Paths of order in a jungle of chaos

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Particle Systems and PDEs XIV

22 June 2026

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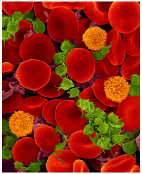
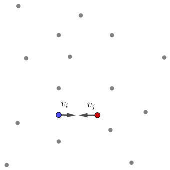
- Systems of many elements are everywhere around us.



The microscopic viewpoint



Typical equations for N elements

$$\frac{d}{dt}x_i(t) = v_i(t),$$

$$\frac{d}{dt}v_i(t) = \frac{1}{N}\sum_{j\neq i} K\left(x_i(t) - x_j(t)\right).$$

The macroscopic viewpoint

These equations dominate the field of *Fluid Dynamics*. Typical equations include the Euler equations and The Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u = -\nabla p + F,$$

$$\nabla \cdot u = 0.$$

The mesoscopic viewpoint

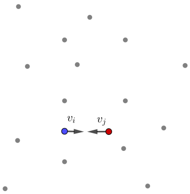


Figure: The full system

$\longrightarrow$

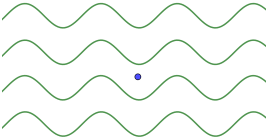


Figure: The "average phenomenon".

- The unknowns of these equations are probability densities/measures which represent an average element of the system.

The Boltzmann equation

- A fundamental kinetic equation

$$\partial_t f(t,x,v) + v \cdot \nabla f(t,x,v) = Q(f,f)(t,x,v),$$

with

$$Q(f,f)(x,v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B\left(|v - v_*|, \sigma\right) \left(f\left(x, v'\right) f\left(x, v'_*\right) - f\left(x, v\right) f\left(x, v_*\right)\right) dv_* d\sigma$$

where

- v and v' are the post-collision velocities.
- v_* and v'_* are the pre-collision velocities.
- v, v' and v_*, v'_* are connected via the conservation of momentum and energy. This gives rise to the scattering parameter σ.
- B represents the intensity of the collision.

Mean field limits – from "average" micro to meso

- In the late 1950's a new approach to consider systems of many elements emerged.
- In this approach, one replaces the full many element system with a "probabilistic" model which consists of a single equations for the possibility that the system is in particular configuration

$$\frac{d}{dt}x_i(t) = v_i(t)$$

$$\frac{d}{dt}v_i(t) = \frac{1}{N}\sum_{j\neq i} K\left(x_i(t) - x_j(t)\right)$$

$$\longrightarrow$$

$$\partial_t F_N(Z_1, \dots, Z_N) = \mathcal{L}_N\left(F_N(Z_1, \dots, Z_N)\right),$$

where $F_N(Z_1, \dots, Z_N)$ is the possibility to find the elements in configuration $(Z_1, \dots, Z_N)$.

- We can also start from the "probabilistic model" on the right hand side (mean field model).
- We usually consider systems of indistinguishable elements in mean field models, i.e. we assume that

$$F_N(Z_1, \dots, Z_N) = F_N\left(Z_{\sigma(1)}, \dots, Z_{\sigma(N)}\right)$$

for any permutation of $\{1, \dots, N\}$, σ.

Mean field limits – BBGKY hierarchy

- The equation for the probability density or measure of the system, usually known as *the master equation*, gives us the ability to see how k average elements behave for any $1 \leq k \leq N$ by considering its marginals, $\{F_{N,k}\}_{k=1,\dots,N}$.
- The hierarchy of equations for $\{F_{N,k}\}_{k=1,\dots,N}$, which will usually be entangled, is known as the BBGKY[1] hierarchy.
- This does not simplify the problem enough for us to be able to consistently solve it.

[1] Bogoliubov–Born–Green–Kirkwood–Yvon.

Mean field limits – asymptotic correlation

- To be able to "close" the BBGKY hierarchy, the mean field approach includes an additional ingredient – an asymptotic correlation relation that expresses the *emerging phenomena we expect to get* when the number of elements involved is "large".
- For example, if we model dilute gas we expect that particles will rarely meet and as such when the number of them in the system is large any given number of particles are "almost" independent

$$
\begin{aligned}
F_{N,1}(Z_1) &\underset{N \text{ large}}{\approx} f(Z_1), \\
&\vdots \\
F_{N,k}(Z_1,\dots,Z_k) &\underset{N \text{ large}}{\approx} f(Z_1)\cdot\dots\cdot f(Z_k).
\end{aligned}
$$

  where f is a function that represents the limiting average element (mean field limit).
- This correlation is known as *molecular chaos*, or just *chaos*.

Mean field limits – what do we do?

- Armed with the master equation and the asymptotic correlation, one can consider the BBGKY hierarchy and find equations for the mean field limit.
- This, together with the asymptotic correlation, can give us an intuition to how the entire system behaves.
- For instance in the case of chaos we'd like to think that

$$F_N \approx f^{\otimes N}$$

but this is a delicate matter.

Kac's model

- Kac's model is a many particle model which, in some sense, pioneered the mean filed limit approach.
- Kac's model consists of N indistinguishable particles with one dimensional velocity that undergo random collision. Due to the diluteness of the gas, the collisions involve only two random particles. If the $i-$th and $j-$th particles collided, then their velocities (v_i, v_j) change to

$$\begin{pmatrix} v_i(\theta) \\ v_j(\theta) \end{pmatrix} = \begin{pmatrix} v_i \cos\theta + v_j \sin\theta \\ -v_i \sin\theta + v_j \cos\theta \end{pmatrix}.$$

  where θ is a random angle of "scattering".
- The collision time is given by a Poisson stream.

Kac's model cont.

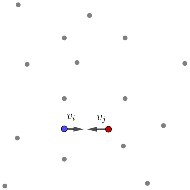


Figure: Pre-collision.

$\longrightarrow$

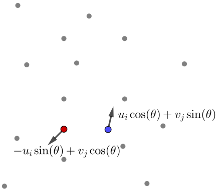


Figure: Post-collision.

- Kac's master equation is given by

$$\partial_t F_N\left(\boldsymbol{V}_N, t\right) = \mathcal{L}_N F_N = N(\mathcal{Q}_N - I)F_N\left(\boldsymbol{V}_N, t\right),$$

with $\boldsymbol{V}_N = (v_1, \ldots, v_N) \in \mathbb{S}^{N-1}\left(\sqrt{N}\right)$,

$$\mathcal{Q}_N F\left(\boldsymbol{V}_N\right) = \frac{1}{\begin{pmatrix} N \\ 2 \end{pmatrix}} \sum_{i<j} \frac{1}{2\pi} \int_0^{2\pi} F_N\left(R_{i,j,\theta} \boldsymbol{V}_N\right) d\theta,$$

and where $R_{i,j,\theta} \boldsymbol{V}_N$ is identical to $\boldsymbol{V}_N$ in all entries but the $i-$th and $j-$th ones which are given by $v_i\left(\theta\right)$ and $v_j\left(\theta\right)$ respectively.

From Kac to Boltzmann

- One can show that if the scattering angles are distributed uniformly then the evolution of $F_{N,1}$ is given by

$$\partial_t F_{N,1}(v) = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \left(F_{N,2}(v(\theta), w(\theta)) - F_{N,2}(v, w) \right) d\theta dw,$$

- Kac has shown that under the assumption that chaoticity holds for the initial data, it propagates in time and taking N to infinity in the above yields the Boltzmann-Kac equation

$$\partial_t f(v) = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \left(f(v(\theta)) f(w(\theta)) - f(v) f(w) \right) d\theta dw.$$

Chaos? Really?

- In recent decades the mean field limit approach permeated into the realms of biology (herds and swarms), chemical interactions (chemotaxis) and even sociology (consensus models).
- While the mean filed limit approach considers various settings the *only* asymptotic correlation which was considered until recently was that of chaoticity.
- This seems to be inappropriate in models that have a tendency for adherence such as biological swarming.
- In a series of two papers, Carlen, Chatelin, Degond, and Wennberg have constructed an animal based model which, after appropriate scaling, breaks chaoticity.

Choose the Leader (CL) model

- The CL model consists of N animals who move in a planar domain and whose velocities are of magnitude 1.
- At a random time (given by a Poisson stream) a random pair of animals meets and one of them, chosen at random, adapts its velocity to the second up to a small amount of "noise". Replacing the velocities with their relative angle by using the identity $v = e^{i\theta}$ we can express the above interaction as

$$(\theta_i, \theta_j) \longrightarrow (\theta_i, \theta_i + \mathscr{Z})$$

where $\mathscr{Z}$ is an independent random variable with values in $[-\pi, \pi]$, distributed in accordance to a given probability density function g which is concentrated around $\theta = 0$.

The CL model's master equation

- The state space can can be identified with $\mathcal{T}^N = [-\pi, \pi]^N$ (with the appropriate boundary identification).
- The master equation for the probability density of the ensemble on $\left(\mathcal{T}^N, \frac{d\theta_1 \dots d\theta_N}{(2\pi)^N}\right)$, F_N, is given by

$$\partial_t F_N(\theta_1, \dots, \theta_N) = \frac{2\lambda}{N-1} \sum_{i<j} \left\{ \frac{1}{2} g(\theta_i - \theta_j) \left([F_N]_{\widetilde{j}} \left(\theta_1, \dots, \widetilde{\theta}_j, \dots, \theta_N\right) + [F_N]_{\widetilde{i}} \left(\theta_1, \dots, \widetilde{\theta}_i, \dots, \theta_N\right) \right) - F_N(\theta_1, \dots, \theta_N) \right\}.$$

where

$$[F_N]_{\widetilde{j}} \left(\theta_1, \dots, \widetilde{\theta}_j, \dots, \theta_N\right) = \int_{-\pi}^{\pi} F_N(\theta_1, \dots, \theta_N) \frac{d\theta_j}{2\pi}.$$

What do we expect?

- From the description of the model we expect that as time passes more animals will meet and strong mutual correlations will form.
- The emergence of these correlation, however, may take time which depends on N.
- Carlen et al. have shown that chaos does propagate for every *fixed* time interval but is broken when we rescale our time variable as well as the noise intensity g, making it more concentrated around $\theta = 0$ as N increases.

Rescaling the interaction

- To make sure that each pair of animals met (and as such correlations are formed) we will rescale time by a factor of the number of animals in the system.
- The interaction rescaling is s motivated from a standard scaling on the line – restricted to $[-\pi,\pi]$: Given a symmetric probability density on $(\mathbb{R}, dx)$, g, and a scaling parameter ε we define the rescaled and restricted probability density on $\left(\mathcal{T}, \frac{d\theta}{2\pi}\right)$, g_ε, by

$$g_\varepsilon(\theta) = \frac{1}{\varepsilon \widetilde{g}_\varepsilon} g\left(\frac{\theta}{\varepsilon}\right)$$

where

$$\widetilde{g}_\varepsilon = \frac{1}{2\pi}\int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} g(x)dx.$$

The rescaled CL model

- Rescaling the time by N and letting the scaling parameter of the interaction probability density depend on N, i.e. $\varepsilon = \varepsilon_N$, the rescaled CL model's master equation reads as

$$\partial_t F_N(\theta_1, \dots, \theta_N) = \frac{2\lambda N}{N-1} \sum_{i<j} \Bigg\{ \frac{1}{2} g_{\varepsilon_N}(\theta_i - \theta_j) \Big([F_N]_{\widetilde{j}} \Big(\theta_1, \dots, \widetilde{\theta}_j, \dots, \theta_N\Big) + [F_N]_{\widetilde{i}} \Big(\theta_1, \dots, \widetilde{\theta}_i, \dots, \theta_N\Big) \Big) - F_N(\theta_1, \dots, \theta_N) \Bigg\}. \tag{1}$$

- The relation between the time and interaction scaling drives the emerging phenomena.

The BBGKY hierarchy

- The BBGKY hierarchy of our rescaled equation is

$$\partial_t F_{N,k}\left(\theta_1,\ldots,\theta_k\right) = \frac{2\lambda N}{N-1}\sum_{i<j\leq k}\left\{\frac{1}{2}g_{\varepsilon_N}\left(\theta_i-\theta_j\right)\right.$$

$$\left.\left(F_{N,k-1}\left(\theta_1,\ldots,\widetilde{\theta}_j,\ldots,\theta_k\right)+F_{N,k-1}\left(\theta_1,\ldots,\widetilde{\theta}_i,\ldots,\theta_k\right)\right)-F_{N,k}\left(\theta_1,\ldots,\theta_k\right)\right\}$$

$$\sum_{i\leq k}\frac{1}{2}\left\{\int_{\mathcal{T}} g_{\varepsilon_N}\left(\theta_i-\theta_{k+1}\right)F_{N,k}\left(\theta_1,\ldots,\widetilde{\theta}_i,\ldots,\theta_{k+1}\right)\frac{d\theta_{k+1}}{2\pi}-F_{N,k}\left(\theta_1,\ldots,\theta_k\right)\right\}$$

- In particular

$$\partial_t F_{N,1}\left(\theta_1,t\right) = \lambda N\left(\int_{-\pi}^{\pi} g_{\varepsilon_N}\left(\theta_1-\theta\right)F_{N,1}\left(\theta,t\right)\frac{d\theta}{2\pi}-F_{N,1}\left(\theta_1,t\right)\right).$$

- The convolution term indicates that Fourier analysis will be a good route to take.

Identifying the correct interaction scaling cont.

- We find that

$$\frac{d}{dt}\widehat{F_{N,1}}(n,t) = \lambda N\left(\widehat{g_{\varepsilon_N}}(n) - 1\right)\widehat{F_{N,1}}(n,t), \qquad n \in \mathbb{Z},$$

which implies

$$\widehat{F_{N,1}}(n,t) = e^{-\lambda N\left(1-\widehat{g_{\varepsilon_N}}(n)\right)t}\widehat{F_{N,1}}(n,0), \qquad n \in \mathbb{Z}.$$

- It is straight forward to show that

$$\widehat{g_{\varepsilon_N}}(n) = 1 + \frac{m_2}{2}\varepsilon_N^2 n^2 + O\left(\varepsilon_N^3 |n|^3\right),$$

where $m_2 = \int_{\mathbb{R}} x^2 g(x)dx$.
- Together with the expression for $F_{N,1}$ we deduce three possible scalings.

The three regimes of scaling

A simple Fourier analysis shows that the time and interaction scaling produce three possible regimes:

(i) $\underline{N\varepsilon_N^2 \underset{N\to\infty}{\longrightarrow} \infty}$: The rescaled interactions act slowly in the rescaled time.

(ii) $\underline{N\varepsilon_N^2 = 1}$: The rescaled interactions and rescaled time are (diffusely) "balanced".

(iii) $\underline{N\varepsilon_N^2 \underset{N\to\infty}{\longrightarrow} 0}$: The rescaled interactions act fast in the rescaled time.

Slow interactions

Chaos is expected in this case.

Definition

Let $\mathcal{X}$ be a Polish space. We say that a sequence of symmetric probability measures, $\mu_N \in \mathscr{P}\left(\mathcal{X}^N\right)$ with $N \in \mathbb{N}$, is μ_0-chaotic for some probability measure $\mu_0 \in \mathscr{P}\left(\mathcal{X}\right)$ if for any $k \in \mathbb{N}$

$$\Pi_k\left(\mu_N\right) \underset{N\to\infty}{\overset{\text{weak}}{\longrightarrow}} \mu_0^{\otimes k},$$

where $\Pi_k\left(\mu_N\right)$ is the $k-$the marginal of μ_N

Fast interactions

- There will be strong deviation from chaoticity in this case as correlation build very fast.
- We expect that the animals will align perfectly, i.e.

$$F_{N,k}\left(\theta_1,\ldots,\theta_k\right)\approx f(\theta_1)\delta\left(\theta_2-\theta_1\right)\ldots\delta\left(\theta_k-\theta_1\right),$$

  where δ is the Dirac measure.
- f represents how a random leader amongst the k elements behaves, and the δ measures tell us that all other elements perfectly align themselves to the random leader.
- We call the above *order*.

The notion of order

Definition (E. '24)

Let $\mathcal{X}$ be a Polish space with a group operation $+$. We say that a sequence of symmetric probability measures, $\mu_N \in \mathscr{P}\left(\mathcal{X}^N\right)$ with $N \in \mathbb{N}$, is μ_0-ordered for some probability measure $\mu_0 \in \mathscr{P}\left(\mathcal{X}\right)$ if for any $k \in \mathbb{N}$

$$\Pi_k\left(\mu_N\right)\left(\theta_1, \dots, \theta_k\right) \underset{N\to\infty}{\overset{\text{weak}}{\longrightarrow}} \mu_0\left(\theta_1\right)\prod_{i=2}^{k}\delta\left(\theta_i - \theta_1\right),$$

where δ is the delta measure concentrated at the additive zero.

Theorem (E. 24')

Let $\{F_N(t)\}_{N\in\mathbb{N}} \in \mathscr{P}\left(\mathcal{T}^N\right)$ *be the family of solutions to the rescaled CL equation. Assume that* $\lim_{N\to\infty} N\varepsilon_N^2 = 0$*. and that* $\{F_{N,k}(0)\}_{k=1,\dots,N,\ N\in\mathbb{N}}$ *converges weakly to* $\{f_k\}_{k\in\mathbb{N}} \in \mathscr{P}\left(\mathcal{T}^k\right)$*. Then for any* $t>0$ $\{F_{N,k}(t)\}_{k=1,\dots,N,\ N\in\mathbb{N}}$ *converges weakly to* $\{f_k(t)\}_{k\in\mathbb{N}} \in \mathscr{P}\left(\mathcal{T}^k\right)$ *which satisfies*

$$\lim_{t\to\infty} f_k(\theta_1,\dots,\theta_k,t) = f_1(\theta_1)\prod_{i=2}^{k}\delta(\theta_i-\theta_1).$$

Moreover, if $\{F_N(0)\}_{N\in\mathbb{N}}$ *is* f_1*−ordered then for all* $t>0$

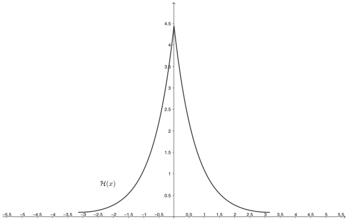
$$f_k(\theta_1,\dots,\theta_k,t) = f_1(\theta_1)\prod_{i=2}^{k}\delta(\theta_i-\theta_1).$$

"Balanced" interactions

- The critical case between slow and fast interactions acts as a "phase transition" – no chaos or order are exhibited.
- One can show that in that case

$$\lim_{t\to\infty}\lim_{N\to\infty} F_{N,2}\left(\theta_1,\theta_2,t\right) = \mathscr{H}\left(\theta_2-\theta_1\right)$$

where $\mathscr{H}$ is *somewhat* concentrated around 0.



Partial order

- A more nuanced notion of alignment is needed – one that allows for deviation from absolute adherence.

Definition (E. & Jiang '25)

Let $\mathcal{X}$ be a Polish space with a continuous group operation $+$ and its inverse. We say that a sequence of symmetric probability measures, $\mu_N \in \mathscr{P}\left(\mathcal{X}^N\right)$ with $N \in \mathbb{N}$, is $\{(\eta_k, \nu_{k-1})\}_{k\in\mathbb{N}}$ –partially ordered where $\eta_k \in \mathscr{P}(\mathcal{X})$ for any $k \in \mathbb{N}$ and $\nu_k \in \mathscr{P}\left(\mathcal{X}^k\right)$ are even and symmetric for any $k \in \mathbb{N}$ if

$$\Pi_k(\mu_N)(\theta_1, \dots, \theta_k) \underset{N\to\infty}{\overset{\text{weak}}{\longrightarrow}} \begin{cases} \eta_1(\theta_1), & k = 1, \\ \frac{1}{k}\sum_{i=1}^{k} \eta_k(\theta_i)\,\nu_{k-1}\left(\theta_1 - \theta_i, \dots, \widetilde{\theta_i - \theta_i}, \dots, \theta_k - \theta_i\right), & k \geq 2, \end{cases}$$

for any $k \in \mathbb{N}$.

- η_k represents how a random leader is distributed amongst the k elements, and ν_{k-1} represents how the other elements deviate from it.

The rise of partial order

Theorem (E. & Jiang 25')

Let $\{F_N(t)\}_{N\in\mathbb{N}} \in \mathscr{P}\left(\mathcal{T}^N\right)$ *be the family of solutions to the rescaled CL equation. Assume that* $N\varepsilon_N^2 = 1$*, that the interaction generating function* g *has a finite moment of order greater or equal to 3, and that* $\{F_{N,k}(0)\}_{k=1,\dots,N,\ N\in\mathbb{N}}$ *converges weakly to* $\{f_k\}_{k\in\mathbb{N}} \in \mathscr{P}\left(\mathcal{T}^k\right)$*. Then for any* $t>0$ $\{F_{N,k}(t)\}_{k=1,\dots,N,\ N\in\mathbb{N}}$ *converges weakly to* $\{f_k(t)\}_{k\in\mathbb{N}} \in \mathscr{P}\left(\mathcal{T}^k\right)$ *which satisfies*

$$\lim_{t\to\infty} f_k(\theta_1,\dots,\theta_k,t) = f_{k,\infty}(\theta_1,\dots,\theta_k)$$

where $f_{k,\infty} \in \mathscr{P}\left(\mathcal{T}^k\right)$ *is given by*

$$f_{k,\infty}(\theta_1,\dots,\theta_k) = \begin{cases} \frac{d\theta_1}{2\pi}, & k=1, \\ \left(\frac{1}{k}\sum_{l=1}^{k} \nu_{k-1}\left(\theta_1-\theta_l,\dots\widetilde{\theta_l-\theta_l},\dots,\theta_k-\theta_l\right)\right)\frac{d\theta_1\dots d\theta_k}{(2\pi)^k}, & k\geq 2. \end{cases}$$

where ν_k *is an even and symmetric probability measure on* $\mathcal{T}^k$ *which is absolutely continuous with respect to* $\frac{d\theta_1\dots d\theta_k}{(2\pi)^k}$*.*

Partial order is delicate

- In general, partial order does not propagate.
- We have explicit rate of convergence of $F_{N,k}(\theta_1, \ldots, \theta_k, t)$ to $f_{k,\infty}(\theta_1, \ldots, \theta_k)$. It is of the form

$$C_k e^{-\alpha t} + \frac{D_k}{\sqrt{N}}.$$

Future prospects

Order and partial order are new notions (2024 and 2025). There is *a lot* to explore. For instance:

- Suitability in other many element models.
- Equivalent definition – probabilistic interpretation.
- Is it possible to identify order and partial order in the microscopic setting?

Thank you for your attention!