Long time stability for large systems of interacting agents

Particle Systems and PDEs XIV, Institut de Mathématiques de Toulouse

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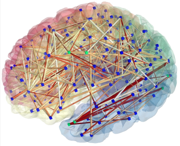
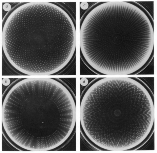
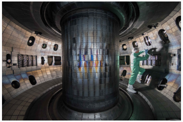
Multi-scale systems and applications


(biological neural network)


(Petri dish)


(Tokamak)

Microscopic

$x_i(t),\ v_i(t),\ \ i = 1, \ldots, N$

ODEs/SDEs

$N \to \infty$

Mesoscopic

$f(t, x, v) \geq 0$

kinetic PDEs

$\int \cdot \ \mathrm{d}v$

Macroscopic

$\rho(t, x),\ u(t, x)$

fluid PDEs

$N \to \infty$

mathematical challenges $\begin{cases} \text{systems dimension} \sim N \\ \text{interaction structure} \\ \text{multiple parameters} \end{cases}$

Goal : asymptotic analysis of the limit $N \to +\infty$ uniformly in time

We consider the system

$$\begin{cases} \mathrm{d}X_i^N = \displaystyle\sum_{j=1}^{N} \underbrace{K^N(X_i^N - X_j^N)}_{\text{interactions}} \mathrm{d}t + \underbrace{\sqrt{2}\mathrm{d}W_i}_{\text{diffusion}}, \quad i \in \{1, \dots, N\} \\ \mathcal{L}(X_1^N, \dots, X_N^N)(t = 0) \; = \; \rho_N(t = 0) \end{cases}$$

$X_i^N(t) \in \mathbb{T}^d$: position of the i-th agent at time $t \geq 0$.

- Interactions are typically singular. Ex:

$$K(x) \; = \; \pm \frac{x}{|x|^d}, \quad x \in \mathbb{R}^d \quad (\text{Keller-Segel or Coulombian}, \;\; \pm div_x K \; \leq \; 0)$$

$$K(x) \; = \; \alpha \frac{x^\perp}{|x|^2}, \quad x \in \mathbb{R}^2 \quad (\text{Biot-Savart}, \;\; div_x K \; = \; 0)$$

- Mean-field regime : $\quad K^N \sim \dfrac{1}{N} K \quad$ and $\quad \rho_N(t = 0) \sim \rho_0^{\otimes N} \quad$ as $N \gg 1$

Goal : Formal analysis of the interacting agents system as $N \gg 1$.

$$\begin{cases} \mathrm{d}X_i^N = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} K(X_i^N - X_j^N)\mathrm{d}t + \sqrt{2}\mathrm{d}W_i\,, \quad i \in \{1,\ldots,N\} \\ \mathcal{L}(X_1^N,\ldots,X_N^N)(t=0) \;=\; \rho_0^{\otimes N} \end{cases}$$

$X_i^N(t) \in \mathbb{T}^d$: position of the i-th agent at time $t \geq 0$.

Propagation of chaos : the agent distribution remains i.i.d. at $t > 0$ as $N \gg 1$

$$\rho_k^N(t, x_1, \ldots, x_k) = \mathcal{L}(X_1^N, \ldots, X_k^N) \underset{N\to+\infty}{\longrightarrow} \rho(t,x_1)\rho(t,x_2)\cdots\rho(t,x_k)\,, \quad t \geq 0\,, \quad N \geq k \geq 1$$

where ρ solves the following drift diffusion equation

$$\begin{cases} \partial_t \rho(t,x) + \nabla_x \cdot \left(\displaystyle\int \underbrace{K(x - x^\star)}_{\text{interactions}} \rho(t,x^\star)\,\mathrm{d}x^\star\, \rho(t,x) - \underbrace{\nabla_x \rho(t,x)}_{\text{diffusion}} \right) = 0\,, \quad x \in \mathbb{T}^d\,, \\ \rho(t=0) = \rho_0\,. \end{cases}$$

Questions of interest

Convergence analysis of

$$\rho_k^N(t) = \mathcal{L}(X_1^N, \dots, X_k^N) \underset{N \to +\infty}{\longrightarrow} \rho(t)^{\otimes k}$$

if $\rho_N \sim \rho_0^{\otimes N}$ at $t = 0$, where

$$\partial_t \rho + \nabla_x \cdot (\rho\, K \star \rho - \nabla_x \rho) = 0 , \quad x \in \mathbb{T}^d .$$

There are several difficulties :

- explicit convergence rates in N
  - optimal rate is $O(N^{-2})$ in $L\,log\,L$ (D. Lacker '23)
- structure of the kernel K
  - Biot-Savart kernel ($d = 2$): L^1 convergence (N. Fournier, M. Hauray, S. Mischler '14)
  - Keller-Segel kernel ($d = 2$): weak convergence (N. Fournier, B. Jourdain '17)
  - Biot-Savart kernel ($d = 2$): rate $O(N^{-1})$ in $L\,log\,L$ (P.-E. Jabin, Z. Wang '18)
  - Keller-Segel/Coulomb kernel ($d = 2$) : explicit rate in $L\,log\,L$ (D. Bresch, *et al* '19 and '20)
  - non attractive Coulomb-Riesz kernel ($d \leq 2$, no diffusion) : weak convergence (M. Duerinckx '16)
  - non attractive Coulomb-Riesz kernel ($d \geq 1$) : weak convergence (S. Serfaty '20)
- uniform rate as $t \gg 1$
  - Biot-Savart law ($d = 2$): rate $O(N^{-1})$ in $L\,log\,L$ (A. Guillin *et al* '25)
  - Keller-Segel kernel ($d = 2$) : rate in $L\,log\,L$ (A. Chodron de Courcel *et al* '25 and '25)

Goal : Uniform estimates as $t \gg 1$ and $N \gg 1$.

Theorem (A. Béjar-López, A.B., P.-E. Jabin, J. Soler ('24))

We consider an interaction kernel

$$K \in W^{-\frac{2}{d+2}, d+2}(\mathbb{T}^d),$$

and we suppose that at $t = 0$, *the* k*-th marginal* ρ_k^N *satisfies*

$$\left\|\rho_k^N(t=0)\right\|_{L^2} \lesssim k^{\alpha k} \qquad \forall 1 \le k \le N, \quad \forall N \in \mathbb{N}^\star,$$

for some $\alpha > 0$. *There exists* $\beta > \alpha$ *such that*

$$\sup_{t \ge 0} \left\|\rho_k^N(t)\right\|_{L^2} \lesssim k^{\beta k} \qquad \forall 1 \le k \le N, \quad \forall N \in \mathbb{N}^\star.$$

Comments :

(i) Uniform in time estimates

(ii) Strong norms

(iii) General singular kernels

Goal : Uniform estimates as $t \gg 1$ and $N \gg 1$.

Comments regarding the assumptions on K :

(i) general assumptions on the structure of K ;

(ii) applies to power laws : K lies in $W^{-\frac{2}{d+2},d+2}$ if

$$K \in \mathcal{C}^\infty(\mathbb{T}^d \setminus \{0\}), \quad \text{and} \quad |K(x)| \underset{x\to 0}{\sim} \frac{1}{|x|^\alpha}, \quad \alpha < 1,$$

thanks to the Sobolev injection $L^p \hookrightarrow W^{-\frac{2}{d+2},d+2}$, $p > d$;

(iii) the result extends to $K \in L^d$;

(iv) applies to more singular kernels than L^d: $K_\alpha \in W^{-\frac{2}{d+2},d+2}(B(0,\varepsilon)) \setminus L^d(B(0,\varepsilon))$ if

$$K_\alpha(x) = \frac{x}{|x|^2\,|\ln|x||^{1-\alpha}}$$

with $d = 2$ and $1/2 < \alpha < 3/4$.

High temperature regime

Objectif : Extension to kernels in H^{-1} when $\sigma \gtrsim \|\rho_k\|^{1/k}$

$$\mathrm{d}X_i = \frac{1}{N}\sum_{j=1}^{N} \underbrace{K(X_i - X_j)}_{\text{interactions}}\,\mathrm{d}t + \underbrace{\sqrt{\sigma}\,\mathrm{d}W_i}_{\text{diffusion}}, \quad i \in \{1,\dots,N\}\,.$$

Theorem (A. Béjar-López, A.B., P.-E. Jabin, J. Soler ('24))

We consider an interaction kernel which satisfies

$$K \in H^{-1}(\mathbb{T}^d)\,, \quad \textit{and} \quad (\mathrm{div}_x K)_- \in L^\infty(\mathbb{T}^d)$$

and we suppose that at $t = 0$*, the* k*-th marginal* ρ_k^N *satisfies*

$$\left\|\rho_k^N(t=0)\right\|_{L^2} \lesssim R^k \qquad \forall\, 1 \le k \le N\,, \quad \forall\, N \in \mathbb{N}^\star\,,$$

for some $0 < R < C\sigma$*. For all* $\tilde{R} > R$*, it holds*

$$\sup_{t\ge 0} \left\|\rho_k^N(t)\right\|_{L^2} \lesssim \tilde{R}^k \qquad \forall\, 1 \le k \le N\,, \quad \forall\, N \in \mathbb{N}^\star\,.$$

Comments :

(i) the result covers the repulsive Coulombian and Biot-Savart kernels in dimension $d \le 5$.

Uniform in time mean-field estimates

Goal : Convergence estimates uniform in $t \gg 1$.

It holds (P.-E. Jabin and Z. Wang, '18):

$$\|\rho_k^N(t) - \rho^{\otimes k}(t)\|_{L^1} \lesssim_k N^{-1/2} e^{Ct},$$

if $K \in W^{-1,\infty}$ and $div_x(K) = 0$ and $\rho^N(t=0) = \rho_0^{\otimes N}$.

Corollary (P.-E. Jabin, Z. Wang ('18) and A. B.-L., A.B., P.-E. J., J. S. ('24))

Under the assumptions of one of the previous theorem, we also suppose

$$K \in W^{-1,\infty}, \quad et \quad div_x(K) = 0, \quad and \quad \rho^N(t=0) = \rho_0^{\otimes N},$$

then we have for all $1 \leq p < 2$

$$\left\|\rho_k^N(t) - \rho^{\otimes k}(t)\right\|_{L^p} \lesssim_k \frac{e^{-\beta t}}{N^\gamma} \quad as \quad (t, N) \to +\infty,$$

where ρ *is the solution to the mean-field model*

$$\partial_t \rho + \nabla_x \cdot (\rho K \star \rho - \nabla_x \rho) = 0, \quad and \quad \rho(t=0) = \rho_0.$$

Comments :

(i) Uniform in time and explicit convergence rates ;

(ii) convergence in strong norms L^p, $1 < p < 2$.

These results are optimal in general

see also Chodron De Courcel *et al* ('23)

Proposition (A. B.-L., A.B., P.-E. J., J. S. ('24))

We consider the Kuramoto kernel

$$K(x) = -\sin(x)\,, \quad \forall x \in \mathbb{T}\,.$$

for sufficiently small $\sigma > 0$, *there exists* $\rho_0 \in \mathcal{C}^2 \cap \mathcal{P}(\mathbb{T})$ *and* $\eta > 0$ *such that*

$$\sup_{t \geq 0} \left\| \rho_1^N(t) - \rho(t) \right\|_{L^1} \geq \eta\,, \quad \forall N \geq 1,$$

where $\rho^N(t=0) = \rho_0^{\otimes N}$ *and where* ρ *solves*

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, K \star \rho - \nabla_x \rho) = 0\,, \\ \rho(t=0) = \rho_0\,. \end{cases}$$

Goal : uniform estimates in $N \gg 1$ and $t \gg 1$ for the system

$$\mathrm{d}X_i = \frac{1}{N}\sum_{j=1}^{N} K(X_i - X_j)\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_i\,, \quad i \in \{1,\dots,N\}\,,$$

under the assumption $K \in L^d(\mathbb{R}^d)$ et $\mathrm{div}_x(K) = 0$

Strategy: long time dynamics of the marginals $\rho_k(t, x_1, \dots, x_k) = \mathcal{L}(X_1, \dots, X_k)$.

BBGKY hierarchy

The sequence $(\rho_k)_{1\leq k\leq N}$ satisfies

$$\partial_t \rho_k + \frac{1}{N}\sum_{i,j=1}^{k} \mathrm{div}_{x_i}[K(x_i - x_j)\rho_k] + \frac{N-k}{N}\sum_{i=1}^{k} \mathrm{div}_{x_i}\int K(x_i - x_{k+1})\rho_{k+1}\mathrm{d}x_{k+1} = \Delta_{X^k}\rho_k\,.$$

When $\mathrm{div}_x(K) = 0$, the L^2 estimate may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_k\|_{L^2}^2 \leq k\left\|\int K(\,\cdot\, - x_{k+1})\rho_{k+1}\mathrm{d}x_{k+1}\right\|_{L^2}^2 - \|\nabla_{X^k}\rho_k\|_{L^2}^2$$

⚠ the estimate on ρ_k features ρ_{k+1} .

TWO KEY POINTS

- Interpolation: if $K \in L^{\frac{2}{\theta}}$ et $\theta \in (0,1)$, the interaction term satisfies

$$\left\| \int K(\cdot - x_{k+1}) \rho_{k+1} \mathrm{d}x_{k+1} \right\|_{L^2} \leq \min \left\{ \begin{array}{l} \|K\|_{L^\infty} \|\rho_k\|_{L^2} \\ \\ \|K\|_{L^2} \|\rho_{k+1}\|_{L^2} \end{array} \right. \leq \|K\|_{L^{\frac{2}{\theta}}} \|\rho_k\|_{L^2}^{1-\theta} \|\rho_{k+1}\|_{L^2}^{\theta} \,,$$

- optimal Sobolev inequality: for $\rho_k \in H^1 \cap \mathcal{P}(\mathbb{R}^{dk})$, it holds

(G. Talenti '76, T. Aubin '76, O. Druet '76, S. Gallot '83, E. Hebey & M. Vaugon '95,....)

$$\|\rho_k\|_{L^2}^{1+\frac{2}{dk}} \leq \|\rho_k\|_{L^{2^*_k}} \leq \omega_k \|\nabla_{X^k} \rho_k\|_{L^2} \,, \quad \text{where} \quad \omega_k \underset{k \to +\infty}{\sim} \frac{C}{\sqrt{k}} \,.$$

Therefore, the L^2 estimate of ρ_k may be written :

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} X_k \leq k \|K\|_{L^{\frac{2}{\theta}}} X_{k+1}^{\theta} X_k^{1-\theta} - C\, k\, X_k^{1+\frac{2}{dk}} \,, \qquad k \in \{1, \ldots, N\} \,,$$

where $X_k = \|\rho_k\|_{L^2}^2$.

Conclusion in the simplified case

Our estimate of $X_k = \|\rho_k\|_{L^2}^2$ may be written:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} X_k \leq k \, \|K\|_{L^{\frac{2}{\theta}}} X_{k+1}^{\theta} X_k^{1-\theta} - C\, k\, X_k^{1+\frac{2}{dk}}, \qquad k \in \{1, \dots, N\}.$$

Homogeneity: we suppose formally that $\rho_k = \rho^{\otimes k}$, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} X^k \leq k \, \|K\|_{L^{\frac{2}{\theta}}} X^{(k+1)\theta + k(1-\theta)} - C\, k\, X^{k+\frac{2}{d}}, \qquad k \in \{1, \dots, N\},$$

whom solutions remain bounded if

$$(k+1)\theta + k(1-\theta) = k + 2/d,$$

that is, $\theta = 2/d$ and $K \in L^d$.

- Extension to second order kinetic systems ;

  $|K(x)| \sim |x|^{-\alpha}$, $\alpha < 1$, without diffusion : weak convergence (M. Hauray, P.-E. Jabin '15), $K \in L^2$ (D. Bresch, M. Duerinckx, P.-E. Jabin '24)

  the linear structure is well understood (F. Bouchut '93)

We consider the system

$$\begin{cases} \mathrm{d}X_i^N = \underbrace{V_i^N}_{\text{transport}} \mathrm{d}t \\ \mathrm{d}V_i^N = \displaystyle\sum_{j=1}^N \underbrace{K^N(X_i^N - X_j^N)}_{\text{interactions}} \mathrm{d}t + \underbrace{\sqrt{2}\mathrm{d}W_i}_{\text{diffusion}}, \quad i \in \{1, \dots, N\} \end{cases}$$

$X_i^N(t) \in \mathbb{T}^d$: position of the i-th particle at time $t \geq 0$.

$V_i^N(t) \in \mathbb{R}^d$: velocity of the i-th particle at time $t \geq 0$.