilon,H})_\Omega \text{ (+ BC)} \quad \forall v_{\varepsilon,H} \in V^1_{\varepsilon,H}$$

  where $V^1_{\varepsilon,H} := \mathrm{Span}\{(\varphi^n_\varepsilon)_n\}$

  $\rightsquigarrow$ the linear system to solve has a reduced size

- Error estimate: assuming $\mathbb{A}_\varepsilon$ periodic and adequate regularity,

$$\|\boldsymbol{\nabla}(u_\varepsilon - u_{\varepsilon,H})\|_{0,\Omega} \leqslant c(\rho)\left(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}}\right)\mathcal{N}_\Omega(u_0)$$

The Heterogeneous Multi-scale Method (HMM) [E and Engquist, 03]

$\rightsquigarrow$ compute an approximation of u_0 by means of local averages of $\mathbb{A}_\varepsilon$

Oscillatory basis functions

Let $\mathcal{T}_H$ be an admissible mesh, with mehsize $H > \varepsilon$.

For any $T \in \mathcal{T}_H$, we denote by $(\Phi^i_T)_{1 \leqslant i \leqslant \mathrm{N}^k_d}$ a set of basis functions of $\mathbb{P}^k_d(T)$, and for any $F \in \mathcal{F}_H$, we let $(\Phi^j_F)_{1 \leqslant j \leqslant \mathrm{N}^k_{d-1}}$ be a set of basis functions of $\mathbb{P}^k_{d-1}(F)$.

Cell-based basis functions $(\varphi^i_{\varepsilon,T})_{1 \leqslant i \leqslant \mathrm{N}^k_d}$

For $T \in \mathcal{T}_H$, and $1 \leqslant i \leqslant \mathrm{N}^k_d$, $\varphi^i_{\varepsilon,T} \in H^1(T)$ solves the local problem

$$\inf \left\{ \int_T \left[\frac{1}{2} \mathbb{A}_\varepsilon \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\varphi - \Phi^i_T\, \varphi \right], \varphi \in H^1(T),\ \Pi^k_F \varphi = 0\ \forall F \in \mathcal{F}_T \right\}.$$

Face-based basis functions $(\varphi^j_{\varepsilon,F})_{1 \leqslant j \leqslant \mathrm{N}^k_{d-1}}$

For $F \in \mathcal{F}_H$ (if $F \in \mathcal{F}^{\mathrm{i}}_H$ with $F \subseteq \partial T_1 \cap \partial T_2$ we let $\mathcal{P} := \{1,2\}$, if $F \in \mathcal{F}^{\mathrm{b}}_H$ with $F \subseteq \partial T_1 \cap \partial\Omega$ we let $\mathcal{P} := \{1\}$), $p \in \mathcal{P}$, and $1 \leqslant j \leqslant \mathrm{N}^k_{d-1}$, $\varphi^j_{\varepsilon,F,T_p} \in H^1(T_p)$ solves the local problem

$$\inf \left\{ \int_{T_p} \left[\frac{1}{2} \mathbb{A}_\varepsilon \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\varphi \right], \varphi \in H^1(T_p),\ \Pi^k_F \varphi = \Phi^j_F,\ \Pi^k_\sigma \varphi = 0\ \forall \sigma \in \mathcal{F}_{T_p} \backslash \{F\} \right\}.$$

We introduce $\varphi^j_{\varepsilon,F}$ defined on $\bigcup_{p \in \mathcal{P}} T_p$ s.t. $\varphi^j_{\varepsilon,F|T_p} := \varphi^j_{\varepsilon,F,T_p}$ for all $p \in \mathcal{P}$.

Fine-scale approximation space

For any $T \in \mathcal{T}_H$, we introduce the space

$$V_{\varepsilon,T}^{k+1} := \left\{ v_\varepsilon \in H^1(T) \mid \mathrm{div}(\mathbb{A}_\varepsilon \boldsymbol{\nabla} v_\varepsilon) \in \mathbb{P}_d^k(T),\ \mathbb{A}_\varepsilon \boldsymbol{\nabla} v_\varepsilon \cdot \boldsymbol{n}_{\partial T} \in \mathbb{P}_{d-1}^k(\mathcal{F}_T) \right\},$$

of dimension $\left(\mathrm{N}_d^k + \mathrm{card}(\mathcal{F}_T) \times \mathrm{N}_{d-1}^k\right)$.

Characterization of $V_{\varepsilon,T}^{k+1}$

$$\boxed{V_{\varepsilon,T}^{k+1} = \mathrm{Span}\left\{ (\varphi_{\varepsilon,T}^i)_{1 \leqslant i \leqslant \mathrm{N}_d^k}, (\varphi_{\varepsilon,F|T}^j)_{F \in \mathcal{F}_T,\, 1 \leqslant j \leqslant \mathrm{N}_{d-1}^k} \right\}}$$

$\rightsquigarrow$ generalizes the ideas of MsFEM *à la* Crouzeix–Raviart [Le Bris, Legoll and Lozinski, 13&14] to arbitrary approximation orders and general element shapes

Approximation in $V_{\varepsilon,T}^{k+1}$

Let $\pi_{\varepsilon,T}^{k+1}(u_0) \in V_{\varepsilon,T}^{k+1}$ such that $\int_T \pi_{\varepsilon,T}^{k+1}(u_0) = \int_T u_\varepsilon$ and

$$\begin{cases} -\mathrm{div}(\mathbb{A}_\varepsilon \boldsymbol{\nabla} \pi_{\varepsilon,T}^{k+1}(u_0)) = -\mathrm{div}(\mathbb{A}_0 \boldsymbol{\nabla} \Pi_T^{k+1} u_0) \in \mathbb{P}_d^k(T) & \text{in } T, \\ \mathbb{A}_\varepsilon \boldsymbol{\nabla} \pi_{\varepsilon,T}^{k+1}(u_0) \cdot \boldsymbol{n}_{\partial T} = \mathbb{A}_0 \boldsymbol{\nabla} \Pi_T^{k+1} u_0 \cdot \boldsymbol{n}_{\partial T} \in \mathbb{P}_{d-1}^k(\mathcal{F}_T) & \text{on } \partial T. \end{cases}$$

$$\|\mathbb{A}_\varepsilon^{1/2} \boldsymbol{\nabla}(u_\varepsilon - \pi_{\varepsilon,T}^{k+1}(u_0))\|_{0,T} \lesssim \|\mathbb{A}_\varepsilon^{1/2} \boldsymbol{\nabla}(u_\varepsilon - \mathcal{L}_\varepsilon^1(u_0))\|_{0,T} + \beta^{1/2} \rho^{1/2} \left(\sqrt{\varepsilon} + H_T^{k+1} + \sqrt{\frac{\varepsilon}{H_T}} \right) \mathcal{N}_T(u_0)$$

Discrete unknowns [SAME SLIDE]

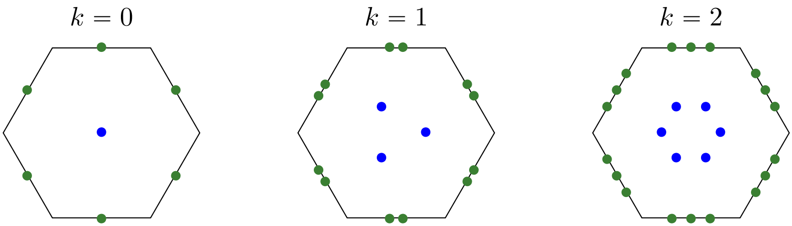


Figure : DoFs associated with local unknowns, $d = 2$

Local hybrid set of unknowns

$$\boxed{\underline{\mathrm{U}}_T^k := \mathbb{P}_d^k(T) \times \mathbb{P}_{d-1}^k(\mathcal{F}_T)}$$

Local reduction operator

$\underline{\mathrm{I}}_T^k : H^1(T) \to \underline{\mathrm{U}}_T^k$ s.t., for any $v \in H^1(T)$, $\underline{\mathrm{I}}_T^k v := \left(\Pi_T^k v, \Pi_{\partial T}^k v\right)$, with $(\Pi_{\partial T}^k v)_{|F} := \Pi_F^k v$ for all $F \in \mathcal{F}_T$

Multi-scale reconstruction operator

Local multi-scale reconstruction operator $p^{k+1}_{\varepsilon,T} : \underline{\mathrm{U}}^k_T \to V^{k+1}_{\varepsilon,T}$

For $\underline{\mathrm{v}}_T = (\mathrm{v}_T, \mathrm{v}_{\partial T}) \in \underline{\mathrm{U}}^k_T$, $p^{k+1}_{\varepsilon,T}\underline{\mathrm{v}}_T \in V^{k+1}_{\varepsilon,T}$ s.t. $\int_T p^{k+1}_{\varepsilon,T}\underline{\mathrm{v}}_T = \int_T \mathrm{v}_T$ solves, for all $w_\varepsilon \in V^{k+1}_{\varepsilon,T}$,

$$\boxed{\left(\mathbb{A}_\varepsilon \boldsymbol{\nabla} p^{k+1}_{\varepsilon,T}\underline{\mathrm{v}}_T, \boldsymbol{\nabla} w_\varepsilon\right)_T = -(\mathrm{v}_T, \mathrm{div}(\mathbb{A}_\varepsilon \boldsymbol{\nabla} w_\varepsilon))_T + (\mathrm{v}_{\partial T}, \mathbb{A}_\varepsilon \boldsymbol{\nabla} w_\varepsilon \cdot \boldsymbol{n}_{\partial T})_{\partial T}}$$

Computation

Requires to invert a SPD matrix of size $\left(\mathrm{N}^k_d + \mathrm{card}(\mathcal{F}_T) \times \mathrm{N}^k_{d-1}\right)$

Approximation properties of $p^{k+1}_{\varepsilon,T}\underline{\mathrm{I}}^k_T$

- $\|\mathbb{A}_\varepsilon^{1/2}\boldsymbol{\nabla}(u_\varepsilon - p^{k+1}_{\varepsilon,T}\underline{\mathrm{I}}^k_T u_\varepsilon)\|_{0,T} = \inf_{w_\varepsilon \in V^{k+1}_{\varepsilon,T}} \|\mathbb{A}_\varepsilon^{1/2}\boldsymbol{\nabla}(u_\varepsilon - w_\varepsilon)\|_{0,T}$
- $\|\mathbb{A}_\varepsilon^{1/2}\boldsymbol{\nabla}(u_\varepsilon - p^{k+1}_{\varepsilon,T}\underline{\mathrm{I}}^k_T u_\varepsilon)\|_{0,T} \leqslant \|\mathbb{A}_\varepsilon^{1/2}\boldsymbol{\nabla}(u_\varepsilon - \pi^{k+1}_{\varepsilon,T}(u_0))\|_{0,T}$
- $\|u_\varepsilon - p^{k+1}_{\varepsilon,T}\underline{\mathrm{I}}^k_T u_\varepsilon\|_{0,T} \lesssim \alpha^{-1/2} H_T \|\mathbb{A}_\varepsilon^{1/2}\boldsymbol{\nabla}(u_\varepsilon - p^{k+1}_{\varepsilon,T}\underline{\mathrm{I}}^k_T u_\varepsilon)\|_{0,T}$

Stabilization

$$\boxed{a_{\varepsilon,T}(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) := (\mathbb{A}_\varepsilon \boldsymbol{\nabla} p^{k+1}_{\varepsilon,T}\underline{\mathrm{u}}_T, \boldsymbol{\nabla} p^{k+1}_{\varepsilon,T}\underline{\mathrm{v}}_T)_T + j_{\varepsilon,T}(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T)}$$

Local stabilization $j_{\varepsilon,T} : \underline{\mathrm{U}}^k_T \times \underline{\mathrm{U}}^k_T \to \mathbb{R}$

For any $\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T \in \underline{\mathrm{U}}^k_T$,

$$j_{\varepsilon,T}(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) := \sum_{F \in \mathcal{F}_T} \frac{\alpha}{H_F} \left(\Pi^k_F (q^{k+1}_{\varepsilon,T}\underline{\mathrm{u}}_T - \mathrm{u}_F), \Pi^k_F (q^{k+1}_{\varepsilon,T}\underline{\mathrm{v}}_T - \mathrm{v}_F) \right)_F,$$

where $q^{k+1}_{\varepsilon,T}\underline{\mathrm{w}}_T := \mathrm{w}_T + (p^{k+1}_{\varepsilon,T}\underline{\mathrm{w}}_T - \Pi^k_T p^{k+1}_{\varepsilon,T}\underline{\mathrm{w}}_T)$

Approximation properties

$$j_{\varepsilon,T}(\underline{\mathrm{I}}^k_T u_\varepsilon, \underline{\mathrm{I}}^k_T u_\varepsilon)^{1/2} \lesssim \|\mathbb{A}_\varepsilon^{1/2} \boldsymbol{\nabla}(u_\varepsilon - p^{k+1}_{\varepsilon,T}\underline{\mathrm{I}}^k_T u_\varepsilon)\|_{0,T}$$

Discrete problem

Global hybrid set of unknowns

$$\boxed{\underline{\mathrm{U}}_H^k := \mathbb{P}_d^k(\mathcal{T}_H) \times \mathbb{P}_{d-1}^k(\mathcal{F}_H)}$$

Discrete problem

Find $\underline{\mathrm{u}}_{\varepsilon,H} \in \underline{\mathrm{U}}_{H,0}^k$ s.t.

$$a_{\varepsilon,H}(\underline{\mathrm{u}}_{\varepsilon,H}, \underline{\mathrm{v}}_H) = (f, \mathrm{v}_{\Omega_H})_\Omega \qquad \text{for all } \underline{\mathrm{v}}_H = (\mathrm{v}_{\Omega_H}, \mathrm{v}_{\partial\mathcal{T}_H}) \in \underline{\mathrm{U}}_{H,0}^k$$

with $a_{\varepsilon,H}(\underline{\mathrm{u}}_{\varepsilon,H}, \underline{\mathrm{v}}_H) := \sum_{T\in\mathcal{T}_H} a_{\varepsilon,T}(\underline{\mathrm{u}}_{\varepsilon,T}, \underline{\mathrm{v}}_T)$

Stability

$$\alpha\left(\|\boldsymbol{\nabla}\mathrm{v}_T\|_{0,T}^2 + \sum_{F\in\mathcal{F}_T} H_F^{-1}\|\Pi_F^k(\mathrm{v}_T - \mathrm{v}_F)\|_{0,F}^2\right) \lesssim a_{\varepsilon,T}(\underline{\mathrm{v}}_T, \underline{\mathrm{v}}_T)$$

Guaranteed error estimate

Theorem (Energy-norm error estimate)

Assume $\mathbb{A}$ is Hölder continuous on $\mathbb{R}^d$, and $u_0 \in H^{k+2+\delta}(\Omega)$ with $\delta = 1$ if $k = 0$, $\delta = 0$ if $k \geqslant 1$ (we also assume $\mathcal{T}_H$ quasi-uniform for simplicity).

$$\|\mathbb{A}_\varepsilon^{1/2}(\nabla u_\varepsilon - \nabla_H p_{\varepsilon,\Omega_H}^{k+1} \underline{\mathrm{u}}_{\varepsilon,H})\|_{0,\Omega} \lesssim \beta^{1/2} \rho \left(\sqrt{\varepsilon} + H^{k+1} + \sqrt{\frac{\varepsilon}{H}} \right) \|u_0\|_{k+2+\delta,\Omega}$$

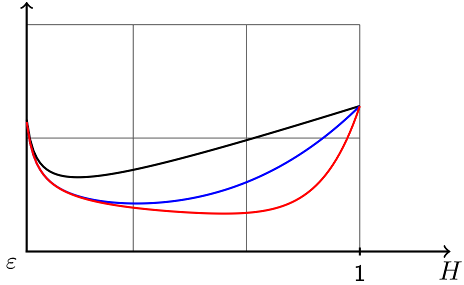


Figure : Overall behavior of the energy-norm error as a function of H, for $k = 0$, $k = 2$ and $k = 8$

Outline

Context

HHO for mildly heterogeneous diffusion

Multi-scale HHO

HHO in a nutshell

Features of HHO (1/2)

Discontinuous skeletal method

- arbitrary order ($k \geqslant 0$) of approximation
- applicable to general meshes (including tetra/hexa-hedra!)
- dimension-independent construction
- optimal ($k+1$) energy- and ($k+2$) L^2-norms error estimates for arbitrary element shapes for all (admissible) orders of cell unknowns

Attractive computational features

- design from primal formulation (SPD matrix in coercive case)
- reduced number of DoFs (in 3D, $\mathrm{N}_{\mathrm{DoFs}}^{\mathrm{HHO}} \approx \frac{1}{2}k^2\mathrm{card}(\mathcal{F}_h)$, to compare with $\mathrm{N}_{\mathrm{DoFs}}^{\mathrm{DG}} \approx \frac{1}{6}k^3\mathrm{card}(\mathcal{T}_h)$) and compact (face-based) stencil
- polynomial basis functions
- no user-dependent parameter to tune
- (fully parallelizable) offline/online structure of computations
- easily implementable in working HDG codes

Features of HHO (2/2)

Physical fidelity

- local conservativity
- reproduction of desirable continuum properties (integration by parts formulas, symmetries, kernels of operators)
- robustness w.r.t. physical parameters in various situations: heterogeneous/anisotropic diffusion, quasi-incompressible linear elasticity, advection-dominated transport...

THANK YOU FOR YOUR ATTENTION

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