Continuous Auction Models

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Outline

Setting: Repeated Auctions

Classical auction: few individuals bid for a unique good. **This paper:**

- one auctioneer, many seller agents
- successive auction rounds
- motivated by *renewable-energy markets*
- continuous-time, continuous-space scaling limit

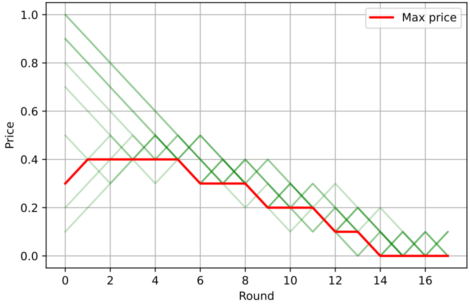


Figure: PSK model: p=0.3, n=10, γ=0.1. Green: bid trajectories. Red: max-price q_t^γ.

n sellers, bids configuration $B = [b_1, \cdots, b_n : \; b_i \in \{0, \gamma, 2\gamma, \ldots, 1\}]$, where $\gamma = 1/m$, $m \in \mathbb{N}$.

Auction rule — one round

- Fraction $p = k/n$ awarded to the lowest np bids $\Rightarrow$ *np winners*, *$n(1-p)$ losers*
- **Winners** increase bid by γ; **Losers** decrease bid by γ
- *Myopic strategy*: each bidder reacts only to win/lose

One-step operator: let $b'_1(B) \leq \ldots \leq b'_n(B)$ be the ordered bids, we define

$$T^\gamma B := \left[b'_i(B) + \gamma\left(\mathbb{1}\{0 < i \leq np\} - \mathbb{1}\{1 > i > np\} \right) : i = 1, \ldots, n \right], \qquad T^\gamma_t B := (T^\gamma)^{\lfloor t/\gamma \rfloor} B$$

Max price: $q^\gamma_t = b'_{np}(T^\gamma_t B)$ — largest winning bid at time t.

Periodicity around the Max Price

Lemma 1 (Bulk periodicity)

Let all bids $\in$ three consecutive sites, then:

(a) Bulk *(no boundary interaction):*

$$(T^\gamma)^K B = B + \gamma \frac{w-\ell}{w+\ell} \cdot K$$

where the period K is given by

$$K = \frac{w+\ell}{\gcd(w,\ell)}$$

(b) Boundary: $[0^{(\ell)}, \gamma^{(w)}]$ *stationary for* $w<\ell$; $[(1-\gamma)^{(\ell)}, 1^{(w)}]$ *stationary for* $w>\ell$.

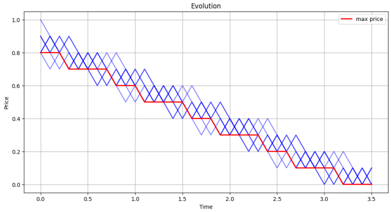


Figure: w=3, ℓ=5: period K=8. Darker: superposed bids.

Continuous Model — Setup and Heuristics

Data:

- μ: probability measure on $[0,1]$; $F = \text{CDF}$
- $p \in (0,1)$: winner fraction
- $q_0 = F^{-1}(p)$: initial max price

Heuristics:

- bids $< q_t$ (*winners*): velocity $+1$
- bids $> q_t$ (*losers*): velocity -1
- w_t, ℓ_t: winning/losing mass *at* q_t
- max-price velocity: $u_t = \dfrac{w_t - \ell_t}{w_t + \ell_t}$

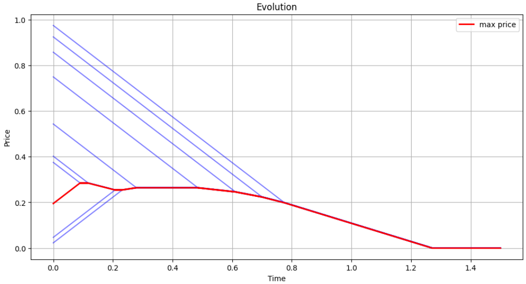


Figure: Continuous model, 10 delta measures, p=0.3. Upon collision with q_t: bids coalesce.

The ODE System

For $p \in (0,1)$ and initial measure μ with CDF F:

$$\frac{dq_t}{dt} = u_t \cdot \mathbb{1}\{q_t \in (0,1)\}$$

$$w_t = p - F\left((q_t - t)^-\right), \qquad \ell_t = F(q_t + t) - p$$

$$u_t = \frac{w_t - \ell_t}{w_t + \ell_t}, \qquad q_0 = F^{-1}(p)$$

Bid distribution at time t

$$d(\mu\mathcal{T}_t)(x) = \delta_{q_t}(x)\ \underbrace{\mu([q_t - t,\ q_t + t))}_{\text{mass accumulated at } q_t} + d\mu(x-t)\,\mathbb{1}\{x < q_t\} + d\mu(x+t)\,\mathbb{1}\{x > q_t\}$$

Define:

$$G(x) := \int_0^x F(z)\,dz = \int_0^x (x-y)\,d\mu(y)$$

Existence and Uniqueness

Let

$$\mathcal{G}_t(x) := G(x+t) - G(x-t) = \int_{x-t}^{x+t} F(z)\,dz = \int_0^1 \Psi_{x,t}(z)\,d\mu(z)$$

where $\Psi_{x,t}(z) := (x+t-z)_+ - (x-t-z)_+$

Lemma 2

For all $t \geq 0$, the equation $\mathcal{G}_t(x) = 2pt$ with $x \in [q_0 - t, q_0 + t]$ has a unique solution $x_t = \mathcal{G}_t^{-1}(2pt)$.

Theorem 3 (Existence & Uniqueness)

The ODE system has a unique solution $(q_t)_{t \geq 0}$:

$$q_t = \begin{cases} \mathcal{G}_t^{-1}(2pt) & \mathcal{G}_t^{-1}(2pt) \in (0,1) \\ 1 & \mathcal{G}_t^{-1}(2pt) > 1 \\ 0 & \mathcal{G}_t^{-1}(2pt) < 0 \end{cases}$$

Proof Idea: Existence and Uniqueness

By the lemma, for every $t \geq 0$ there is a unique $q_t \in [q_0 - t, q_0 + t]$ such that $\mathcal{G}_t(q_t) = 2pt$. Differentiate this identity:

$$\frac{d}{dt}\mathcal{G}_t(q_t) = 2p.$$

Since

$$\partial_t \mathcal{G}_t(q_t) = F(q_t + t) + F((q_t - t)^-), \qquad \partial_x \mathcal{G}_t(q_t) = F(q_t + t) - F((q_t - t)^-),$$

and

$$F(q_t + t) = p + \ell_t, \qquad F((q_t - t)^-) = p - w_t,$$

we get

$$2p = (2p + \ell_t - w_t) + (w_t + \ell_t)\dot{q}_t.$$

Therefore

$$\dot{q}_t = \frac{w_t - \ell_t}{w_t + \ell_t}.$$

Uniqueness

Then any solution of the ODE satisfies $\mathcal{G}_t(q_t) = 2pt$. By the lemma, this determines q_t uniquely.

The Baricenter Formula

Define the baricenter of μ on $(\alpha, \beta]$:

$$\mathrm{b}[\alpha, \beta] := \frac{1}{F(\beta) - F(\alpha^-)} \int_{(\alpha,\beta]} z\, d\mu(z)$$

Proposition 4 (Baricenter decomposition)

The unique solution of the ODE system satisfies, for all $t \geq 0$:

$$q_t = \mathrm{b}[q_t - t,\ q_t + t] + t\,\frac{w_t - \ell_t}{w_t + \ell_t}$$

Physical interpretation

- $\mathrm{b}[q_t - t,\ q_t + t]$: baricenter at time 0 of bids that have hit q_t by time t
- Second term: net momentum displacement of that mass
- *Total momentum is conserved at each collision*

Proof of Equivalence: Theorem 2 $\Leftrightarrow$ Proposition 3

Integration by parts on $\mathcal{G}_t(q_t) = \int_{q_t-t}^{q_t+t} F(z)\,dz$:

$$\int_{q_t-t}^{q_t+t} F(z)\,dz = \big[z\,F(z)\big]_{q_t-t}^{q_t+t} - \int_{(q_t-t,\,q_t+t]} z\,d\mu(z)$$

$$= (q_t+t)\,\underbrace{F(q_t+t)}_{=p+\ell_t} - (q_t-t)\,\underbrace{F((q_t-t)^-)}_{=p-w_t} - (w_t+\ell_t)\,\mathrm{b}[q_t-t,\,q_t+t]$$

Setting $= 2pt$ and simplifying:

$$q_t(w_t+\ell_t) + t(\ell_t-w_t) - (w_t+\ell_t)\,\mathrm{b}[q_t-t,\,q_t+t] = 0$$

$$\Longrightarrow \quad \boxed{q_t = \mathrm{b}[q_t-t,\,q_t+t] + t\,\frac{w_t-\ell_t}{w_t+\ell_t}}$$

Theorem 2 ($\mathcal{G}_t(q_t) = 2pt$) $\Longleftrightarrow$ Proposition 3 (baricenter formula), via integration by parts using $F(q_t+t) = p+\ell_t$ and $F((q_t-t)^-) = p-w_t$.

Explicit Solution: Uniform Measure

$F(x) = x$, $p \leq \frac{1}{2}$: **If** $p \leq \frac{1}{4}$**:**

$$q_t = \begin{cases} p & 0 \leq t \leq p \\ -t + 2\sqrt{pt} & p \leq t \leq 4p \\ 0 & t \geq 4p \end{cases}$$

If $\frac{1}{4} < p < \frac{1}{2}$**:**

$$q_t = \begin{cases} p & 0 \leq t \leq p \\ -t + 2\sqrt{pt} & p \leq t \leq \frac{1}{4p} \\ (2p-1)t + \frac{1}{2} & \frac{1}{4p} \leq t \leq \frac{1}{2(1-2p)} \\ 0 & t \geq \frac{1}{2(1-2p)} \end{cases}$$

If $p = \frac{1}{2}$**:** $\quad q_t \equiv \frac{1}{2}$.

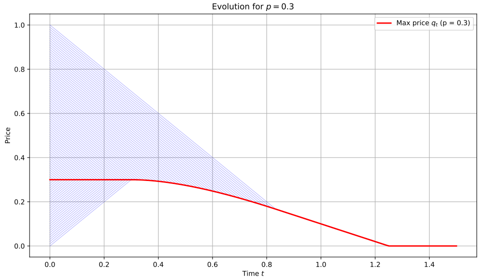


Figure: Continuous model, uniform initial measure, $p = 0.3$.

Delta Measures: Ballistic Dynamics

Theorem 5 (Evolution of delta measures)

For $\mu = \frac{1}{n}\sum_{b \in B} \delta_b$, *the system has a unique solution* $(B_t, q_t)_{t \geq 0}$:

- *Each bid travels ballistically until hitting* q_t
- $\dot{q}_t = u_t$: *piecewise constant, right-continuous at collisions*
- q_t: *continuous and piecewise linear*
- *Finite total number of collisions*

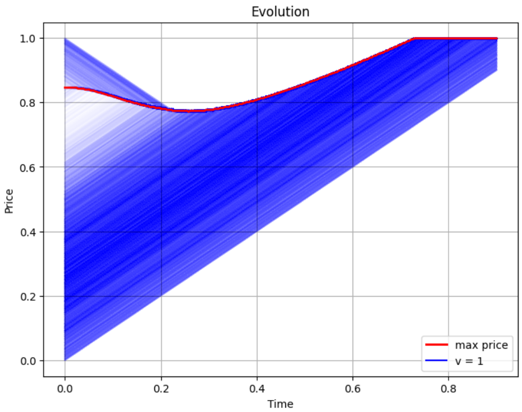


Figure: Sinusoidal initial density, p=0.85. Red: max-price q_t. Bids coalesce upon collision.

Learning parameter $\gamma = 1/m$, $m \in \mathbb{N}$; bids in $I^\gamma := [0,1] \cap \gamma\mathbb{N}$; $w = np$ winners, $\ell = n(1-p)$ losers. **One-step operator:**

$$T^\gamma B := \left[b_i'(B) + \gamma\big(\mathbb{1}\{0 < i \leq np\} - \mathbb{1}\{1 > i > np\}\big) : i = 1, \dots, n\right]$$

$$T_t^\gamma B := (T^\gamma)^{\lfloor t/\gamma \rfloor} B, \qquad q_t^\gamma := b_{np}'(T_t^\gamma B)$$

Key quantities

$$F(x) = \tfrac{1}{n}\textstyle\sum_i \mathbb{1}\{b_i \leq x\}, \quad \mathcal{G}_t(x) = \int_{x-t}^{x+t} F(z)\, dz, \quad \mathrm{b}[\alpha, \beta] = \frac{\sum_i b_i \mathbb{1}_{b_i \in (\alpha,\beta]}}{\sum_i \mathbb{1}_{b_i \in (\alpha,\beta]}}$$

Proposition 6 (Max-price evolution, discrete)

At time $t \in \gamma\mathbb{N}$

$$\left| q_t^\gamma - \mathrm{b}[q_t^\gamma - t,\, q_t^\gamma + t] - \frac{(p - F((q_t^\gamma - t)^-)) - (F(q_t^\gamma + t) - p)}{F(q_t^\gamma + t) - F((q_t^\gamma - t)^-)} \cdot t \right| \leq 2\gamma$$

$$\left| \mathcal{G}_t(q_t^\gamma) - 2pt \right| \leq 2\gamma$$

Idea of the proof: Once a bid's distance from q_t^γ falls below 2γ, it stays within 2γ of the max price forever. Hence the baricenter of bids near q_t^γ lies within 2γ of q_t^γ.

Why exact identity fails in the discrete model

- **Continuous (Thm. 2):** bids coalesce *exactly* at $q_t \Rightarrow \mathcal{G}_t(q_t) = 2pt$ holds *exactly*
- **Discrete:** bids cluster within 2γ of q_t^γ, but exact coalescence is impossible on a lattice of step γ

Convergence Theorem

Let $B^\gamma := [\gamma\lfloor b_i/\gamma\rfloor : b_i \in B]$ be the γ-discretization of B (n = number of bids).

Theorem 7

Let $\mathcal{T}_t B$ (continuous) and $T_t^\gamma B^\gamma$ (discrete) be the respective evolutions, with max prices q_t and q_t^γ. Then:

$$\sup_{t\geq 0}\left|q_t^\gamma - q_t\right| \leq (3n+2)\gamma$$

$$\sup_{b\in B}\sup_{t\geq 0}\left|b_t^\gamma - b_t\right| \leq (3n+2)\gamma$$

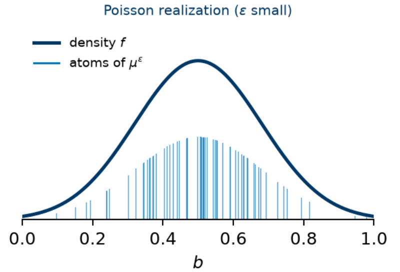
- Each of the n bids contributes $O(\gamma)$ error per collision event, there are at most n collisions, from which follows the bound $3n+2$.
- The theorem provides a rigorous justification of the continuous model as scaling limit

Setup: Poisson Initial Bids

- $\mu(db) = f(b)\,db$: probability measure on $[0,1]$, CDF F
- N^ε: Poisson process on $[0,1]$ with intensity $\varepsilon^{-1}\mu$
- Empirical measure and CDF:

$$\mu^\varepsilon := \varepsilon \sum_{b\in N^\varepsilon} \delta_b, \qquad F^\varepsilon(x) = \varepsilon \sum_{b\in N^\varepsilon} \mathbb{1}\{b \le x\}$$

- $(q_t^\varepsilon)_{t\ge 0}$: max-price evolution for μ^ε
- $(q_t)_{t\ge 0}$: deterministic limit for μ



Atoms of μ^ε concentrate where f is large. As $\varepsilon \to 0$, $\mu^\varepsilon \to \mu$ a.s.

Integrated functions

$$G^\varepsilon(x) = \int_0^x F^\varepsilon(y)\,dy, \quad G(x) = \int_0^x F(y)\,dy, \quad \mathcal{G}_t^\varepsilon(x) = G^\varepsilon(x+t) - G^\varepsilon(x-t)$$

Theorem 8 (LLN for the max price)

For all fixed $t \geq 0$:

$$\lim_{\varepsilon \to 0} q_t^\varepsilon = q_t \qquad \textit{almost surely}$$

Proof sketch:

1. LLN for Poisson processes: $F^\varepsilon(x) \to F(x)$ and $G^\varepsilon(x) \to G(x)$ a.s., uniformly in x
2. Hence $\sup_x |\mathcal{G}_t^\varepsilon(x) - \mathcal{G}_t(x)| \to 0$ a.s.
3. Both $q_t = \mathcal{G}_t^{-1}(2pt)$ and $q_t^\varepsilon = (\mathcal{G}_t^\varepsilon)^{-1}(2pt)$
4. Continuity of the inversion map:

$$|q_t^\varepsilon - q_t| = \left|(\mathcal{G}_t^\varepsilon)^{-1}(2pt) - \mathcal{G}_t^{-1}(2pt)\right| \to 0 \quad \text{a.s.}$$

Gaussian Fluctuations — Functional CLTs

Rescaled fluctuation processes:

$$\zeta^{F^\varepsilon}(x) := \varepsilon^{-1/2}(F^\varepsilon(x) - F(x)), \quad \zeta^{G^\varepsilon}(x) := \varepsilon^{-1/2}(G^\varepsilon(x) - G(x))$$

Proposition 9 (Functional CLT for F^ε)

$\zeta^{F^\varepsilon} \xrightarrow{\text{law}} \zeta^F$ in $D[0,1]$ (Skorokhod J_1): ζ^F is a centred Gaussian process with covariances

$$\mathbb{E}[\zeta^F(x)\zeta^F(y)] = F(\min(x,y))$$

i.e. $\zeta^F(x) = B(F(x))$, B = standard Brownian motion on $[0,\infty)$.

Proposition 10 (Functional CLT for G^ε)

$\zeta^{G^\varepsilon} \xrightarrow{\text{law}} \zeta^G$ in $C[0,1]$ (sup norm): ζ^G is a centred Gaussian process with covariances

$$\mathrm{Cov}(\zeta^G(x), \zeta^G(y)) = \int_0^{x\wedge y} (x-z)(y-z)\, f(z)\, dz.$$

follows from the continuous mapping Theorem, since $\zeta^{G^\epsilon}(x) = \int_0^x \zeta^{F^\epsilon}(y)\, dy$.

Fluctuation Field of $\mathcal{G}_t$

Corollary 11

Let $\zeta^{\mathcal{G}^\varepsilon}(x,t) := \varepsilon^{-1/2}(\mathcal{G}_t^\varepsilon(x) - \mathcal{G}_t(x))$. *As* $\varepsilon \to 0$*:*

$$\zeta^{\mathcal{G}^\varepsilon}(x,t) \xrightarrow{\text{law}} \zeta^{\mathcal{G}}(x,t) := \zeta^{G}(x+t) - \zeta^{G}(x-t)$$

Centered Gaussian with covariance:

$$\mathrm{Cov}\big(\zeta^{\mathcal{G}}(x,s),\ \zeta^{\mathcal{G}}(y,t)\big) = \int_0^1 \Psi_{x,s}(z)\,\Psi_{y,t}(z)\,\mu(dz)$$

where $\Psi_{x,t}(z) := (x+t-z)_+ - (x-t-z)_+$.

Proof tool

$\Phi_t(\psi)(x) := \psi(x+t) - \psi(x-t)$ is Lipschitz on $C[0,1]$.

Apply the *Continuous Mapping Theorem* to $\zeta^{G^\varepsilon} \xrightarrow{\text{law}} \zeta^{G}$.

CLT for the Max Price — Main Result

Theorem 12 (Gaussian fluctuations of q_t^ε)

Let $c_t := F(q_t + t) - F(q_t - t)$*. Then:*

$$\varepsilon^{-1/2}(q_t^\varepsilon - q_t) \xrightarrow{\text{law}} \zeta_t^q := -\frac{\zeta^{\mathcal{G}}(q_t,\, t)}{c_t}$$

The process $\{\zeta_t^q\}_{t\geq 0}$ *is centered Gaussian with*

$$\boxed{\mathrm{Cov}(\zeta_t^q,\, \zeta_s^q) = \frac{1}{c_t\, c_s} \int_0^1 \Psi_{q_t,t}(z)\, \Psi_{q_s,s}(z)\, \mu(dz)}$$

Proof: Functional Delta Method

Apply the Functional Delta Method to the map $\phi : g \longmapsto g^{-1}(2pt)$.

Step 1. We have already established:

$$\varepsilon^{-1/2}\big(\mathcal{G}_t^{\varepsilon}(\cdot) - \mathcal{G}_t(\cdot)\big) \xrightarrow{\text{law}} \zeta^{\mathcal{G}}(\cdot, t) \quad \text{in } C[0,1].$$

Step 2. The map ϕ is Hadamard differentiable at $\mathcal{G}_t$, with derivative in the direction $h \in C[0,1]$:

$$\phi'_{\mathcal{G}_t}(h) = -\frac{h(\mathcal{G}_t^{-1}(2pt))}{\frac{d}{dx}\mathcal{G}_t(\mathcal{G}_t^{-1}(2pt))} = -\frac{h(q_t)}{c_t}, \qquad c_t := \frac{d}{dx}\mathcal{G}_t(q_t) = F(q_t + t) - F(q_t - t).$$

Step 3. The Functional Delta Method gives:

$$\begin{aligned} \varepsilon^{-1/2}(q_t^{\varepsilon} - q_t) &= \\ &= \varepsilon^{-1/2}\left((\mathcal{G}_t^{\epsilon})^{-1}(2pt) - (\mathcal{G}_t)^{-1}(2pt)\right) \xrightarrow{\text{law}} \phi'_{\mathcal{G}_t}(\zeta^{\mathcal{G}}(\cdot, t)) = -\frac{\zeta^{\mathcal{G}}(q_t,\, t)}{c_t}. \end{aligned}$$

[van der Vaart & Wellner 1996]

Multi-Velocity Model — Setup

Generalization: bidder i has individual velocity $v_i \in \mathbb{N}$.

- Winner: $b_i \mapsto b_i + \gamma v_i$
- Loser: $b_i \mapsto b_i - \gamma v_i$
- $v_i = 1 \Rightarrow$ original PSK model

Main result

Under periodicity, the max-price velocity equals

$$u = \frac{w - \ell}{w + \ell} \cdot h$$

$$\text{where} \qquad h := \frac{n}{\sum_{i=1}^{n} \frac{1}{v_i}}$$

is the harmonic mean of $\{v_i\}$ at q_t.

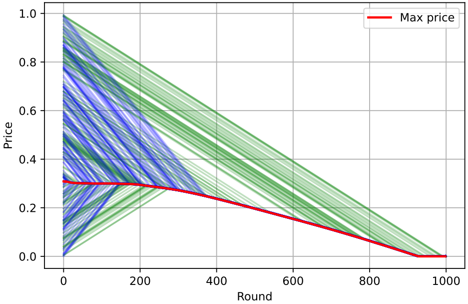


Figure: Two-velocity model.

Max-Price Velocity = Harmonic Mean

Proposition 13

If $(T^\gamma)^\kappa B = B + N\gamma$ for some $N \in \mathbb{Z}$, $\kappa \in \mathbb{N}$ (periodic motion), then:

$$u = \frac{N}{\kappa} = \frac{w - \ell}{\sum_{i=1}^{n} \frac{1}{v_i}} = \frac{w - \ell}{w + \ell} \cdot h, \qquad h := \frac{n}{\sum_{i=1}^{n} \frac{1}{v_i}}$$

h = *harmonic mean of the velocities* $\{v_i\}$.

Key idea: bidder i must satisfy $(w_i(\kappa) - \ell_i(\kappa))v_i = N$ and $w_i(\kappa) + \ell_i(\kappa) = \kappa$, giving $w_i = \frac{1}{2}(\kappa + N/v_i)$ and $\ell_i = \frac{1}{2}(\kappa - N/v_i)$. Summing over i:

$$(w - \ell)\kappa = \sum_{i=1}^{n}(w_i(\kappa) - \ell_i(\kappa)) = N \sum_{i=1}^{n} \frac{1}{v_i} \implies \frac{N}{\kappa} = \frac{w - \ell}{\sum_i \frac{1}{v_i}}. \qquad \square$$

Two Velocities: Periodicity vs. Separation

$v_i \in \{\nu_-, \nu_+\}$, $\nu_- < \nu_+$; n_-, n_+ bidders of each type; $h = n/(n_-/\nu_- + n_+/\nu_+)$.

Theorem 14 (Periodicity $\Leftrightarrow$ bounded velocity)

The following are equivalent:

1. *Motion attains a periodic increment state*
2. $\dfrac{|w - \ell|}{n} \cdot h \leq \nu_-$

- **Condition holds:** slow and fast bids coexist near q_t
- **Condition fails:** slow bids separate from q_t; drift ballistically (baricenter of fast bids grows faster than ν_-)

Ballistic Price Evolution

Theorem 15

Assume B is in the bulk regime: $|b_i - b_j| = O(\gamma)$ for all i, j. Then, uniformly in γ and $t \geq 0$:

$$\sup_{t\geq 0} \frac{1}{\gamma}|q_t^\gamma - q_t| < \infty, \qquad \sup_t \frac{1}{\gamma}|b_i^\gamma(t) - b_i(t)| < \infty$$

where

$$q_t = \begin{cases} q_0 + \dfrac{w-\ell}{n}\, h\, t & \text{if } \dfrac{|w-\ell|}{n} h \leq \nu_- \\ q_0 + \dfrac{w-\ell-n_- \operatorname{sign}(w-\ell)}{n}\, \nu_+\, t & \text{otherwise} \end{cases}$$

$$b_i(t) := \begin{cases} q_t & \text{if } \dfrac{|w-\ell|}{n} h \leq \nu_- \text{ or } v_i = \nu_+ \\ q^0 + \operatorname{sign}(w-\ell)\nu_- t & \text{otherwise} \end{cases}$$

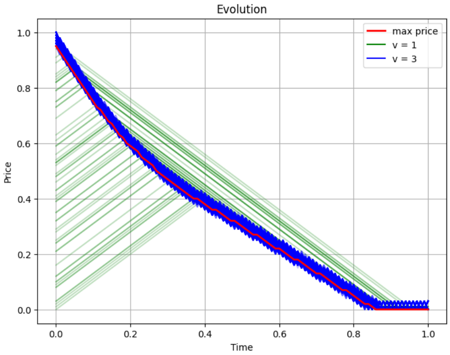


Figure: Two velocities with separation.

Continuous Multi-Velocity Model

ODE system for $B = [(b_1, v_1), \ldots, (b_n, v_n)]$, $v_i \in \{\nu_-, \nu_+\}$:

$$h_t = \frac{w_t + \ell_t}{\frac{1}{n}\sum_i \frac{1}{v_i} \mathbb{1}\{b_i(t) = q_t\}}, \qquad \frac{dq_t}{dt} = u_t$$

$$u_t = \begin{cases} \dfrac{w_t - \ell_t}{w_t + \ell_t} \cdot h_t, & \left|\dfrac{w_t - \ell_t}{w_t + \ell_t} \cdot h_t\right| \le \nu_- \\[2ex] \dfrac{w_t - \ell_t - \operatorname{sign}(w_t - \ell_t)\, m_t^-}{m_t^+} \cdot \nu_+ & \left|\dfrac{w_t - \ell_t}{w_t + \ell_t} \cdot h_t\right| > \nu_- \end{cases}$$

$$m_t^{\pm} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{b_i(t) = q_t\} \mathbb{1}\{v_i = \nu_{\pm}\}$$

$$\frac{db_i}{dt} = \begin{cases} +v_i & b_i(t) < q_t \\ u_t & b_i(t) = q_t,\ |u_t| \le v_i \quad \text{(stays on } q_t\text{)} \\ \operatorname{sign}(u_t) v_i & b_i(t) = q_t,\ |u_t| > v_i \quad \text{(separates)} \\ -v_i & b_i(t) > q_t \end{cases}$$

Convergence: Multi-Velocity Discrete → Continuous

Theorem 16

Let B^γ be the γ-discretization of B (velocities preserved). Then, uniformly in $\gamma \in (0,1)$ and $t \geq 0$:

$$\frac{1}{\gamma}|q_t^\gamma - q_t| < C_1, \qquad \frac{1}{\gamma}|b_i^\gamma(t) - b_i(t)| < C_2 \quad \forall\, i$$

for finite constants $C_1, C_2 > 0$.

Proof: induction over inter-collision intervals $[t_l, t_{l+1}]$:

1. Apply ballistic evolution theorem on each interval
2. Timing error: $|t_1^\gamma - t_1| \leq C'\gamma$ (linear behaviour near collision)
3. Propagate bound across all finitely many collisions

Open problems and possible generalisations

- **More than two velocities:** periodicity and convergence for $k \geq 3$ distinct bidder velocities
- **Max-price fluctuations (multi-velocity):** CLT for q_t^ε in the multi-velocity Poisson model; covariance formula remains open
- **Time-varying winner fraction $p(t)$:** generalisation of the ODE and baricenter formula to a time-dependent competition level
- **Stochasticity in the interaction:** random winner fraction p, random bid updates, or random participation at each round
- **Asymmetric update:** winners increase by γ_+, losers decrease by $\gamma_- \neq \gamma_+$
- **Non-myopic strategies:** sellers with memory; cooperative or competitive interaction between sellers

Thank you!