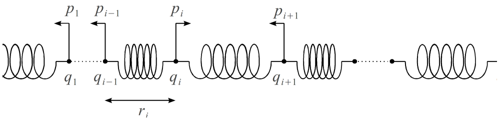
Nonlinear Fluctuations for a Chain of Weakly Anharmonic Oscillators

Stefano Olla
CEREMADE, Université Paris-Dauphine, PSL
Institut Universitaire de France
GSSI, L'Aquila
joint works with Kohei Hayashi and Tomasz Komorowski

PSPDEs XIV
Toulouse
June 22, 2026

Introduction: Historical comments



The Fermi-Pasta-Ulam-Tsingou numerical experiment (1955):

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = \sum_j \left(\frac{p_j}{2} + V(q_j - q_{j-1}) \right), \qquad V(r) = \frac{c_2}{2} r^2 + \frac{c_3}{3} r^3 + \frac{c_4}{4} r^4$$

Fermi wanted to check if the non-linearities ($c_3 \neq 0, c_4 > 0$) will generate equipartition (i.e. 'Thermalization' or 'ergodicity' in a generic sense).
The experiment was done at very low energy, and it 'failed'. But repeating it at higher energies, numerical evidence confirm equipartition at relatively short time.
Questions remain open, and it motivated wide physical and mathematical research in many directions. Completely integrable dynamics (like Toda lattice) provide conterexamples.

Macroscopic transport

The hamiltonian dynamics

$$\dot{q}_j = p_j \qquad \dot{p}_j = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}), \qquad j \in \mathbb{Z}$$

has three locally conserved quantities

$$r_j = q_j - q_{j-1} \text{ (volume)}, \qquad p_j \text{ (momentum)}, \qquad e_j = \frac{p_j^2}{2} + V(r_j) \text{ (energy)},$$

that are the only one for the infinite dynamics, under the *ergodicity* assumption.

Macroscopic transport

The hamiltonian dynamics

$$\dot{q}_j = p_j \qquad \dot{p}_j = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}), \qquad j \in \mathbb{Z}$$

has three locally conserved quantities

$$r_j = q_j - q_{j-1} \text{ (volume)}, \qquad p_j \text{ (momentum)}, \qquad e_j = \frac{p_j^2}{2} + V(r_j) \text{ (energy)},$$

that are the only one for the infinite dynamics, under the *ergodicity* assumption.
The conservation of momentum implies a macroscopic ballistic transport of the three conserved quantities, governed by the compressible Euler equations

$$\partial_t \mathfrak{r} = \partial_x \mathfrak{p}, \qquad \partial_t \mathfrak{p} = \partial_x \tau(\mathfrak{r}, \mathfrak{u}), \qquad \partial_t \mathfrak{e} = \partial_x (\mathfrak{p} \tau(\mathfrak{r}, \mathfrak{u})), \qquad \mathfrak{u} = \mathfrak{e} - \frac{\mathfrak{p}^2}{2}.$$

Intended as law of large number for the corresponding empirical densities (*hydrodynamic limit*), under hyperbolic scaling of space and time (nj, nt for $n \to \infty$).

Equilibrium Fluctuations

Equilibrium distributions are parametrized by tension τ, temperature $T = \beta^{-1}$, momentum P, formally written as

$$\prod_{j\in\mathbb{Z}} \frac{e^{-\beta(e_j+\tau r_j+Pp_j)}}{\mathcal{Z}(\beta,\tau,P)} dr_j\, dp_j.$$

The ergodicity assumption is that these are the only stationary and translation invariant (regular) measures of the dynamics.

Equilibrium Fluctuations

Equilibrium distributions are parametrized by tension τ, temperature $T = \beta^{-1}$, momentum P, formally written as

$$\prod_{j \in \mathbb{Z}} \frac{e^{-\beta(e_j + \tau r_j + P p_j)}}{\mathcal{Z}(\beta, \tau, P)} dr_j \, dp_j .$$

The ergodicity assumption is that these are the only stationary and translation invariant (regular) measures of the dynamics.
We start in equilibrium with $P = 0$ and $\tau = 0$ (for simplicity) and temperature $T = \beta^{-1}$.
Then the initial fluctuations of the three conserved quantities are gaussian (CLT) and evolve deterministically following the linearized Euler equations:

$$\partial_t \tilde{\mathfrak{r}} = \partial_x \tilde{\mathfrak{p}}, \qquad \partial_t \tilde{\mathfrak{p}} = c^2 \partial_x \tilde{\mathfrak{r}}, \qquad \partial_t \tilde{\mathfrak{e}} = 0,$$

$c = \sqrt{\partial_r \tau(r, u)}$ is the *sound velocity*.

Normal Modes

$$\chi^{\pm} = c\tilde{\mathfrak{r}} \pm \tilde{\mathfrak{p}} \qquad \text{two phonon or sound modes}$$

$$\chi^{0} = \tilde{\mathfrak{e}} \qquad \text{energy or heat mode}$$

they evolve deterministically as

$$\chi^{\sigma}(x,t) = \chi^{\sigma}(x + \sigma ct), \qquad \sigma = -1, 0, 1.$$

Phonon modes evolves pallistically with opposite velocities $\pm c$, heat mode does not evolve in the Euler scale.

Beyond Euler time scale?

Numerical evidence since the 90's showed that the energy mode evolves in a superdiffusive time scale (cf. Lepri, Livi, Politi PRL 1997).

Nonlinear Fluctuating Hydrodynamics Theory
Van Beijeren PRL 2012, Spohn JSP 2014
This is a mesoscopic approach to catch the superdiffusive macroscopic broadening of the modes.
Develop Euler equations up to second order and add a dissipative randomness (a gradient of space-time white noises). This is a system of *stochastic Burgers equations*.
From a *mode coupling* analysis of the corresponding correlations

- phonon modes $\Longrightarrow$ KPZ universal scaling function,
- energy mode $\Longrightarrow$ $5/3$-Levy distribution

Beyond Euler time scale?

Numerical evidence since the 90's showed that the energy mode evolves in a superdiffusive time scale (cf. Lepri, Livi, Politi PRL 1997).

Nonlinear Fluctuating Hydrodynamics Theory
Van Beijeren PRL 2012, Spohn JSP 2014
This is a mesoscopic approach to catch the superdiffusive macroscopic broadening of the modes.
Develop Euler equations up to second order and add a dissipative randomness (a gradient of space-time white noises). This is a system of *stochastic Burgers equations*.
From a *mode coupling* analysis of the corresponding correlations

- phonon modes $\Longrightarrow$ KPZ universal scaling function,
- energy mode $\Longrightarrow$ $5/3$-Levy distribution

If $c_3 = 0$ and $\tau = 0$ (even potential)

- phonon modes $\Longrightarrow$ diffusive
- energy mode $\Longrightarrow$ $3/2$-Levy distribution

Mathematical Results

Rigorous mathematical results have been obtained by adding noise directly on the microscopic hamiltonian dynamics. Noise is such that conserves the three modes, destroying all other conserved quantities, for example exchanging momentum between nearest neighbor particles at random times.

- This gives the required ergodicity to the infinite system. (Fritz, Funaki, Lebowitz, PTRF, 1994).
- With such stochastic perturbation, compressible Euler Equations can be proven in the smooth regime (Olla, Varadhan, Yau, CMP 1993; Even, Olla, ARMA, 2014).
- Linearized fluctuations on the Euler scale can be proven with a smoother noise (Olla, Xu, Nonlinearity, 2020, includes also mechanical boundary conditions on the tension).

Mathematical Results beyond the Euler Scale

Beyond Euler scaling it is very hard to obtain mathematical results, even in presence of conservative noise. Results can be obtained for the Harmonic Chain ($c_3 = 0 = c_4$):

- Thermal diffusivity diverges: superdiffusion of the energy (Basile, Bernardin, Olla, CMP 2009).
- energy mode $\Longrightarrow$ 3/2-Levy distribution, and also corresponding fractional non–stationary superdiffusion given by

$$\partial_t \mathfrak{e} = D|\Delta_x|^{3/4}\mathfrak{e} \qquad D = 2^3\gamma^{-1/2}$$

  $\gamma > 0$ rate of the random exchanges. (Jara, Komorowski, Olla, CMP 2015)
- Recentered phonon modes are diffusive (Komorowski, Olla, Nonlinearity, 2016).

Mathematical Results beyond the Euler Scale

Beyond Euler scaling it is very hard to obtain mathematical results, even in presence of conservative noise. Results can be obtained for the Harmonic Chain ($c_3 = 0 = c_4$):

- Thermal diffusivity diverges: superdiffusion of the energy (Basile, Bernardin, Olla, CMP 2009).
- energy mode $\Longrightarrow$ 3/2-Levy distribution, and also corresponding fractional non–stationary superdiffusion given by

$$\partial_t \mathfrak{e} = D|\Delta_x|^{3/4}\mathfrak{e} \qquad D = 2^3\gamma^{-1/2}$$

  $\gamma > 0$ rate of the random exchanges. (Jara, Komorowski, Olla, CMP 2015)
- Recentered phonon modes are diffusive (Komorowski, Olla, Nonlinearity, 2016).

Unlike the Euler equation, here the superdiffusive and diffusive equations depend of the rate γ of the random momentum exchange.

Weak non-linearity

Infinite dynamics ($j \in \mathbb{Z}$), with random nearest neighbor exchanges of velocities with rate γ, in equilibrium with $\tau = 0, P = 0$ and $\beta > 0$.
V is a smooth non-linear and $V(0) = V'(0) = 0$. Then we scale it as $V_\varepsilon(r) = \varepsilon^{-2} V(\varepsilon r)$:

$$V_\varepsilon(r) = \frac{c_2}{2!} r^2 + \frac{c_3}{3!} \varepsilon r^3 + \frac{c_4}{4!} \varepsilon^2 r^4 + O(\varepsilon^3), \qquad c_k = V^{(k)}(0).$$

Weak non-linearity

Infinite dynamics ($j \in \mathbb{Z}$), with random nearest neighbor exchanges of velocities with rate γ, in equilibrium with $\tau = 0, P = 0$ and $\beta > 0$.
V is a smooth non-linear and $V(0) = V'(0) = 0$. Then we scale it as $V_\varepsilon(r) = \varepsilon^{-2} V(\varepsilon r)$:

$$V_\varepsilon(r) = \frac{c_2}{2!} r^2 + \frac{c_3}{3!} \varepsilon r^3 + \frac{c_4}{4!} \varepsilon^2 r^4 + O(\varepsilon^3), \qquad c_k = V^{(k)}(0).$$

Choose $\varepsilon_n = \frac{1}{\sqrt{n}}$.

$$\xi_j^{\pm} = \sqrt{c_2} r_{j+1} \pm p_j, \qquad \xi_j^0 = e_j = \frac{p_j^2}{2} + V_n(r_j), \qquad \bar{\xi}_j^\sigma = \xi_j^\sigma - \mathbb{E}(\xi_j^\sigma).$$

Recentered Fluctuating fields, for $\sigma = -1, 0, 1$

$$\mathcal{X}_t^{\sigma,n}(\varphi) = \frac{1}{\sqrt{n}} \sum_{j \in \mathbb{Z}} \overline{\xi_j^\sigma}(t) \varphi\left(\frac{[j + \sigma t]}{n}\right)$$

Evolution of fluctuating fields

$$\mathcal{X}_t^{\sigma,n}(\varphi) = \frac{1}{\sqrt{n}} \sum_{j\in\mathbb{Z}} \overline{\xi_j^\sigma}(t)\varphi\left(\frac{[j+\sigma\sqrt{c_2}t]}{n}\right)$$

At time 0 the converge in law to the white noises

$$\mathcal{X}_0^{\sigma,n}(\varphi) \underset{n\to\infty}{\longrightarrow} \mathcal{X}_0^\sigma(\varphi), \qquad \mathbb{E}\left[\mathcal{X}_0^\sigma(\varphi)\mathcal{X}_0^{\sigma'}(\varphi)\right] = 0, \text{ if } \sigma \neq \sigma'$$

$$\mathbb{E}\left[(\mathcal{X}_0^\pm(\varphi))^2\right] = \frac{2}{\beta}\|\varphi\|_{L^2}^2, \qquad \mathbb{E}\left[\left(\mathcal{X}_0^0(\varphi)\right)^2\right] = \frac{3}{\beta^2}\|\varphi\|_{L^2}^2.$$

Evolution of fluctuating fields

$$\mathcal{X}_t^{\sigma,n}(\varphi) = \frac{1}{\sqrt{n}} \sum_{j\in\mathbb{Z}} \overline{\xi_j^\sigma}(t)\varphi\left(\frac{[j+\sigma\sqrt{c_2}t]}{n}\right)$$

At time 0 the converge in law to the white noises

$$\mathcal{X}_0^{\sigma,n}(\varphi) \underset{n\to\infty}{\longrightarrow} \mathcal{X}_0^{\sigma}(\varphi), \qquad \mathbb{E}\left[\mathcal{X}_0^{\sigma}(\varphi)\mathcal{X}_0^{\sigma'}(\varphi)\right] = 0, \text{ if } \sigma \neq \sigma'$$

$$\mathbb{E}\left[(\mathcal{X}_0^{\pm}(\varphi))^2\right] = \frac{2}{\beta}\|\varphi\|_{L^2}^2, \qquad \mathbb{E}\left[\left(\mathcal{X}_0^{0}(\varphi)\right)^2\right] = \frac{3}{\beta^2}\|\varphi\|_{L^2}^2.$$

At the hyperbolic time scale nothing moves:

$$\mathcal{X}_{nt}^{\sigma,n}(\varphi) \underset{n\to\infty}{\longrightarrow} \mathcal{X}_0^{\sigma}(\varphi), \qquad \sigma = -1, 0, 1.$$

Non-linear fluctuations of the phonon modes

At diffusive time scale $(n^2 t)$ we prove the convergence to the energy solutions of two uncoupled stochastic Burgers equations with drift:

$$\mathcal{X}^{\pm,n}_{n^2 t}(\varphi) \underset{n\to\infty}{\longrightarrow} \int u^{\pm}(t,x)\varphi(x)dx$$

$$\partial_t u^{\pm} = \frac{\gamma}{4}\partial_x^2 u^{\pm} \pm \frac{c_3}{8c_2^2}\partial_x (u^{\pm})^2 \pm D_V \partial_x u^{\pm} + \sqrt{\gamma\beta^{-1}}\partial_x \dot{W}^{\pm}$$

$$D_V = \frac{2c_2c_4 - c_3^2}{24c_2^3}.$$

Here $\dot{W}^{+}(t,x), \dot{W}^{-}(t,x)$ are independent standard white noises, $\gamma > 0$ is the rate of the random exchanges of velocities.
(work with Kohei Hayashi), arXiv:2510.12922
Notice that the nonlinear term depends on the presence of $c_3 \neq 0$ (asymmetric interaction).

Evolution of the energy mode

with Kohei Hayashi and Tomasz Komorowski *(in progress)*

$$V_\varepsilon(r) = \frac{c_2}{2!} r^2 + \frac{c_3}{3!} \varepsilon r^3 + \frac{c_4}{4!} \varepsilon^2 r^4 + O(\varepsilon^3), \qquad c_k = V^{(k)}(0).$$

With $\varepsilon_n = n^{-a}$, $a > 1/8$ (that include $\varepsilon_n = \frac{1}{\sqrt{n}}$)

$$\mathcal{X}_t^{0,n}(\varphi) = \frac{1}{\sqrt{n}} \sum_{j\in\mathbb{Z}} \overline{e_j}(N^{3/2} t) \varphi\left(\frac{j}{n}\right)$$

still converges to the solution of 3/2-Levy superdiffusion:

$$\partial_t \mathcal{X}_t^0 = -C_{\mathrm{cb}} |\Delta|^{3/4} \mathcal{X}_t^0 + \sqrt{C_{\mathrm{cb}} \beta^{-1}} |\Delta|^{3/8} \dot{W}_t.$$

Open question: $a = 1/8$ seems to be critical (a linear drift term may appear). What happens for $a < 1/8$? In particular $a = 0$???

Stationary energy solutions of SBE

A stationary $\mathcal{S}'(\mathbb{R})$-valued process $u(t,\cdot)$ is an energy solution of the SBE

$$\partial_t u = \gamma\partial_{xx} u + \Lambda\partial_x(u^2) + \sqrt{\gamma\beta^{-1}}\partial_x \dot{W},$$

if for any $\varphi \in \mathcal{S}(\mathbb{R})$ and $t > 0$

- $$\mathcal{A}_t(\varphi) = \lim_{\delta\to 0} \int_0^t ds \int_{\mathbb{R}} u(s, \imath_\delta(x))\partial_x\varphi(x)dx \qquad \text{exists in } L^2,$$

Stationary energy solutions of SBE

A stationary $\mathcal{S}'(\mathbb{R})$-valued process $u(t,\cdot)$ is an energy solution of the SBE

$$\partial_t u = \gamma \partial_{xx} u + \Lambda \partial_x (u^2) + \sqrt{\gamma\beta^{-1}} \partial_x \dot{W},$$

if for any $\varphi \in \mathcal{S}(\mathbb{R})$ and $t > 0$

- $$\mathcal{A}_t(\varphi) = \lim_{\delta \to 0} \int_0^t ds \int_{\mathbb{R}} u(s, \imath_\delta(x)) \partial_x \varphi(x) dx \qquad \text{exists in } L^2,$$
- $\hat{u}(t) = u(T-t)$, $\hat{\mathcal{A}}_t = -(\mathcal{A}_T - \mathcal{A}_{T-t})$,

$$M_t = u(t,\varphi) - u(0,\varphi) - \gamma \int_0^t u(s, \partial_x^2 \varphi) ds + \Lambda \mathcal{A}_t(\varphi)$$
$$\hat{M}_t = \hat{u}(t,\varphi) - \hat{u}(0,\varphi) - \gamma \int_0^t \hat{u}(s, \partial_x^2 \varphi) ds + \Lambda \hat{\mathcal{A}}_t(\varphi)$$

are (forward/backward) martingales with quadratic variation $t\gamma \|\partial_x \varphi\|_{L^2}^2$.

Large literature on SBE from microscopic dynamics

Weak Asymmetry in the dynamics:

- L. Bertini and G. Giacomin. CMP 1997. (with Cole-Hopf mapping)
- P. Gonçalves and M. Jara. ARMA 2014 (energy solutions)
- P. Gonçalves, M. Jara, and S. Sethuraman. AoP, 2015.
- M. Gubinelli and N. Perkowski. Energy solutions of KPZ are unique. JAMS 2018.
- .....

Weak interactions (with strong asymmetry):

- K. Hayashi, SPA 23, J Stat Phys 2024.

Relevant recent works

On some simpler models

- P.Gonçalves, K.Hayashi. CMP, 2023 (on Bernardin-Stoltz dynamics)
- G. Cannizzaro, P. Gonçalves, R. Misturini, and A. Occelli. *From ABC to KPZ*. PTRF, 2024. (2 uncoupled SBE)
- Hugo Da Cunha, Makiko Sasada, *Stationary fluctuations for an exclusion process with mass and energy conservation*, arXiv:2606.01937

This last is an interesting model with two conserved quantities, for certain choice of the parameters of asymmetry are expected two independent SBE's, while for other choice SBE + Levy 5/3 would arise (still to be proven).

Bound on variances

For any stationary Markov process Z_t with invariant measure μ and generator $L = A + S$:

$$\mathbb{E}_\mu \left(\sup_{0 \le t \le T} \left[\int_0^t F(Z_s) ds \right]^2 \right) \le 16 T \int F(-S)^{-1} F \, d\mu$$

Ideas of the proof: 1- correctors to the phonon field

Recall

$$V_{\varepsilon_n}(r) = \frac{c_2}{2!}r^2 + \frac{c_3}{3!}\varepsilon_n r^3 + \frac{c_4}{4!}\varepsilon_n^2 r^4 + O(\varepsilon_n^3), \qquad \varepsilon_n = \frac{1}{\sqrt{n}}$$

$$\xi_j^{\pm} = \sqrt{c_2}\, r_{j+1} \pm p_j, \qquad \xi_j^0 = e_j = \frac{p_j^2}{2} + V_n(r_j), \qquad \bar{\xi}_j^{\sigma} = \xi_j^{\sigma} - \mathbb{E}(\xi_j^{\sigma}), \qquad \sigma = -1, 0, 1$$

recentered phonon fields at diffusive scale:

$$\mathcal{X}_{n^2 t}^{\pm,n}(\varphi) = \frac{1}{\sqrt{n}} \sum_{j\in\mathbb{Z}} \overline{\xi_j^{\pm}}(t)\, \varphi\left(\frac{[j \pm \sqrt{c_2}\, n^2 t]}{n}\right)$$

Ideas of the proof: 1- correctors to the phonon field

Recall

$$V_{\varepsilon_n}(r) = \frac{c_2}{2!}r^2 + \frac{c_3}{3!}\varepsilon_n r^3 + \frac{c_4}{4!}\varepsilon_n^2 r^4 + O(\varepsilon_n^3), \qquad \varepsilon_n = \frac{1}{\sqrt{n}}$$

$$\xi_j^{\pm} = \sqrt{c_2}\, r_{j+1} \pm p_j, \qquad \xi_j^0 = e_j = \frac{p_j^2}{2} + V_n(r_j), \qquad \bar{\xi}_j^{\sigma} = \xi_j^{\sigma} - \mathbb{E}(\xi_j^{\sigma}), \qquad \sigma = -1, 0, 1$$

recentered phonon fields at diffusive scale:

$$\mathcal{X}_{n^2 t}^{\pm,n}(\varphi) = \frac{1}{\sqrt{n}} \sum_{j \in \mathbb{Z}} \overline{\xi_j^{\pm}}(t)\, \varphi\left(\frac{[j \pm \sqrt{c_2}\, n^2 t]}{n}\right)$$

$$\tilde{\xi}_j^{\pm} := \xi_j^{\pm} + \varepsilon_n \frac{c_3}{2 c_2^{3/2}} e_j \qquad \widetilde{\mathcal{X}}_{n^2 t}^{\pm,n}(\varphi) := \frac{1}{\sqrt{n}} \sum_{j \in \mathbb{Z}} \overline{\tilde{\xi}_j^{\pm}}(t)\, \varphi\left(\frac{[j \pm \sqrt{c_2}\, n^2 t]}{n}\right)$$

$$\mathcal{X}_{n^2 t}^{\pm,n}(\varphi) - \widetilde{\mathcal{X}}_{n^2 t}^{\pm,n}(\varphi) \xrightarrow[n \to \infty]{} 0 \qquad \text{in law in } \; D([0,T], \mathbb{R}).$$

Macroscopic time evolution of the phonon fields

$$\tilde{\xi}_j^{\pm} = \sqrt{c_2} r_{j+1} \pm p_j + \varepsilon_n \mathfrak{u} e_j, \qquad \mathfrak{u} = \frac{c_3}{2c_2^{3/2}}, \; D_V = \frac{2c_2 c_4 - c_3^2}{24 c_2^3}.$$

$$(\partial_t + n^2 L)\tilde{\xi}_j^{\pm} = \pm \frac{\gamma\sqrt{c_2}}{2} n^2 \Delta p_j + \nabla\left(\mathfrak{u} p_j \bar{r}_{j+1}\right) \pm D_V C_2 \nabla r_j + \nabla Z_{n,j}^{\pm}(t)$$

where $Z_{n,j}$ will result negligeable terms. Then

$$2p_j = \bar{\xi}_j^{+} - \bar{\xi}_j^{-}, \qquad 2C_2 r_j = \bar{\xi}_j^{+} - \bar{\xi}_j^{-}$$

$$\mathfrak{u} p_j \bar{r}_{j+1} = \frac{c_3}{8c_2^2}\left[(\bar{\xi}_j^{+})^2 - (\bar{\xi}_j^{-})^2\right]$$

In order to close the equation we have to substitute (locally) $p_j \bar{r}_{j+1}$ with the product of the local averages.

Second order Boltzmann-Gibbs principle

$$\overrightarrow{p}_j^{\ell} = \frac{1}{\ell}\sum_{i=0}^{\ell-1} \overline{p}_{j+i}, \qquad \overrightarrow{r}_j^{\ell} = \frac{1}{\ell}\sum_{i=0}^{\ell-1} \overline{r}_{j+i},$$

$$\mathbb{E}_n\left[\sup_{0\le t\le T}\left|\int_0^t \sum_{j\in\mathbb{Z}}\Big(p_j(n^2 s)\overline{r}_{j+1}(n^2 s) - \overrightarrow{p}_j^{\ell}(n^2 s)\,\overrightarrow{r}_j^{\ell}(n^2 s)\Big)\varphi(j/n)\,ds\right|^2\right]$$
$$\lesssim T\left(\frac{\ell}{n} + \frac{1}{\ell}\right)\left(\|\varphi\|^2_{L^2(\mathbb{R})} + \|\varphi'\|^2_{L^2(\mathbb{R})}\right)$$

Second order Boltzmann-Gibbs principle

$$\overrightarrow{p}_j^{\ell} = \frac{1}{\ell}\sum_{i=0}^{\ell-1} \overline{p}_{j+i}, \qquad \overrightarrow{r}_j^{\ell} = \frac{1}{\ell}\sum_{i=0}^{\ell-1} \overline{r}_{j+i},$$

$$\mathbb{E}_n\left[\sup_{0\le t\le T}\left|\int_0^t \sum_{j\in\mathbb{Z}}\Big(p_j(n^2 s)\overline{r}_{j+1}(n^2 s) - \overrightarrow{p}_j^{\ell}(n^2 s)\,\overrightarrow{r}_j^{\ell}(n^2 s)\Big)\varphi(j/n)\,ds\right|^2\right]$$
$$\lesssim T\Big(\frac{\ell}{n}+\frac{1}{\ell}\Big)\Big(\|\varphi\|^2_{L^2(\mathbb{R})}+\|\varphi'\|^2_{L^2(\mathbb{R})}\Big)$$

Two steps:

1. $p_j \sim \overrightarrow{p}_j^{\ell}$: easy by random exchanges of velocities,

Second order Boltzmann-Gibbs principle

$$\overrightarrow{p}_j^{\ell} = \frac{1}{\ell}\sum_{i=0}^{\ell-1} \overline{p}_{j+i}, \qquad \overrightarrow{r}_j^{\ell} = \frac{1}{\ell}\sum_{i=0}^{\ell-1} \overline{r}_{j+i},$$

$$\mathbb{E}_n\left[\sup_{0\le t\le T}\left|\int_0^t \sum_{j\in\mathbb{Z}} \left(p_j(n^2 s)\overline{r}_{j+1}(n^2 s) - \overrightarrow{p}_j^{\ell}(n^2 s)\overrightarrow{r}_j^{\ell}(n^2 s)\right)\varphi(j/n)ds\right|^2\right]$$
$$\lesssim T\left(\frac{\ell}{n} + \frac{1}{\ell}\right)\left(\|\varphi\|^2_{L^2(\mathbb{R})} + \|\varphi'\|^2_{L^2(\mathbb{R})}\right)$$

Two steps:

1. $p_j \sim \overrightarrow{p}_j^{\ell}$: easy by random exchanges of velocities,
2. $r_j \sim \overrightarrow{r}_j^{\ell}$: more work, we have to use the Hamiltonian part of the dynamics.

Handling fields with the wrong velocity: a Riemann-Lebesgue lemma

Fields in the wrong velocity recentering became negligeable:

$$\lim_{n\to\infty} \mathbb{E}_n\left[\sup_{0\le t\le T}\left|\int_0^t \widetilde{\mathcal{X}}^{\pm,n}_{n^2 s}(\varphi_{\mp\sqrt{c_2}n^2 s})^2 ds\right|\right] = 0.$$

Handling fields with the wrong velocity: a Riemann-Lebesgue lemma

Fields in the wrong velocity recentering became negligeable:

$$\lim_{n\to\infty} \mathbb{E}_n\left[\sup_{0\le t\le T}\left|\int_0^t \widetilde{\mathcal{X}}^{\pm,n}_{n^2 s}(\varphi_{\mp\sqrt{c_2}n^2 s})^2 ds\right|\right] = 0.$$

---

then for the nonlinear term in the evolution of $\widetilde{\mathcal{X}}^{+,n}_{n^2 t}$ we have

$$\frac{c_3}{8c_2^2}\int_0^t \left[\widetilde{\mathcal{X}}^{+,n}_{n^2 t}(\varphi_{+\sqrt{c_2}n^2 s})^2 - \widetilde{\mathcal{X}}^{-,n}_{n^2 t}(\varphi_{+\sqrt{c_2}n^2 s})^2\right] ds$$

similarly for the nonlinear term for $\widetilde{\mathcal{X}}^{-,n}_{n^2 t}$

$$\frac{c_3}{8c_2^2}\int_0^t \left[\widetilde{\mathcal{X}}^{+,n}_{n^2 t}(\varphi_{-\sqrt{c_2}n^2 s})^2 - \widetilde{\mathcal{X}}^{-,n}_{n^2 t}(\varphi_{-\sqrt{c_2}n^2 s})^2\right] ds$$

Handling fields with the wrong velocity: a Riemann-Lebesgue lemma

Fields in the wrong velocity recentering became negligeable:

$$\lim_{n\to\infty} \mathbb{E}_n\left[\sup_{0\le t\le T} \left| \int_0^t \widetilde{\mathcal{X}}^{\pm,n}_{n^2 s}(\varphi_{\mp\sqrt{c_2}n^2 s})^2 ds \right| \right] = 0.$$

---

then for the nonlinear term in the evolution of $\widetilde{\mathcal{X}}^{+,n}_{n^2 t}$ we have

$$\frac{c_3}{8c_2^2} \int_0^t \left[\widetilde{\mathcal{X}}^{+,n}_{n^2 t}(\varphi_{+\sqrt{c_2}n^2 s})^2 - \widetilde{\mathcal{X}}^{-,n}_{n^2 t}(\varphi_{+\sqrt{c_2}n^2 s})^2 \right] ds$$

similarly for the nonlinear term for $\widetilde{\mathcal{X}}^{-,n}_{n^2 t}$

$$\frac{c_3}{8c_2^2} \int_0^t \left[\widetilde{\mathcal{X}}^{+,n}_{n^2 t}(\varphi_{-\sqrt{c_2}n^2 s})^2 - \widetilde{\mathcal{X}}^{-,n}_{n^2 t}(\varphi_{-\sqrt{c_2}n^2 s})^2 \right] ds$$

Martingale parts also became orthogonal. So we have uncoupled SBE.

Equipartition of energy

$$\lim_{n\to\infty} \mathbb{E}_n\left[\sup_{0\le t\le T} \left| \int_0^t \sum_{j\in\mathbb{Z}} \left(c_2 \overline{r_j^2}(n^2 s) - \overline{p_j^2}(n^2 s)\right) \partial_x \varphi\left(\tfrac{j}{n}\right) ds \right|^2 \right] = 0.$$