

Thimble decomposition and Wall Crossing Structure for Physical Integrals

Roberta Angius



HIGHER STRUCTURES, MODULI
SPACES AND INTEGRABILITY



Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

Based on

[hep-th: 2506.03252] with **S.L. Cacciatori** and **A. Massidda**

Paris, January 28, 2026

Motivations

Analysis of complex **integrals**
describing generic physical systems

$$I = \int_{\Gamma} F(x_1, x_2, \dots; \mu_1, \mu_2, \dots) dx_1^2 dx_2^2 \dots$$

Direct computation

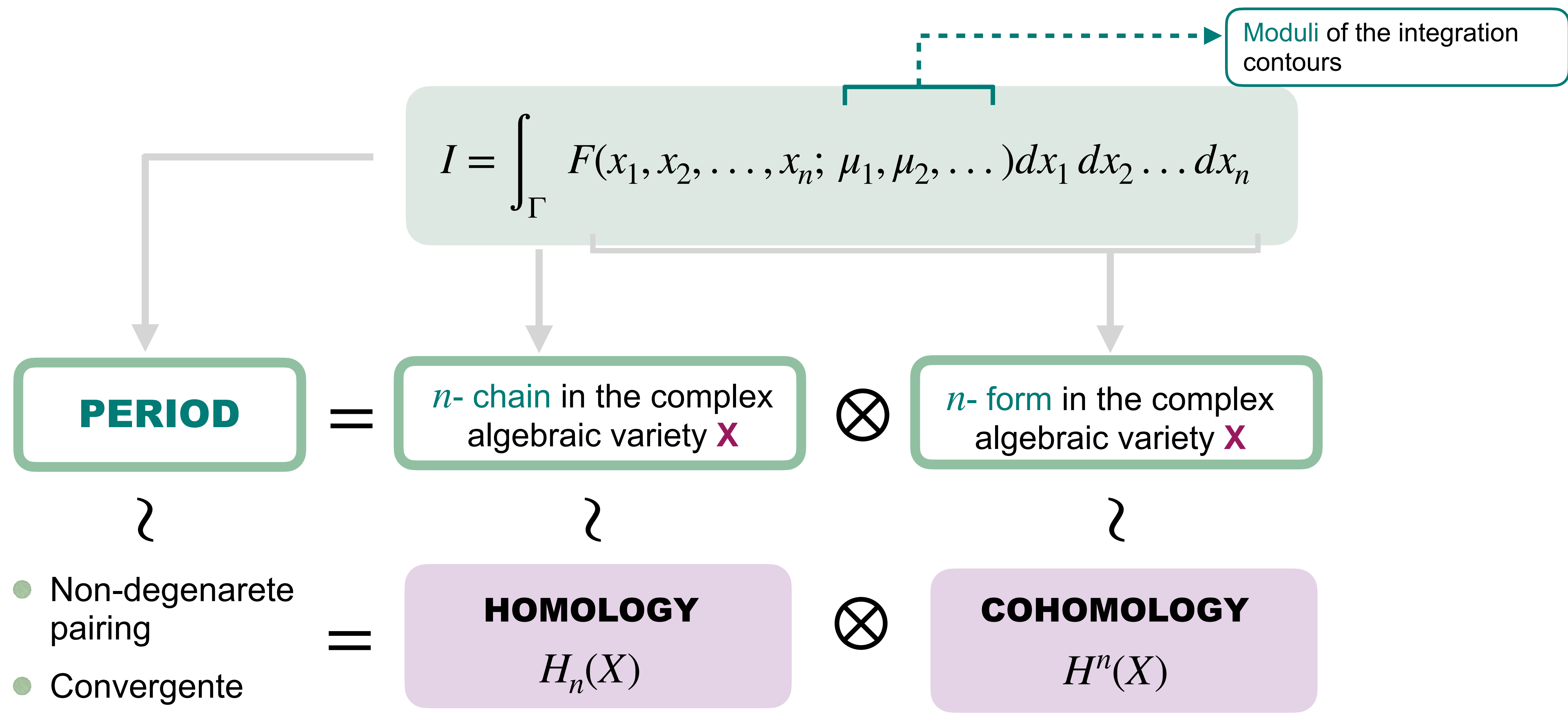
**Resolution of
systems of differential
equations**

**Geometric interpretation
in order to simplify the
integral.**

Periods

[Kontsevich, Zagier 2001]

Values of integrals of algebraically defined differential forms over certain chains in algebraic varieties



The advantages

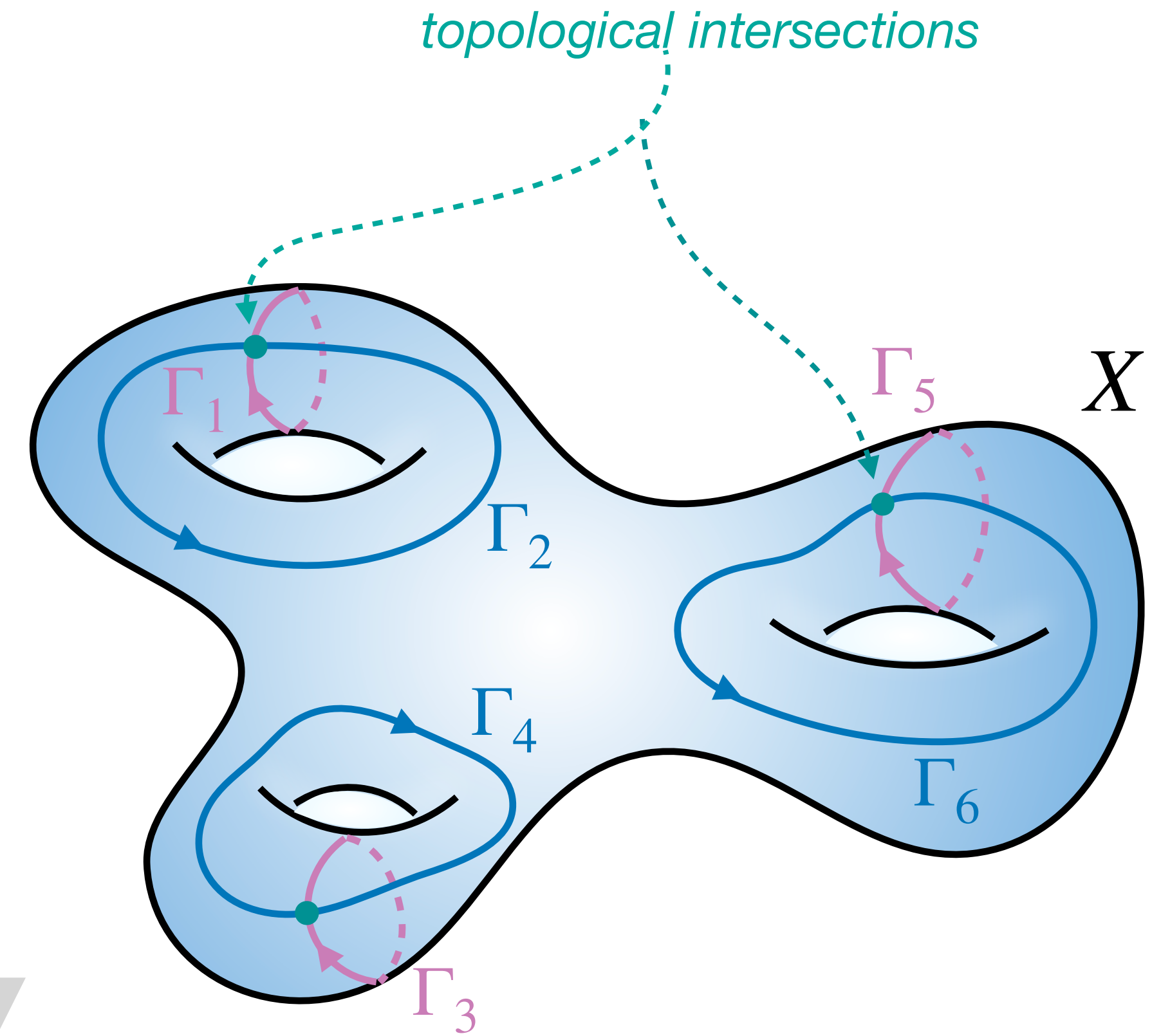
The **value** of the integral **only depends** on the homology class of the integration cycle and the cohomology class of the integration form, but it does **not depend** on the specific representatives.

- The homology and cohomology rings are **finitely** generated

$$H_n(X, \bullet) = \langle \Gamma_1, \dots, \Gamma_N \rangle$$

$$H^n(X, \bullet) = \langle \omega_1, \omega_2, \dots, \omega_N \rangle$$

- There is a non-degenerate **internal product** in homology given by **topological intersection** among cycles and its dual in cohomology



$$I = \int_{\Gamma} \omega(\bar{x}; \bar{\mu}) =$$

Element of a finite dimensional
vector space
equipped
with a **double projection**.

The Master Integrals

Double projection:

- Fix a basis: $H_n(X, \bullet) = \langle \Gamma_1, \dots, \Gamma_N \rangle$
and an internal product: $(\Gamma_i, \Gamma_j) = I_{ij}$ $\left. \vphantom{\begin{array}{l} \text{Fix a basis: } H_n(X, \bullet) = \langle \Gamma_1, \dots, \Gamma_N \rangle \\ \text{and an internal product: } (\Gamma_i, \Gamma_j) = I_{ij} \end{array}} \right\} \rightarrow \Gamma = (\Gamma, \Gamma_1) \Gamma_1 + (\Gamma, \Gamma_2) \Gamma_2 + \dots + (\Gamma, \Gamma_N) \Gamma_N$

$$I = \int_{\Gamma} \omega = (\Gamma, \Gamma_1) \cdot \int_{\Gamma_1} \omega + (\Gamma, \Gamma_2) \cdot \int_{\Gamma_2} \omega + \dots + (\Gamma, \Gamma_N) \cdot \int_{\Gamma_N} \omega$$

Intersection numbers in homology

Master Integrals

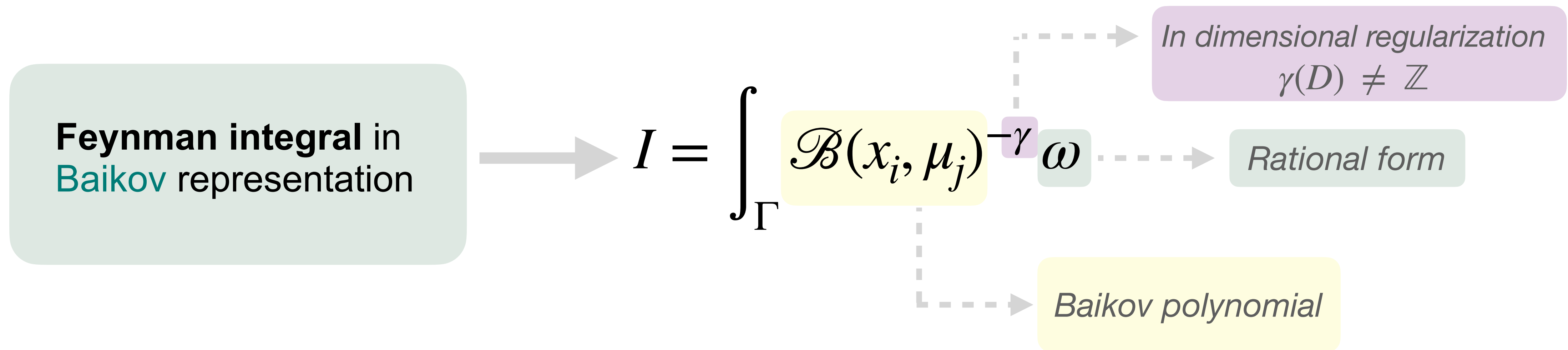
- Fix a basis: $H^n(X, \bullet) = \langle \omega_1, \dots, \omega_N \rangle$
and an internal product: $(\omega_i, \omega_j) = \tilde{I}_{ij}$ $\left. \vphantom{\begin{array}{l} \text{Fix a basis: } H^n(X, \bullet) = \langle \omega_1, \dots, \omega_N \rangle \\ \text{and an internal product: } (\omega_i, \omega_j) = \tilde{I}_{ij} \end{array}} \right\} \rightarrow \omega = (\omega, \omega_1) \omega_1 + (\omega, \omega_2) \omega_2 + \dots + (\omega, \omega_N) \omega_N$

$$I = (\omega, \omega_1) \cdot \int_{\Gamma} \omega_1 + (\omega, \omega_2) \cdot \int_{\Gamma} \omega_2 + \dots + (\omega, \omega_N) \cdot \int_{\Gamma} \omega_N$$

Intersection numbers in cohomology

Master Integrals

The problems

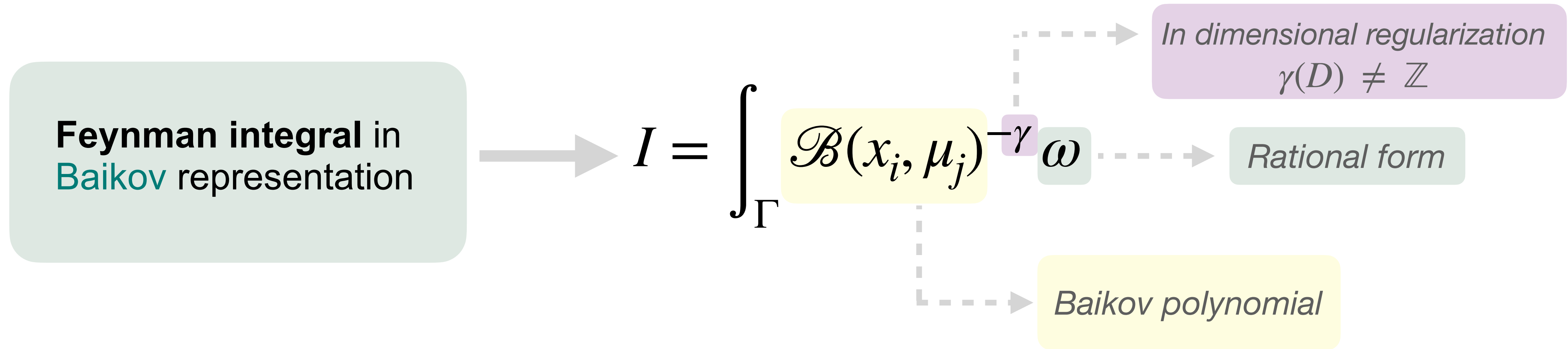


- ▶ Multivalued integral with a potentially complicate monodromy
- ▶ Special values of the parameters at which the manifold X becomes singular
- ▶ ...

Main Task

To indentify the **right** homology/cohomology to define the pairing

The problems



- ▶ In the case of Feynman integrals, this idea to exploit the period interpretation [Pham, 1981/1985] and intersection theory methods to perform the MIs decomposition is not new [Mastrolia, Mizera - 2018].
- ▶ And many computational algorithms has been developed in the meantime:
[Brunello, Chestnov, Crisanti, Fontana, Frallesvig, Gasparotto, Laporta, Mandal, Mastrolia, Matsubara-Heo, Matsumoto, Mattiazzi, Mizera, Munch, Peraro, Pokraka, Takayama ...]
- ▶ However many cases (as limits of singular varieties) are still unsolved.

Plan of the presentation

- Exponential Integrals
- Wall Crossing Structure
- Holomorphic functions
- Multivalued functions
- Final remarks

Exponential Integrals

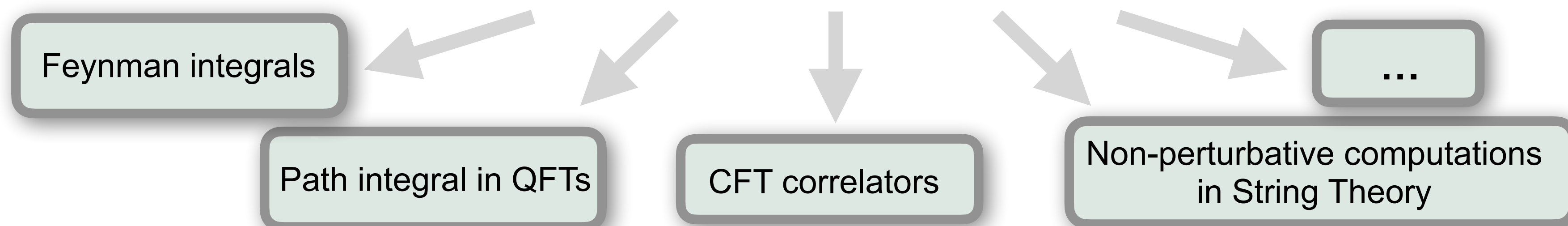
The exponential period map

[Kontsevich, Soibelman - 2024]

- X $\longrightarrow$ n -dim complex algebraic variety
- $f : X \mapsto \mathbb{C}$ $\longrightarrow$ Complex valued function
- μ $\longrightarrow$ Holomorphic volume form over X
- Γ $\longrightarrow$ Open integration chain on $X \setminus D_0$

s.t. $Re(f)|_{\Gamma} : Supp(\Gamma) \mapsto \mathbb{R}$ is a proper map bounded from below

$$\int_{\Gamma} e^{-f} \mu : H_{\bullet}^{Betti, global}((X, D_0), f) \otimes H_{dR, global}^{\bullet}((X, D_0), f) \mapsto \mathbb{C}$$



Generalized Exponential Integral

- Rescaling the function: $f \mapsto \gamma f$ with $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$

Generalized exponential integral:

$$I(\gamma) = \int_{\Gamma} e^{-\gamma f} \mu : H_{\bullet}^{Betti, global}((X, D_0), \gamma f) \otimes H_{dR, global}^{\bullet}((X, D_0), \gamma f) \mapsto \mathbb{C}$$

Wall Crossing Structure:

To study how the structure of the resulting integral depends on γ .

By varying the parameter γ , the **homology** and the **cohomology** groups involved in the pairing can change, and for special values of γ they can even **reduce their dimensionality** leading to a reduced number of MIs.

De Rham Cohomology

$$(X, D_0, f) \quad \text{---} \quad \blacktriangleright$$

The integrand n -form μ is represented by a class $[\mu]$ in the degree n Twisted de Rham group

$$[\mu] \in H_{dR}^n(X, D_0, f)$$

Global Twisted de Rham

$$H_{dR}^\bullet(X, D_0, f) \cong \mathbb{H}^\bullet(X, \Omega_{X, D_0}^\bullet, \nabla_f)$$

Graded abelian group of equivalence classes of closed forms on $X \setminus D_0$ with respect to the differential

$$\nabla_f = d + df \wedge = d + \alpha \wedge$$

where

$$(\Omega_{X, D_0}^\bullet, \nabla_f) : \Omega_{X, D_0}^0 \xrightarrow{\nabla_f} \Omega_{X, D_0}^1 \xrightarrow{\nabla_f} \dots \xrightarrow{\nabla_f} \Omega_{X, D_0}^n$$

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Generalized version:

$$H_{dR, \gamma}^\bullet(X, D_0, f) \cong \mathbb{H}^\bullet(X, \Omega_{X, D_0}^\bullet, \nabla_{\gamma f})$$

with respect to the differential

$$\nabla_f = d + \gamma df \wedge$$

Global Betti Homology

Graded abelian group which captures the topology of chains on $X \setminus D_0$ relative to the level sets of f at infinity:

$$H_{\bullet}^{Betti, global}((X, D_0), f, \mathbb{Z}) \equiv H_{\bullet}(X, D_0 \cup f^{-1}(\infty), \mathbb{Z})$$

- Let us fix $D_0 = \emptyset$ for simplicity.

The integration cycle Γ is a **non-compact n -dim** cycle in the complex variety X with **boundaries** on the subset $\{\mathbf{z} \in X \mid f(\mathbf{z}) = \infty\} \subset X$. The cycle Γ is represented by the class $[\Gamma] \in H_n^{Betti, global}(X, f, \mathbb{Z})$

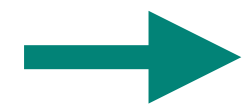
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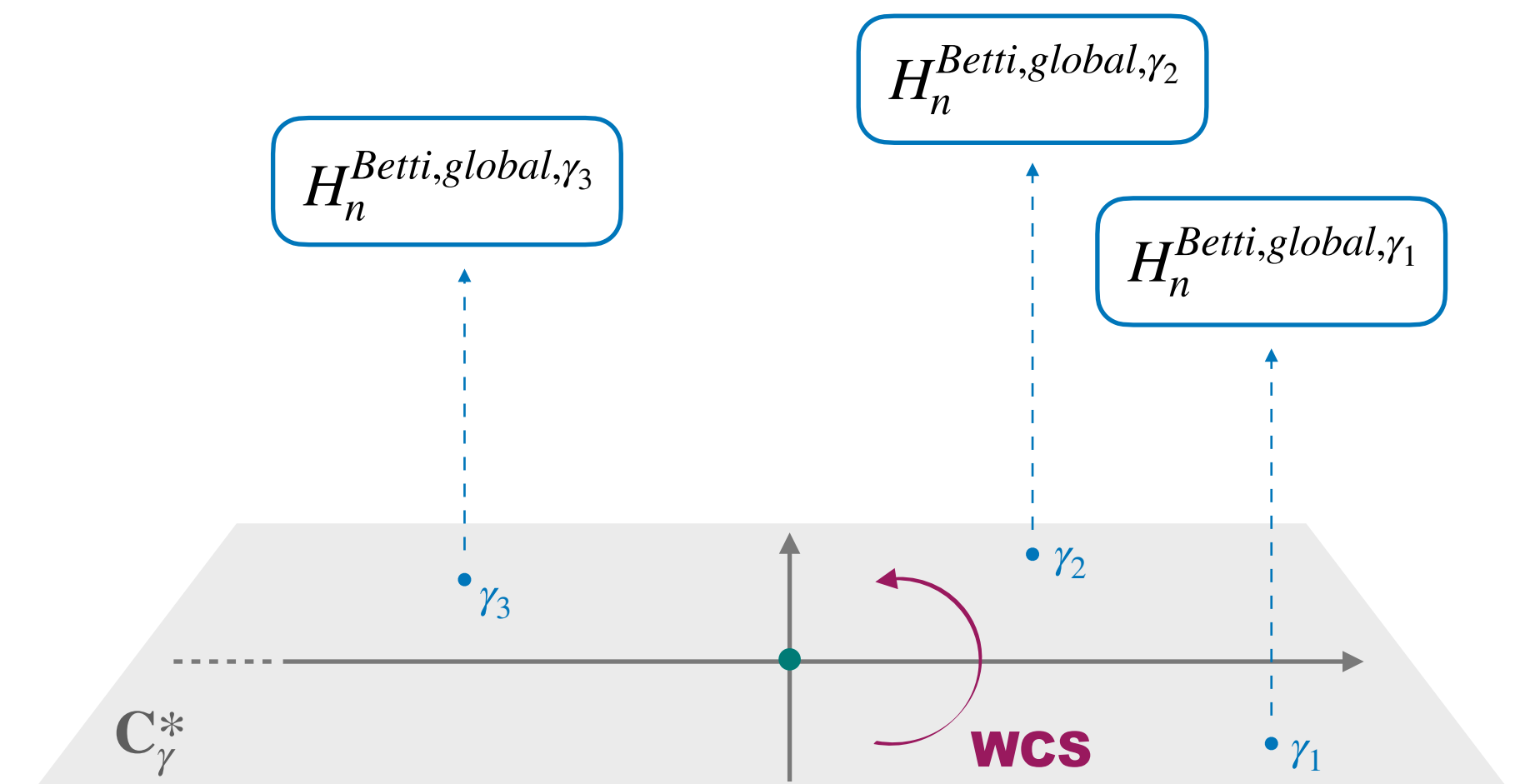
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Generalized version:

$$H_{\bullet}^{Betti,global,\gamma}((X, D_0), f, \mathbb{Z}) \equiv H_{\bullet}(X, D_0 \cup (\gamma f)^{-1}(\infty), \mathbb{Z})$$



Master Integrals decomposition

Given the **exponential integral**, with fixed integrand and fixed $\gamma \in \mathbb{C}^*$

$$I = \int_{\Gamma} e^{-\gamma f} \mu$$

- Define a **basis** of **integration contours** $\{\Gamma_i\}_{i=1}^{N=\dim H_n}$ for the Betti Homology group $H_n^{\text{Betti}, \text{global}, \gamma}(X, f, \mathbb{Z})$
- Define a non-degenerate **internal product** on the group $H_n^{\text{Betti}, \text{global}, \gamma}(X, f, \mathbb{Z})$:

$$(\Gamma_i, \Gamma_j) = c_{ij} \quad \text{-----} \rightarrow \text{Intersection numbers}$$

such that the integration contour Γ can be written in terms of the following **linear combination**:

$$\Gamma = (\Gamma, \Gamma_1) \Gamma_1 + (\Gamma, \Gamma_2) \Gamma_2 + \dots + (\Gamma, \Gamma_N) \Gamma_N$$

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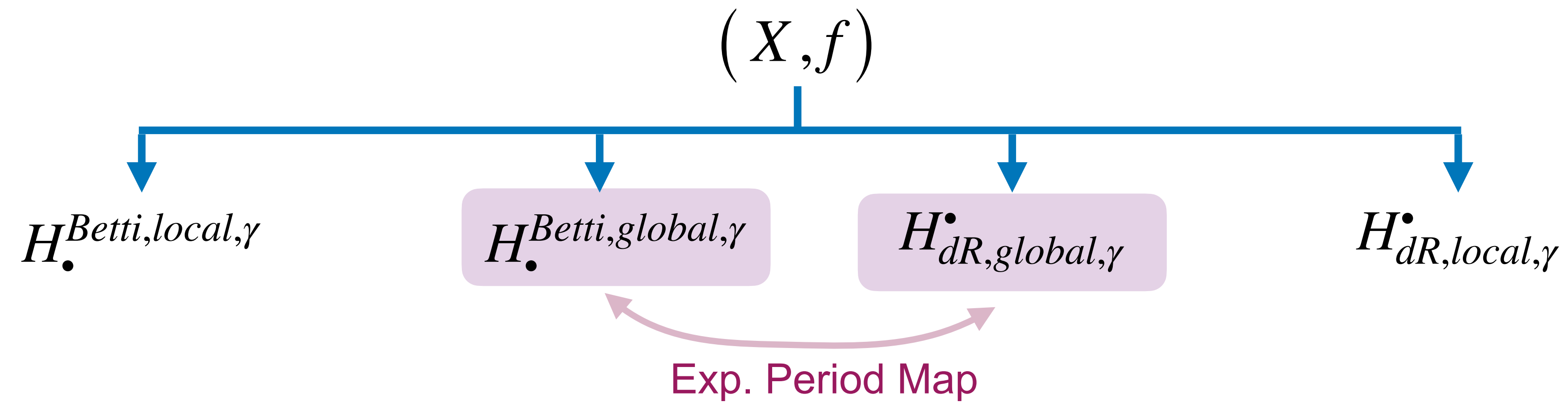
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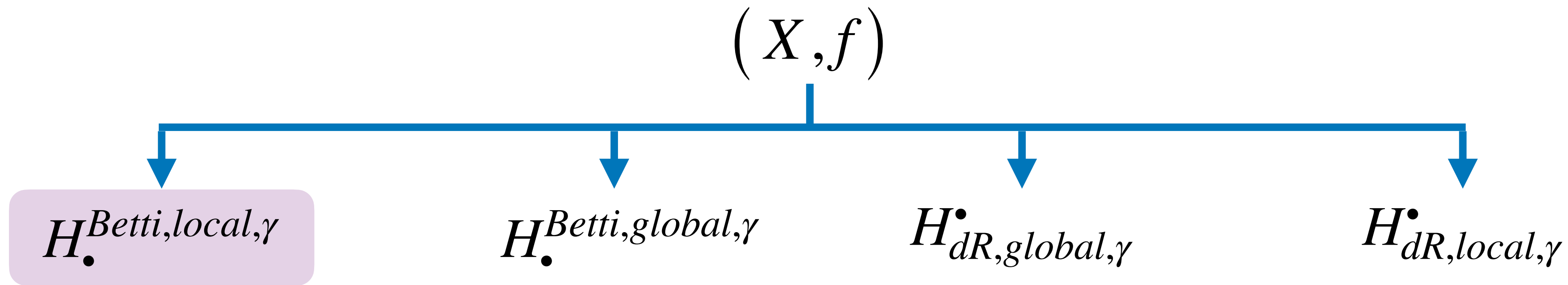
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Local to Global isomorphism

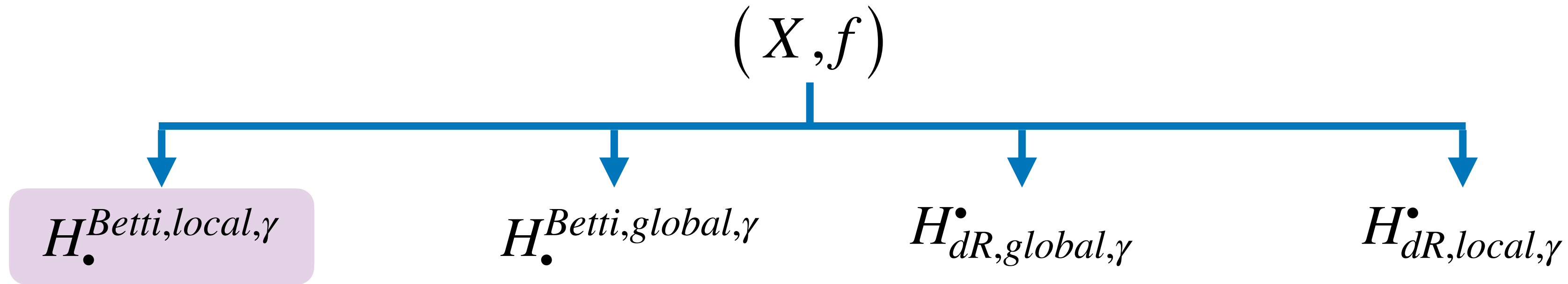


Local to Global isomorphism

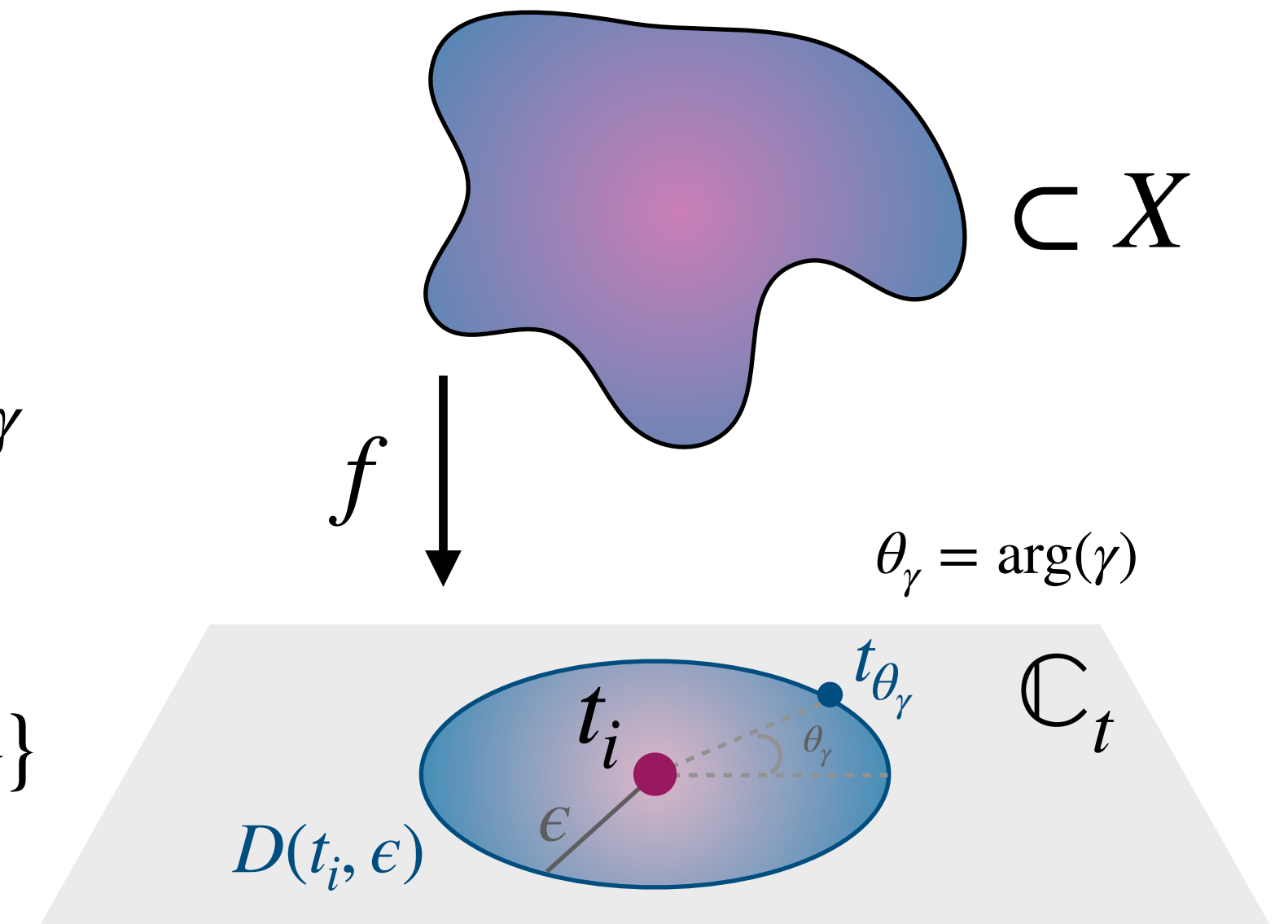


- Let us assume that f :
- Has finitely many non-degenerate **critical points** : $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_N\}$
 - All with distinct **critical values** : $S = \{f(\sigma_1) = t_1, \dots, f(\sigma_N) = t_N\}$

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Local Betti Homology

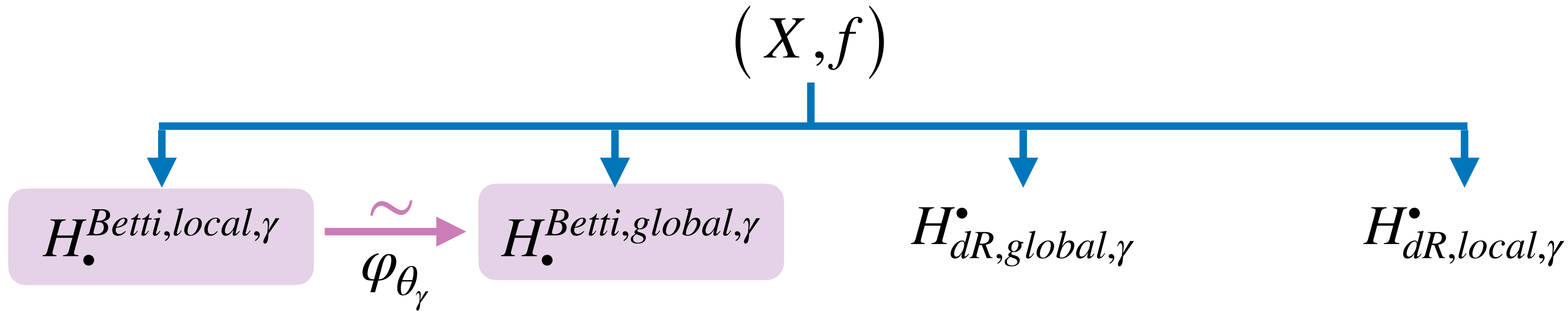
Fixed γ and a small positive number $\epsilon \in \mathbb{R}$, we can associate to each pair (t_i, γ) the following homology group

$$H_{\bullet}^{Betti, local, t_i, \gamma}(X, f) = H_{\bullet} \left(\gamma f^{-1}(D(\gamma t_i, \epsilon)), \gamma f^{-1}(t_{\theta_\gamma}); \mathbb{Z} \right)$$

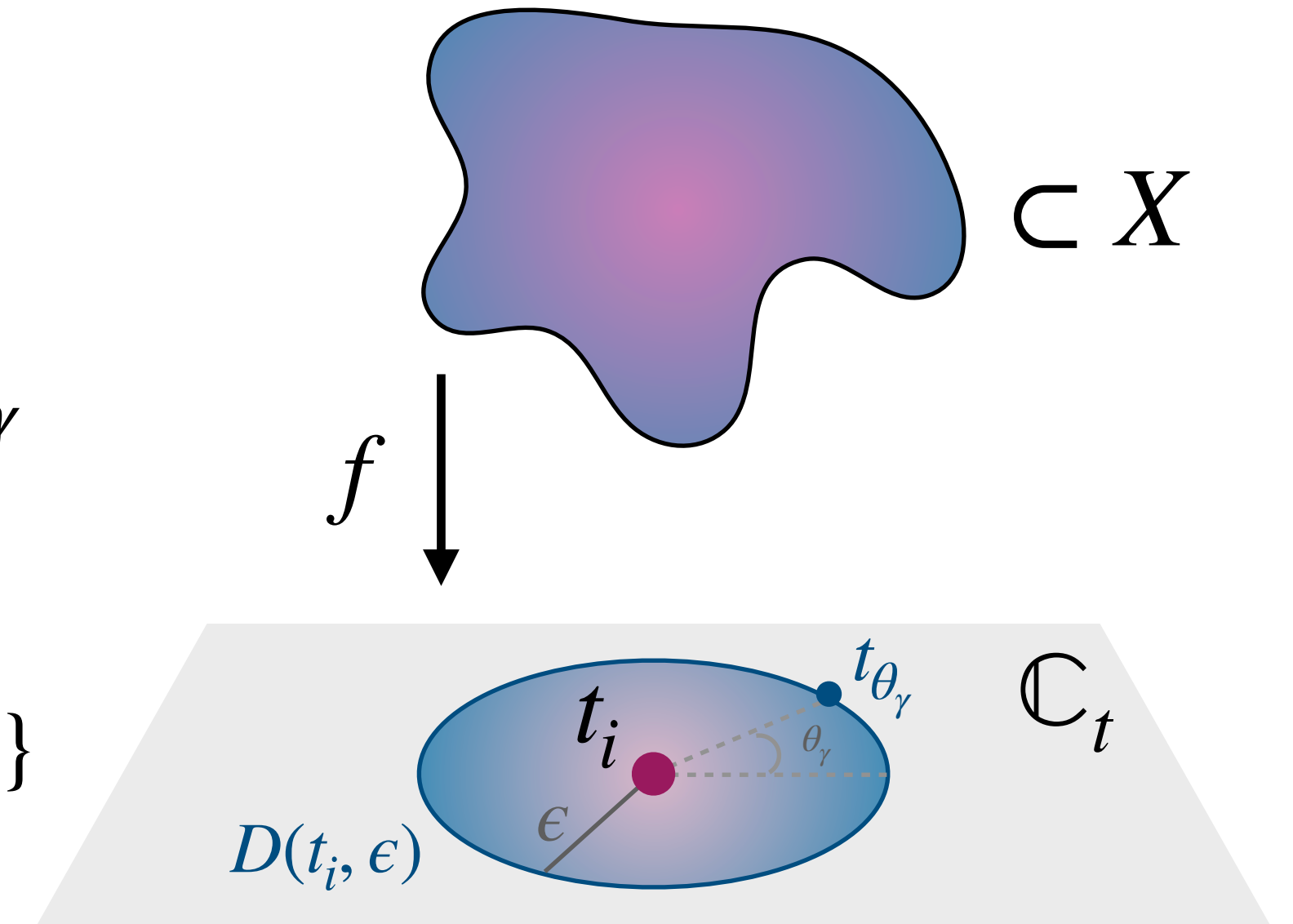
Summing over all the local relative homologies we obtain:

$$H_{\bullet}^{Betti, local, \gamma}(X, f) = \bigoplus_{i=1}^N H_{\bullet}^{Betti, local, t_i, \gamma}(X, f)$$

Local to Global isomorphism



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Local Betti Homology

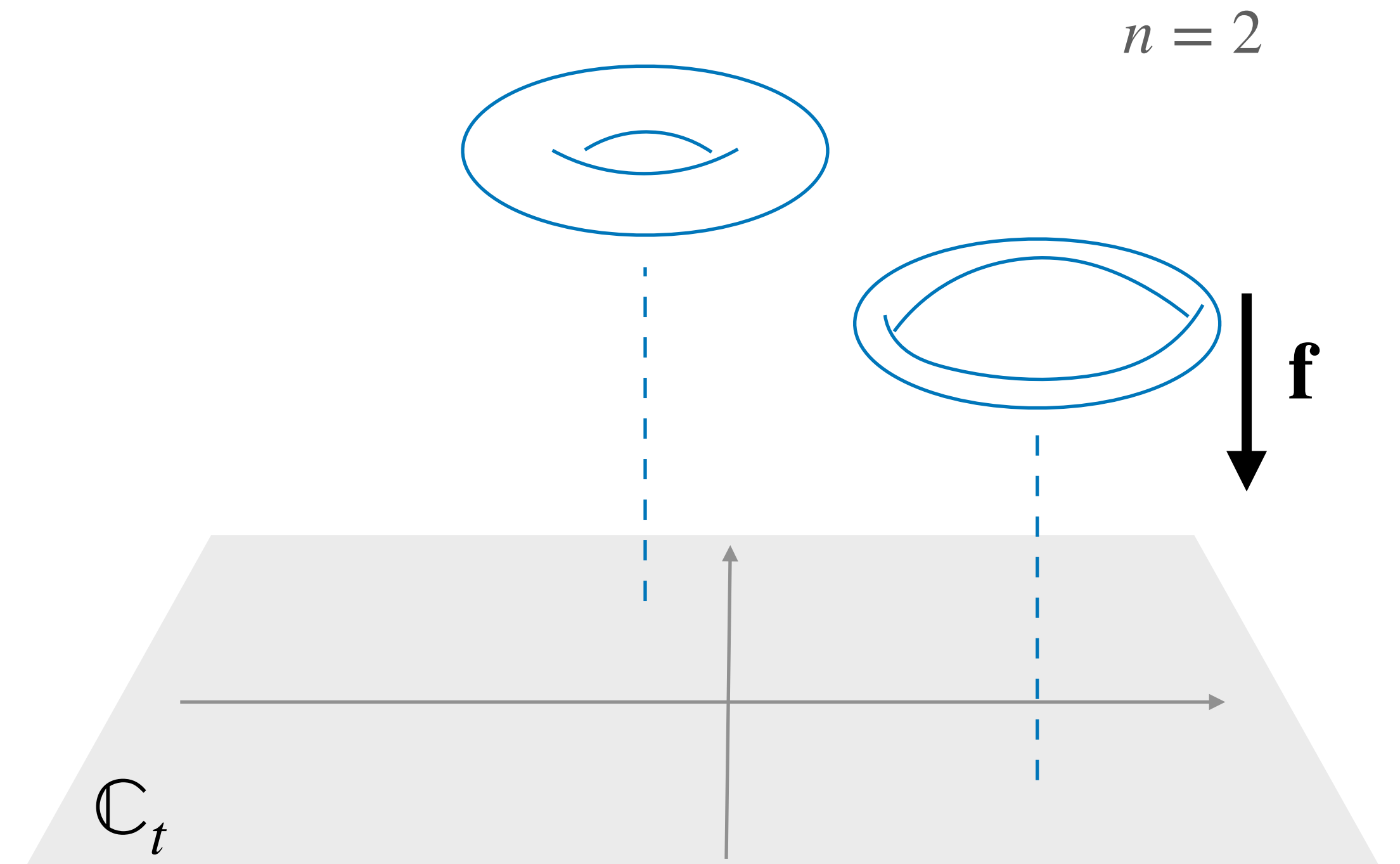
► We can reconstruct the **global** Betti Homology group using **local data**.

① Consider the level sets associated with the function $f: \gamma f(\mathbf{z}) = t \in \mathbb{C}_t$

For each value of t this equation defines a $(n - 1)$ -dim complex **algebraic variety**.

Geometric point of view:

As t varies we have an entire **family** of algebraic varieties with **different sizes** for some of their **internal** $(n - 1)$ -dimensional **cycles**.



Local Betti Homology

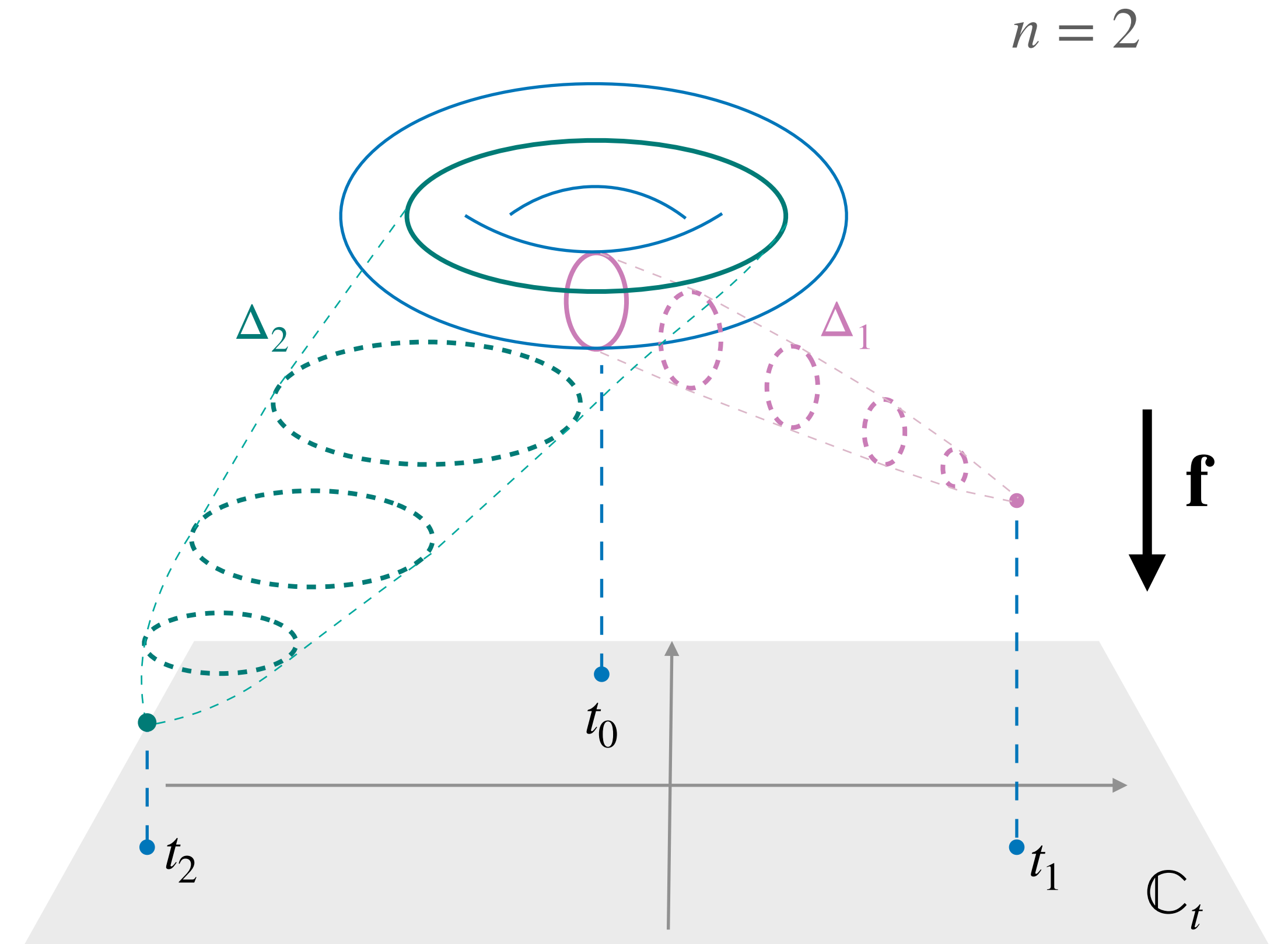
- ② Define the set of critical loci of $f : \Sigma = \{\sigma_i \in X \mid df(\sigma_i) = 0\}$ and the corresponding set of critical values $S = \{f(\sigma_1) = t_1, f(\sigma_2) = t_2, \dots, f(\sigma_N) = t_N\}$ (we are assuming non-degeneracy)

Geometric point of view:

In correspondence of these critical values the algebraic variety on the fiber is singular.

- Some of the internal $(n - 1)$ -cycles shrink to zero size.

Vanishing cycles Δ_i



Local Betti Homology

► For each **critical point** t_i there is a **vanishing cycle** Δ_i .

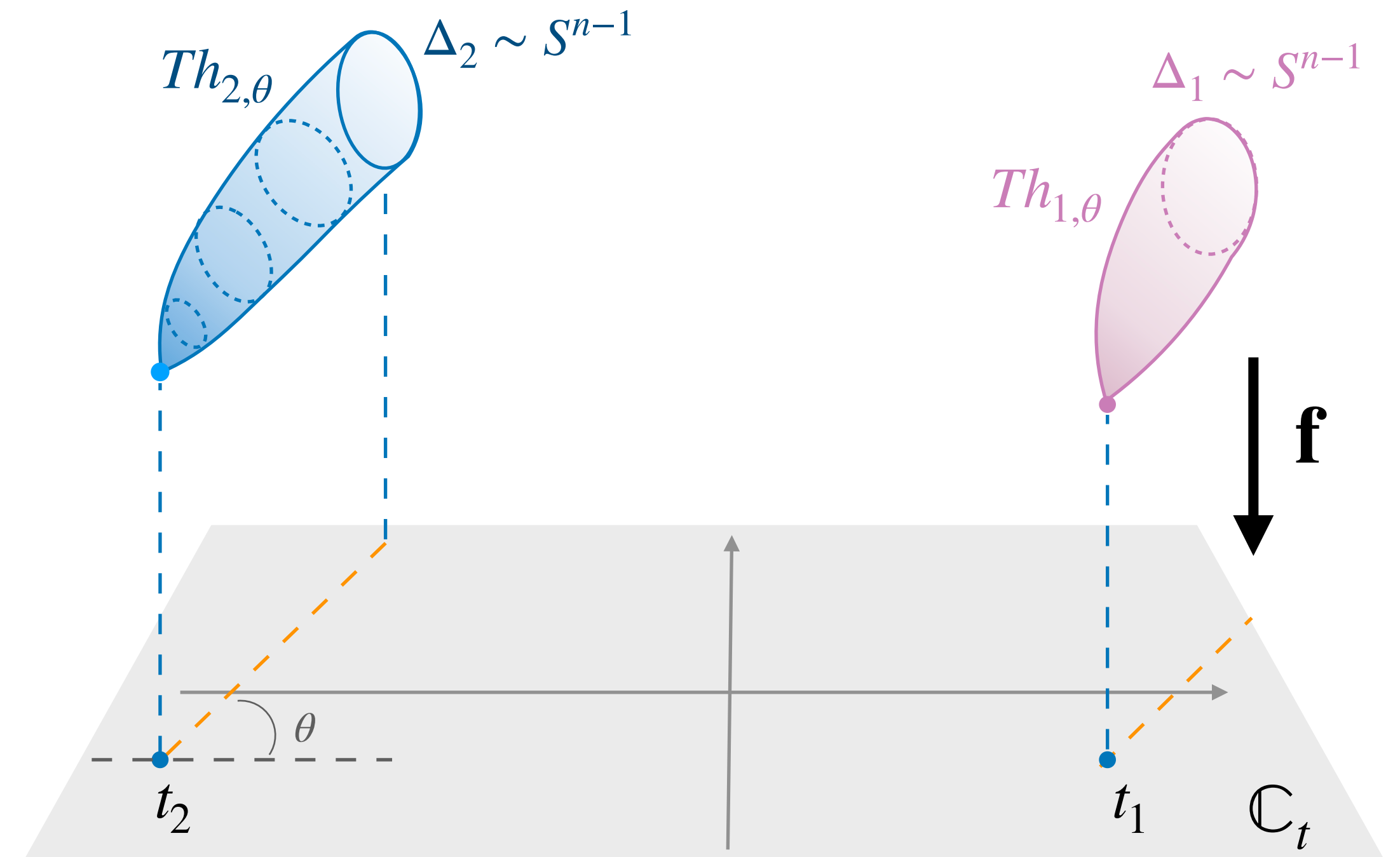
③ Consider the direction $\theta = \arg(\gamma)$ in the t -plane and define the set of thimbles $\{Th_{i,\theta}\}_{i=1}^N$.

Thimble $Th_{i,\theta}$ = **Trace** of the vanishing cycle Δ_i along the direction $\theta = \arg(\gamma)$ starting from the critical point t_i .

► **NOTE:** Thimbles are n -dimensional. They have the right dimension to be integration cycles!

The whole set of **thimbles** constructed along the direction $\arg(\gamma)$ form a **basis** for the **Betti homology**

$$H_n^{Betti,\gamma}(X, f, \mathbb{Z}) \simeq \langle Th_{1,\theta}, Th_{2,\theta}, \dots, Th_{N,\theta} \rangle$$



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Internal product in Betti Homology

► For generic values of the angle $\theta = \arg(\gamma)$, thimbles associated with different critical values never intersect

We can define an intersection pairing:

$$\langle \cdot, \cdot \rangle : H_n^{\text{Betti}, \text{global}, \gamma}(X, f, \mathbb{Z}) \times H_n^{\text{Betti}, \text{global}, e^{i\pi}\gamma}(X, f, \mathbb{Z}) \mapsto \mathbb{Z}$$

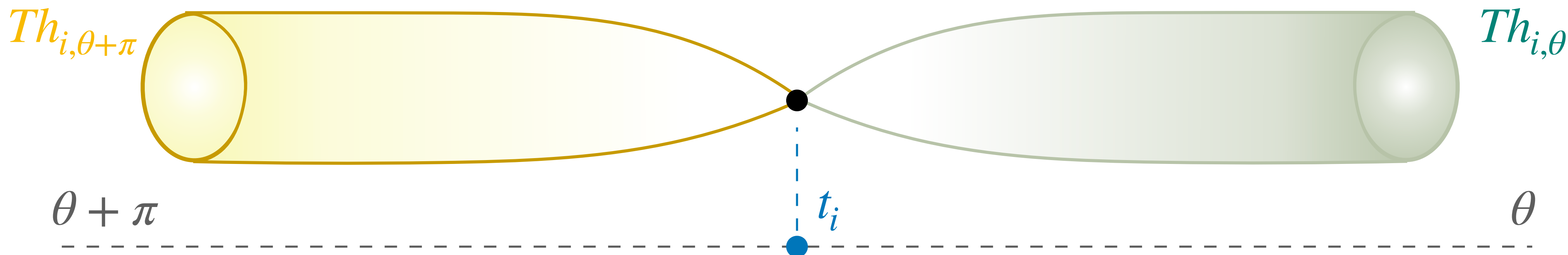
Betti homology

$$\{Th_{i,\theta}\}_{i=1}^N$$

Dual Betti homology

$$\{Th_{i,\theta+\pi}\}_{i=1}^N$$

With respect to these basis the intersection pairing is given by: $\langle Th_{i,\theta}, Th_{j,\theta+\pi} \rangle = \delta_{ij}$



Wall Crossing Structure

Thimbles degeneration

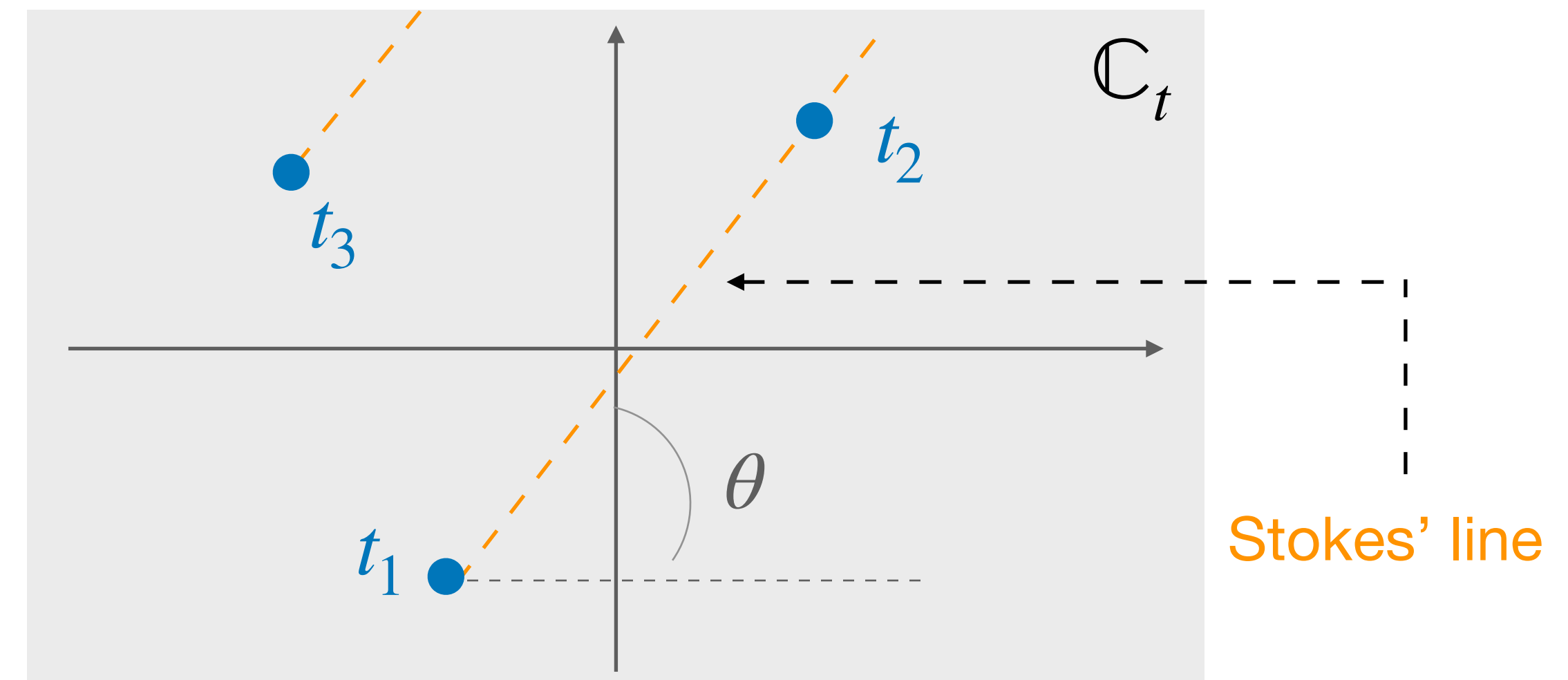
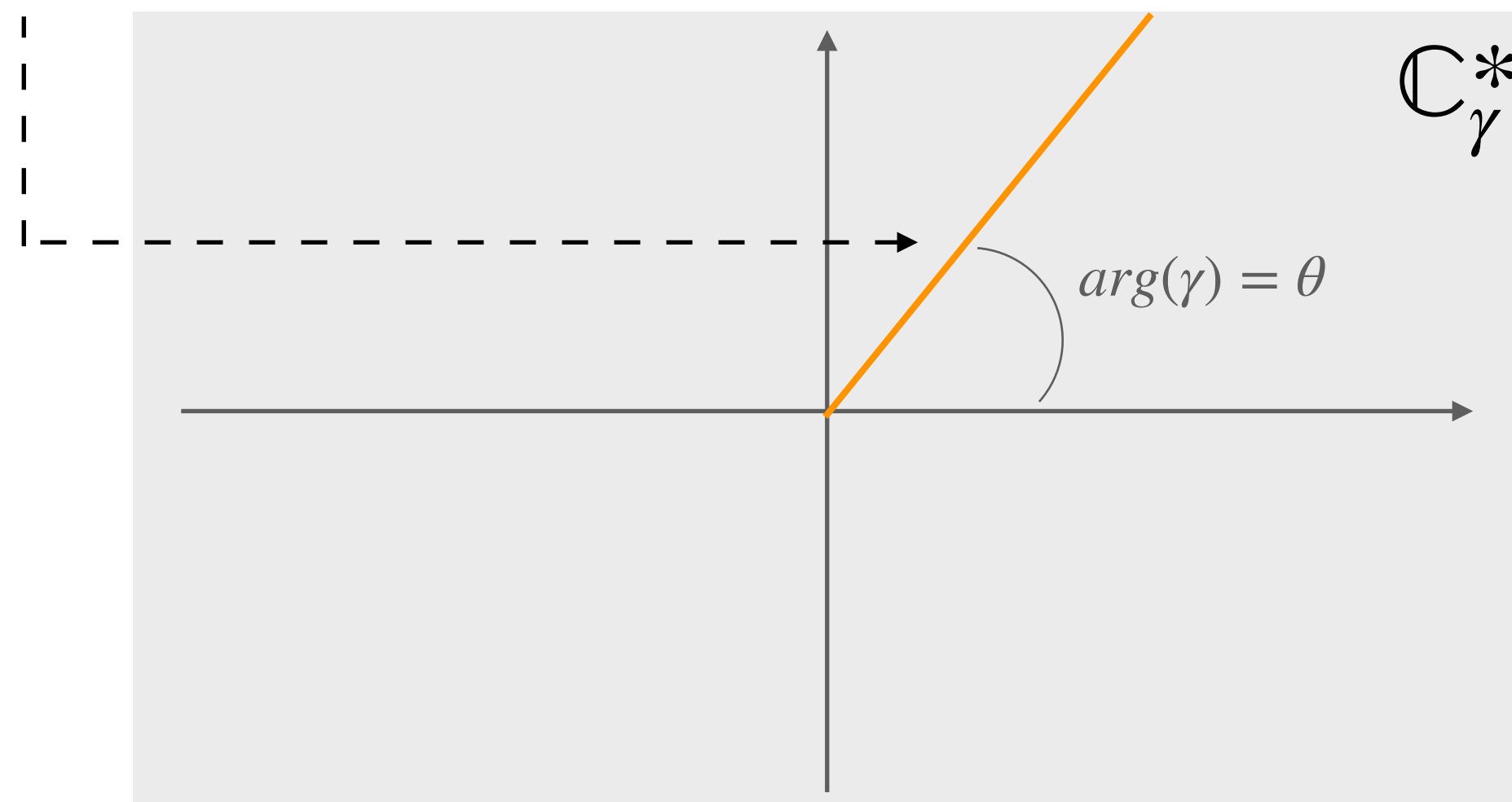
- Problems appear for special values of γ for which the lines with direction $\theta = \arg(\gamma)$ in the plane $\mathbb{C}_t$ intersect two or more critical points.



Degeneration of the corresponding thimbles

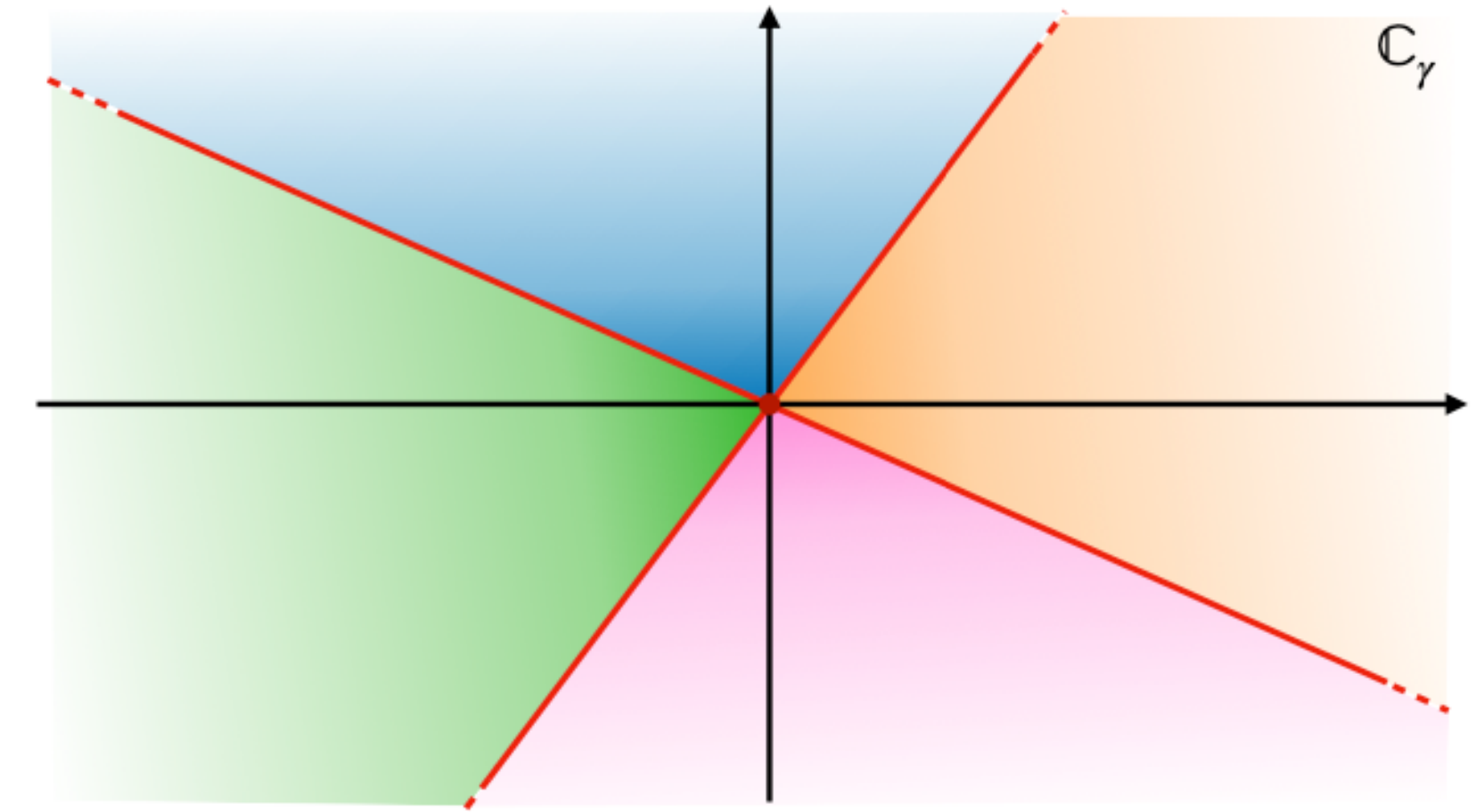
Appearance of **Stokes rays** in the plane $\mathbb{C}_\gamma^*$. In correspondence of this lines we have a **wrong number** of **thimbles**, then a corresponding **wrong computation** of the **number of MIs**.

Stokes' ray



Wall Crossing Structure (WCS)

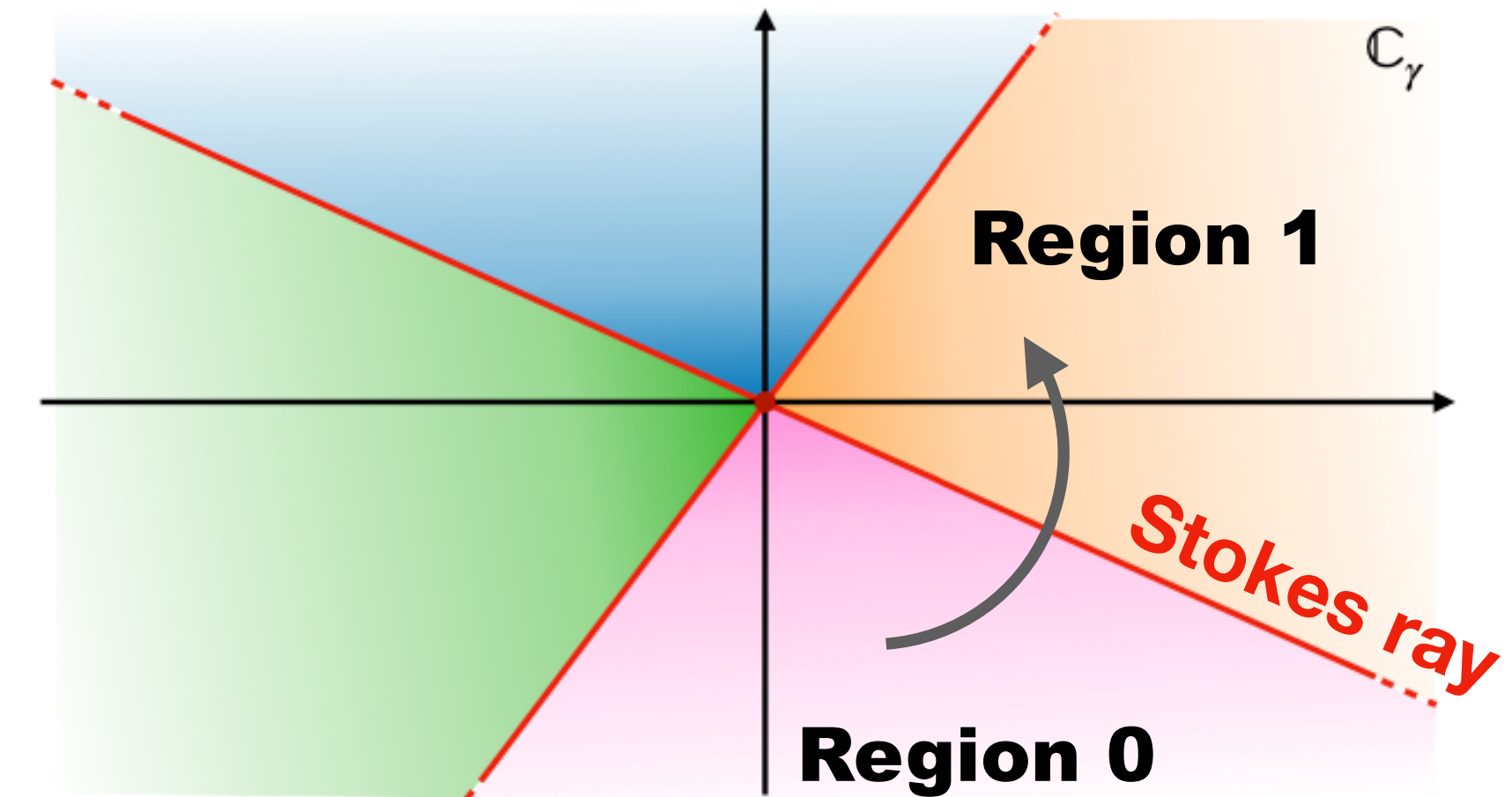
- ▶ Stokes' rays divide the plane $\mathbb{C}_\gamma^*$ in different sectors.
- ▶ Study how the basis of thimbles change when we cross these rays = **WCS**.



Wall Crossing Structure (WCS)

► Stokes' rays divide the plane $\mathbb{C}_\gamma^*$ in different **sectors**.

► Study how the basis of thimbles change when we cross these rays = **WCS**.



► To cross a Stokes ray corresponding to a Stokes line in the $\mathbb{C}_t$ plane connecting the critical values t_i and t_j imposes a **discontinuity jump** for the corresponding thimbles Γ_i and Γ_j described by matrix of the form:

$$\begin{pmatrix} \Gamma_i^{+(1)} \\ \Gamma_j^{+(1)} \end{pmatrix} = \begin{pmatrix} 1 & \Delta_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_i^{+(0)} \\ \Gamma_j^{+(0)} \end{pmatrix} \rightarrow \text{Intersection numbers among the corresponding vanishing cycles}$$

$$\Delta_{ij} = (\pm 1) \Delta_i \circ \Delta_j.$$

IMPORTANT!

Local monodromies around the critical points completely determine these numbers via Picard-Lefschetz theorem:

$$M_j(\Delta_i) = \Delta_i + (-1)^{n(n+1)/2} (\Delta_i \circ \Delta_j) \Delta_j$$

Master Integrals evaluation

Evaluate the integrals along the thimbles in the different sectors and determine which ones are dominant.

Fixed γ  $\mathbf{I} = \begin{pmatrix} I_1(\gamma) \\ I_2(\gamma) \\ \vdots \\ I_N(\gamma) \end{pmatrix}$ with $I_i(\gamma) = \int_{th_{i,\gamma}} e^{-\gamma f} \mu = e^{-\gamma t_i} \int_{th_{i,\gamma}} e^{-\gamma(f-t_i)} \mu$

For $\gamma \mapsto \infty$: $I_i(\gamma) = e^{-\gamma t_i} \sum_{\lambda} c_{i,\lambda} \gamma^{-\lambda-1}$  We need to compute these coefficients

Put $s = f(\mathbf{z}) - t_i$ and write the thimble as $th_{i,\gamma} = \bigcup_{s \geq 0} \Delta_i(s)$

Using the Gelfand-Leray map, the exponential integral becomes

$$I_i(\gamma) = e^{-\gamma t_i} \int_0^\infty ds e^{-\gamma s} vol_{\Delta_i}(s) \quad \text{where} \quad vol_{\Delta_i}(s) = \int_{\Delta_i(s)} \frac{\mu}{ds} \Big|_{\Delta_i(s)}$$

*Volume of the (n-1)-dim
vanishing cycle*



Master Integrals evaluation

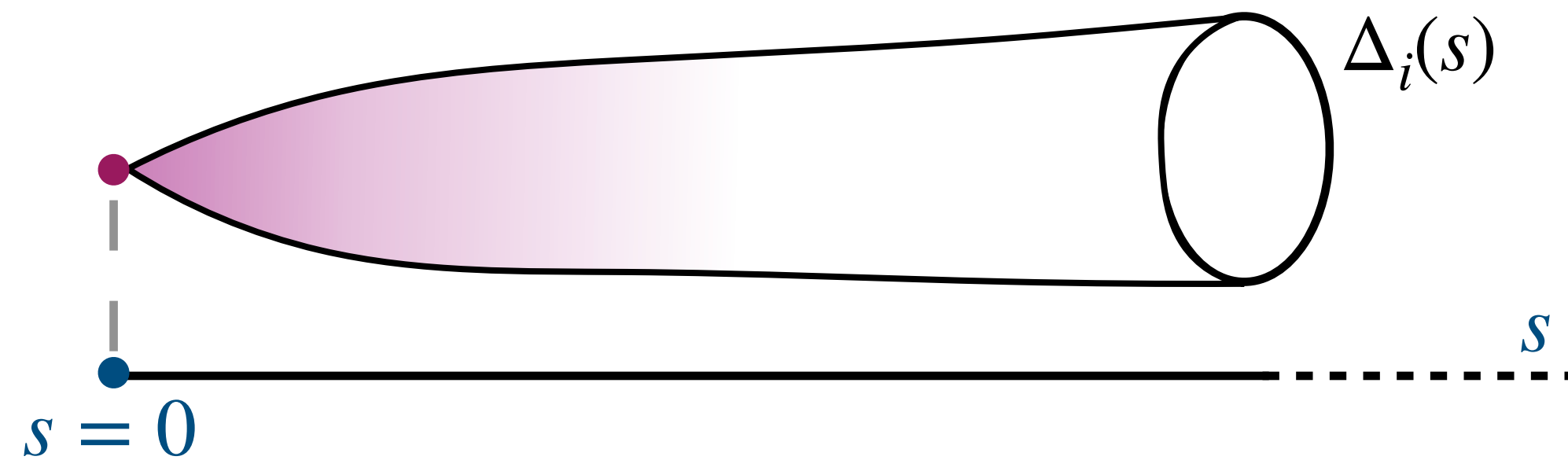
$$vol_{\Delta_i}(s) = \int_{\Delta_i(s)} \frac{\mu}{ds} \Big|_{\Delta_i(s)}$$

Power series expansion

$$vol_{\Delta_i}(s) = \sum_{\lambda} \sum_{0 \leq k \leq k_{max}} a_{\lambda,k} s^{\lambda} (\log s)^k$$

λ : Eigenvalue $e^{2\pi i \lambda}$ of the monodromy operator acting on $H_{n-1}(f^{-1}(s), \mathbb{Z})$ when we turn around $s = 0$.

$k_{max} + 1 =$ Dimension of the Jordan block



Substituting the expansion into the exponential integral:

$$I_i(\gamma) = e^{-\gamma t_i} \sum_{\lambda} \sum_k a_{\lambda,k} \int_0^{\infty} ds e^{-\gamma s} s^{\lambda} (\log s)^k = e^{-\gamma t_i} \sum_{\lambda} \sum_k a_{\lambda,k} \frac{d^k}{d\lambda^k} [\gamma^{-(\lambda+1)} \Gamma(\lambda + 1)]$$

Comparing with the previous expansion:

$$c_{i,\lambda} = \frac{1}{\gamma^{-(\lambda+1)}} \sum_k a_{\lambda,k} \frac{d^k}{d\lambda^k} [\gamma^{-(\lambda+1)} \Gamma(\lambda + 1)] .$$

Closed form for the coefficients !

Holomorphic functions

Holomorphic function

In the simple case in which the function $f(\mathbf{z}) : X \mapsto \mathbb{C}$ is **holomorphic** the previous analysis connects with the study of **Lefschetz thimbles** in (a complexified version of) **Morse theory**.

$$I = \int_{\Gamma} e^{-\gamma f} \mu$$

- $X = \mathbb{C}^n$

- $h = \text{Re}(\gamma f) \longrightarrow$ **Morse function**

- $\Gamma_i \longrightarrow$ **Lefschetz thimbles:**

Solutions of the gradient flow equations:

$$\frac{du^i}{d\tau} = g^{ij} \frac{\partial h}{\partial u^j}$$

Assure convergence of the integral along the paths Γ_i

- $\text{Im}(\gamma f) = \text{const}$
- Γ_i passes through the critical point σ_i
- h monotonically increases along Γ_i

Holomorphic Morse theory

Let us consider the case $X = \mathbb{C}$, and

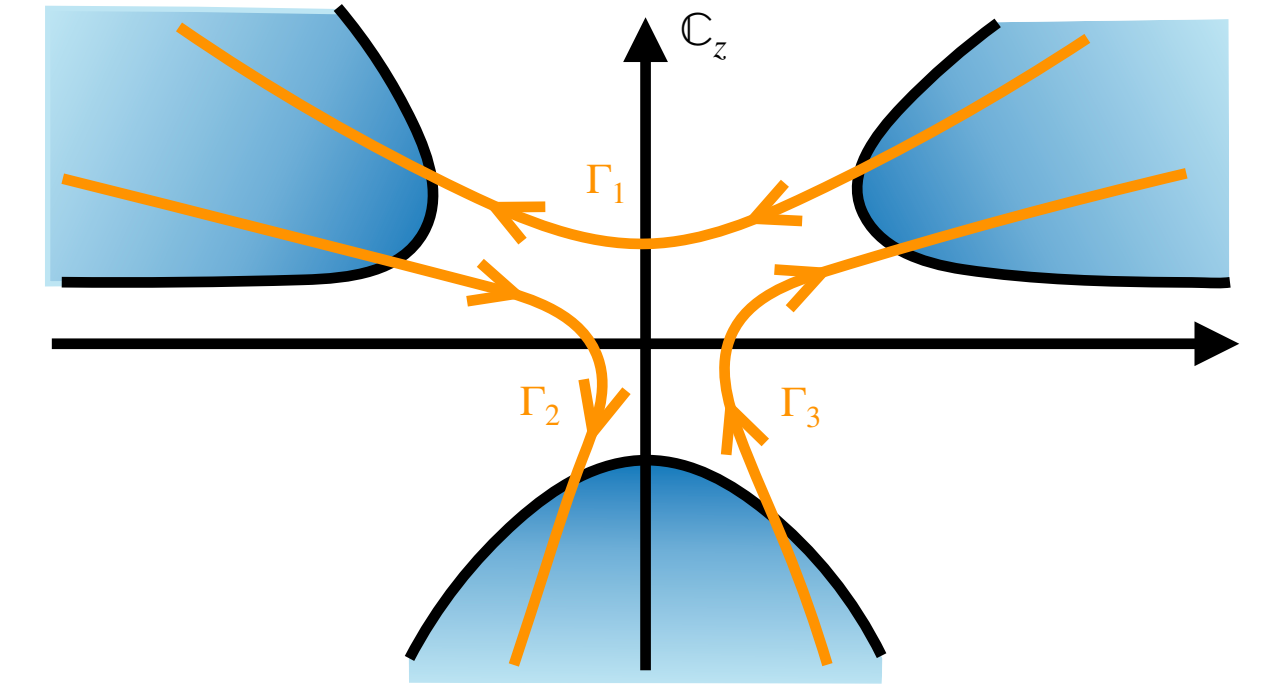
$$I(\gamma) = \int_{\Gamma} e^{-Re(\gamma f(\mathbf{z})) - iIm(\gamma f(\mathbf{z}))} g(\mathbf{z}) d\mathbf{z}$$

Any reasonable cycle Γ that assure convergence:

- $\Gamma \in H_1(\mathbb{C}, D_N, \mathbb{Z})$ where $D_N = \{\mathbf{z} \in \mathbb{C} \mid Re(\gamma f(\mathbf{z})) \leq N, N > > 1\}$
- $Im(\gamma f) \Big|_{\Gamma} = const$

Morse theory applied to this setup, with Morse function $h = Re(\gamma f)$ over the non-compact space $\mathbb{C}$ gives us the information to construct the relative homology

- Betti inequality $rank(rank(H_1(\mathbb{C}, D_N, \mathbb{Z}))) \leq \#crit.points$ (saturated in the case of saddles)
- Gradient flow equations



Example: Pearcey's Integral

$$P(\gamma) = \int_{\Gamma} e^{-\gamma(z^4 + bz^2 + cz + d)} dz$$

Describing the Grand-canonical **partition function** of gauge Skyrme models for nuclear matter.

[Cacciatori, Canfora, Lagos, Muscolino, Vera - '21]

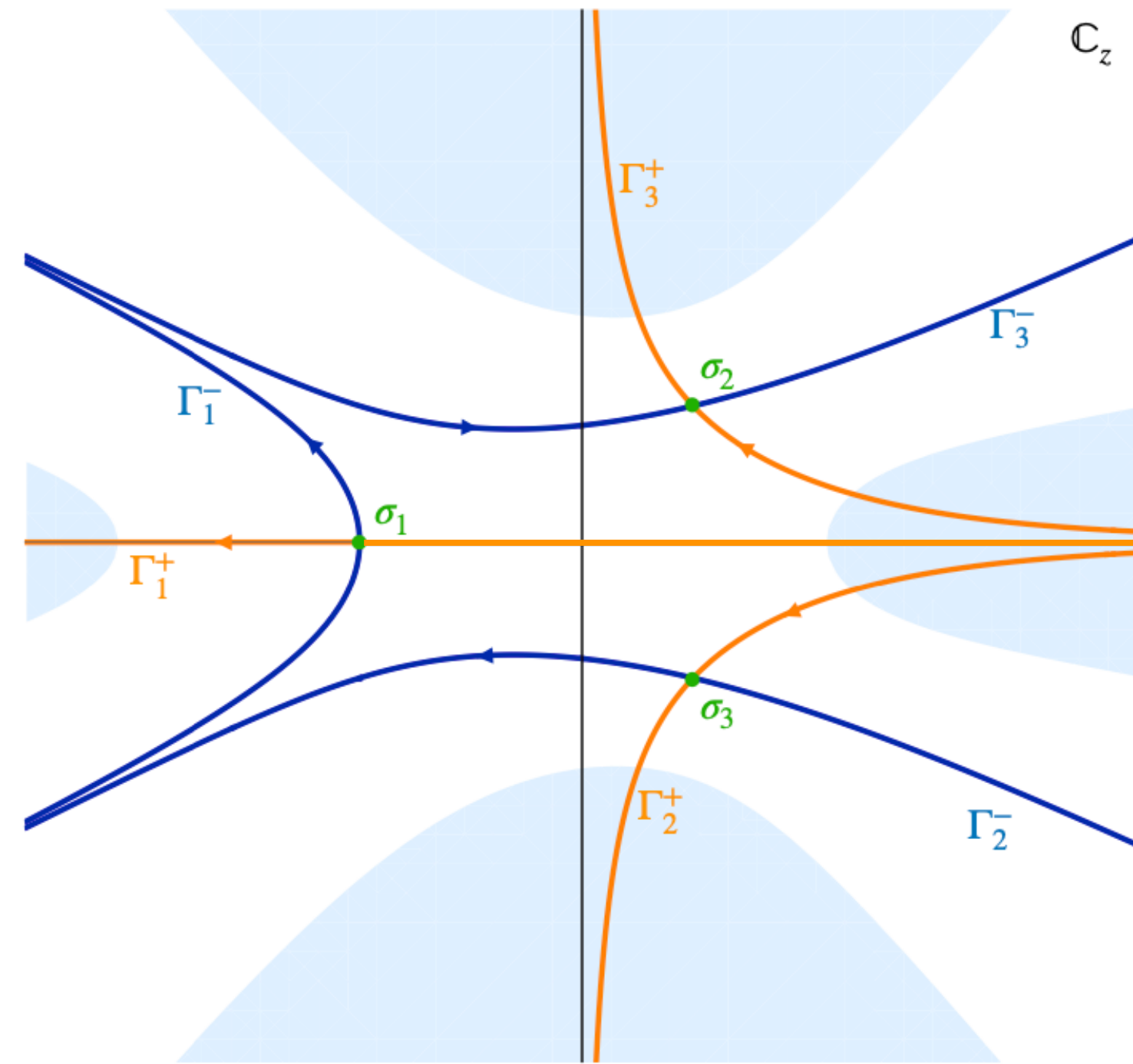
- The algebraic variety is $X = \mathbb{C}$.
- The critical points of f define the set: $\Sigma = \{z \in \mathbb{C} \mid f'(z) = 4z^3 + 2bz + c = 0\}$

According with the sign of the discriminant $\Delta = 8b^3 + 27c^2$ we have three different situations:

$$\Delta \equiv \begin{cases} > 0 & 1 \text{ real and 2 complex conjugate solutions,} \\ < 0 & 3 \text{ real different solutions,} \\ = 0 & 3 \text{ real solutions with at least a multiple root.} \end{cases}$$

Three **different regions** in the **parameter space** (a, b) with different **Betti homologies** and different **WCSs**.

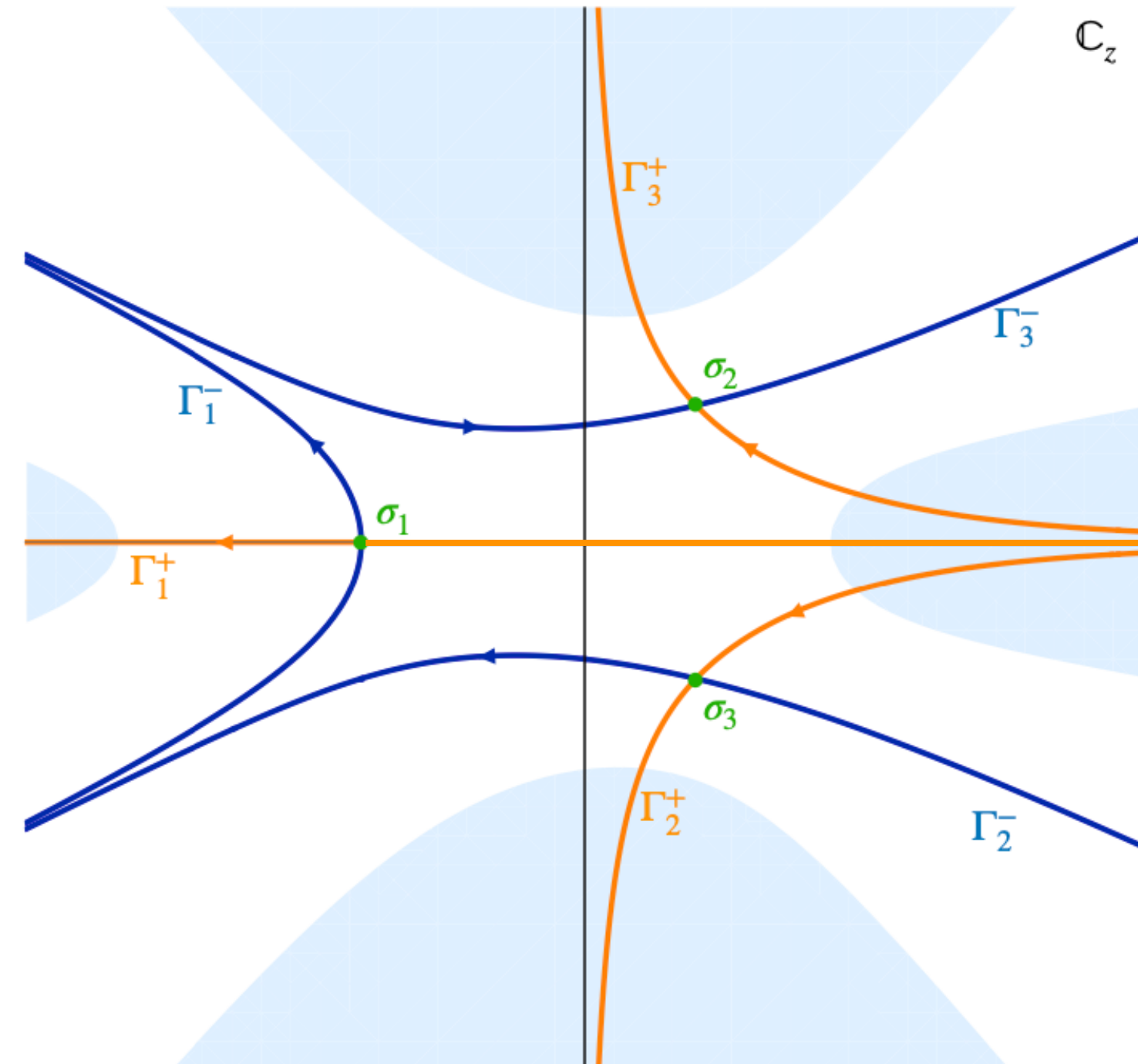
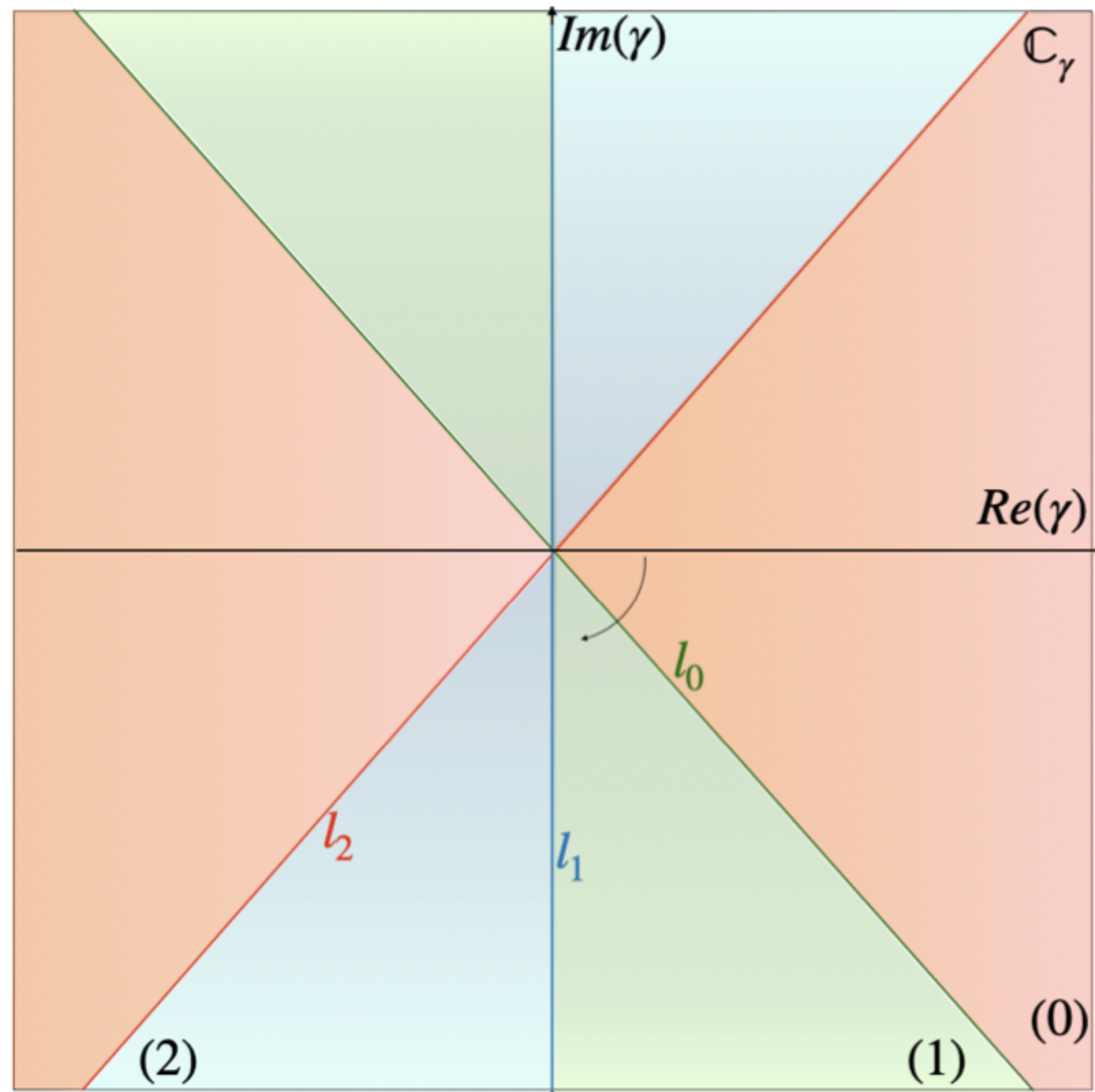
Region $\Delta > 0$



- 3 distinct critical points in $X = \mathbb{C}$
- 3 distinct thimbles $\{\Gamma_i^+\}_{i=1}^3$ for the Betti homology $H_n^{Betti,\gamma}(\mathbb{C}, f, \mathbb{Z})$ and 3 distinct thimbles for the dual homology

$$\dim H_1^{Betti}(\mathbb{C}, f, \mathbb{Z}) = 3$$

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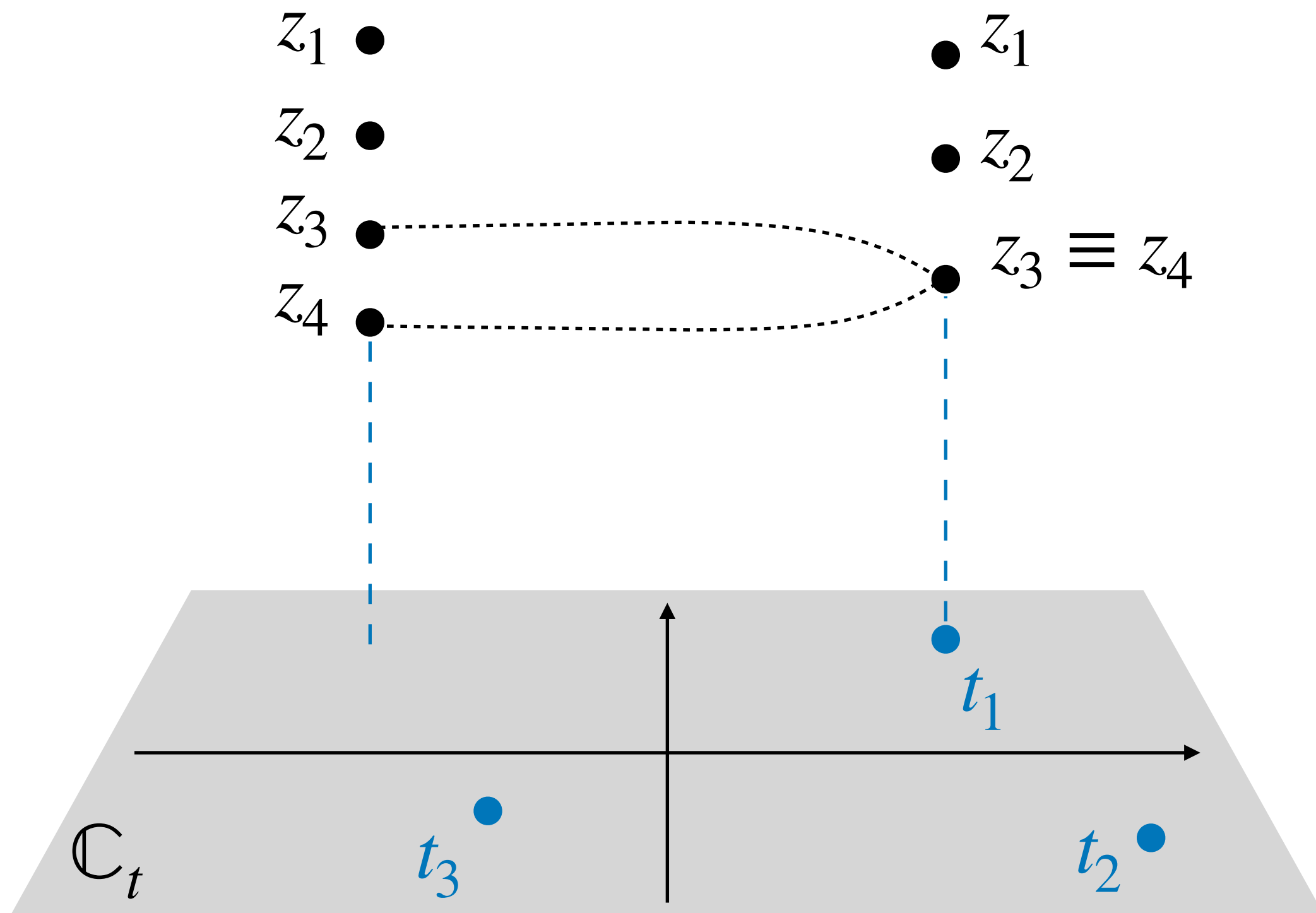
$$\begin{aligned} l_0 : Re(\gamma) &= -\frac{11}{16}\sqrt{\frac{3}{2}}Im(\gamma), & \text{where } Im(\gamma f(z))|_{\sigma_1} &= Im(\gamma f(z))|_{\sigma_2}, \\ l_1 : Re(\gamma) &= 0, & \text{where } Im(\gamma f(z))|_{\sigma_2} &= Im(\gamma f(z))|_{\sigma_3}, \\ l_2 : Re(\gamma) &= \frac{11}{16}\sqrt{\frac{3}{2}}Im(\gamma), & \text{where } Im(\gamma f(z))|_{\sigma_1} &= Im(\gamma f(z))|_{\sigma_3}. \end{aligned}$$

Stokes' rays separating the regions (0), (1) and (2) in the plane $\mathbb{C}_\gamma$



Region $\Delta > 0$

- Fix $\gamma \notin l_i$
- **3 critical points** in Σ corresponding to the **3 critical values** $f(\sigma_1) = t_1, f(\sigma_2) = t_2$ and $f(\sigma_3) = t_3$
- Construct the **fibration** $f: \mathbb{C} \mapsto \mathbb{C}_t$, with **fibers** are the four points $f^{-1}(t) = \{z_1(t), z_2(t), z_3(t), z_4(t)\}$



- Construct the **vanishing cycles**

$$\Delta_1 = \{z_3\} - \{z_4\} \quad \text{and} \quad \Delta_2 = \Delta_3 = \{z_1\} - \{z_4\}$$

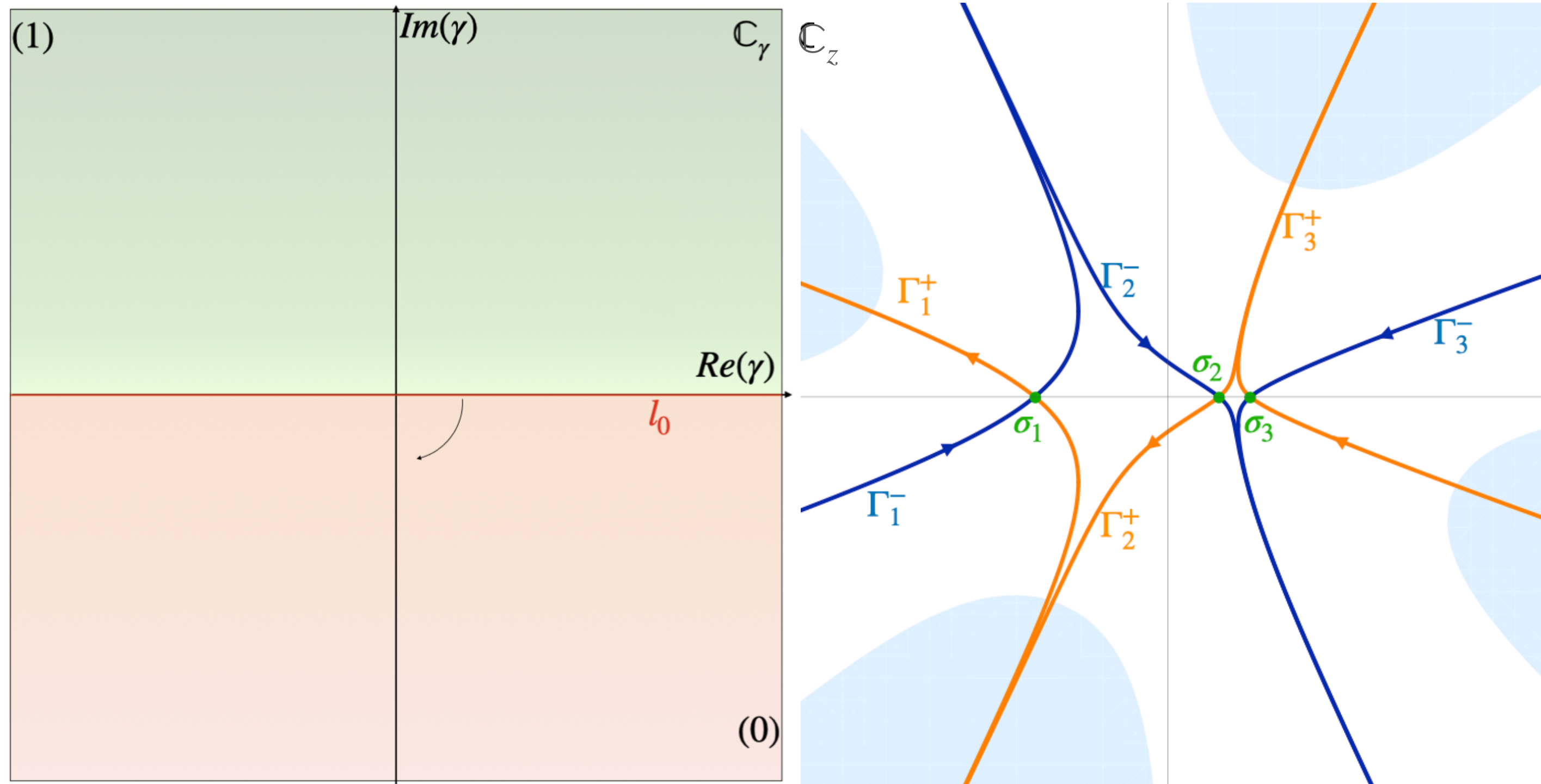
- Compute the **monodromies** around the critical values

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = M_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- Compute the **jump matrices** for the WCS

$$T^{(0)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Region $\Delta < 0$



► $\dim H_1^{\text{Betti}}(\mathbb{C}, f, \mathbb{Z}) = 3$

► 3 vanishing cycles

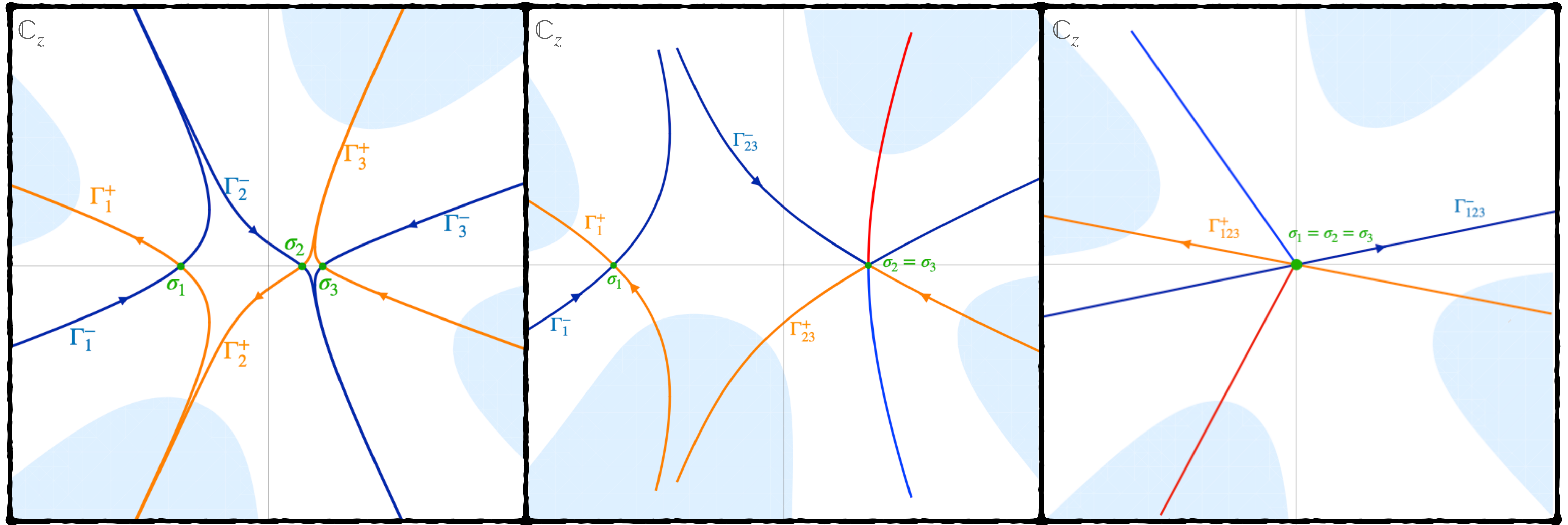
$$\begin{aligned}\Delta_1 &= \{z_1\} - \{z_2\}, \\ \Delta_2 &= \{z_3\} - \{z_4\}, \\ \Delta_3 &= \{z_1\} - \{z_4\}\end{aligned}$$

► Monodromies

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ and } M_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

► 1 Stokes' ray: study the corresponding WCS.

Region $\Delta = 0$



$$H_1(X, D_N, \mathbb{Z}) = \text{span}\{\Gamma_1^+, \Gamma_{23}^+\} \cong \mathbb{Z}^2$$

$$H_1(X, D_N, \mathbb{Z}) = \text{span}\{\Gamma_{123}^+\} \cong \mathbb{Z}$$

$$H_1(X, D_N, \mathbb{Z})^\vee = \text{span}\{\Gamma_1^-, \Gamma_{23}^-\} \cong \mathbb{Z}^2$$

$$H_1(X, D_N, \mathbb{Z})^\vee = \text{span}\{\Gamma_{123}^-\} \cong \mathbb{Z}$$

Multivalued functions

Multivalued function

Let us consider the case in which $f : X \mapsto \mathbb{C}$ is a multivalued function.

→ The main constructions underlying the analysis still work but we need some modifications!

MAIN POTENTIAL PROBLEM

$$H_{dR}^\bullet(X, D_0, f) \cong \mathbb{H}^\bullet(X, \Omega_{X, D_0}^\bullet, \nabla_f) \quad \dashrightarrow \quad \nabla_f = d + \alpha \wedge$$

Additional contribution to the 1-form α defining the **twist** in the de Rham cohomology

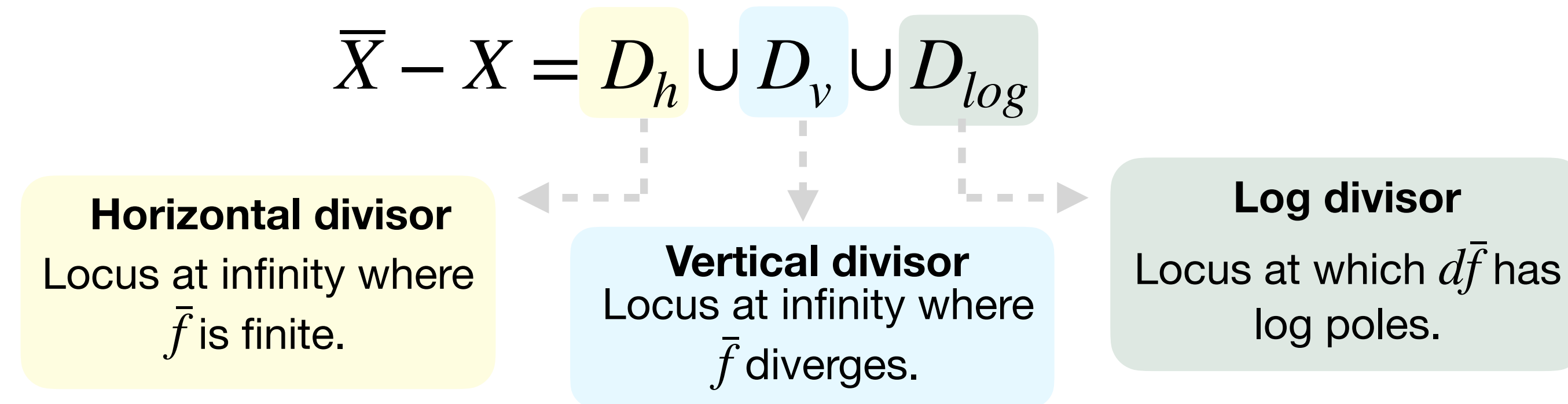
- Holomorphic case: $\alpha = df$ (**exact form**)
- Multivalued case: $\alpha = \alpha_{reg} + \alpha_{log} + \alpha_\infty$ (**closed form**, not necessary exact)



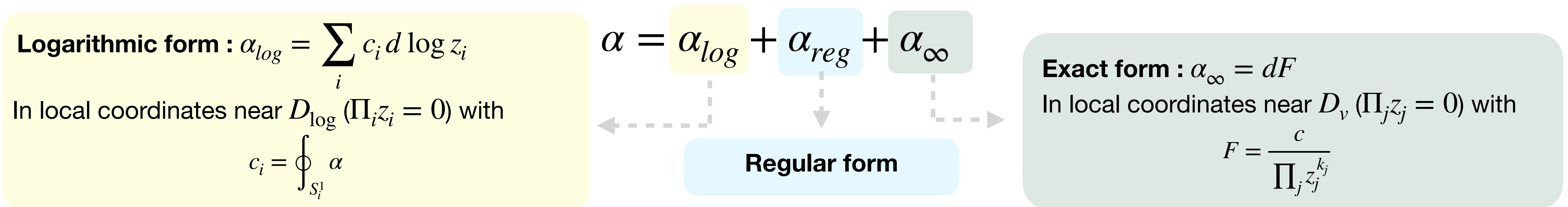
Geometric Manipulations

Choice of a “good” compactification

► Choose a suitable compactification for $X \mapsto \bar{X}$ s.t.:



- (i) Among all the normal crossing divisors, only D_v and D_{log} can have common irreducible components;
- (ii) For any $\mathbf{z} \in \bar{X}$ there exists a small neighbourhood U_z in which the closed meromorphic 1-form α can be locally expressed as:



► Study the behavior of $\bar{f}: \bar{X} \mapsto \mathbb{C}$ along the divisors.

Logarithmic exponent

$$I = \int_{\Gamma} \frac{dx \wedge dy}{[y^2 - x(x-1)(x-\lambda)]^\gamma} = \int_{\Gamma} e^{-\gamma \log[y^2 - x(x-1)(x-\lambda)]} dx \wedge dy = \int_{\Gamma} e^{-\gamma \log \mathcal{B}(x,y;\lambda)} dx \wedge dy$$

- $X = \mathbb{C}^2 \setminus \{\mathcal{B} = 0\}$
- $\bar{X} = \mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1 = X \cup \{\mathcal{B} = 0\} \cup \mathbb{P}^1$
- We study the closed form:
$$d \log \overline{\mathcal{B}} = \frac{2\eta y dy + [y^2 + x^2 + x\lambda(x-2\eta)]d\eta + [-3x^2 - \eta^2\lambda + 2x\eta(1+\lambda)]dx}{y^2\eta - x(x-\eta)(x-\eta\lambda)}$$
- $D_h = D_v = \emptyset$
- $D_{\log} = D_{\overline{\mathcal{B}}} \cup D_{\infty}$ with $D_{\overline{\mathcal{B}}} = \overline{\mathcal{E}}_{\lambda} = \{[x : y : \eta] \in \mathbb{P}^2 \mid \overline{\mathcal{B}} = 0\}$

$$D_{\infty} = \mathbb{P}^1 = \{[x : y : 0] \in \mathbb{P}^2\}$$

$$\alpha = d \log \mathcal{B}$$

Regime $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$

► Critical points

$$\mathcal{L}(\alpha) = \{(x_i, y_i) \mid d \log \mathcal{B} = 0\} = \left\{ \left(\frac{1}{3} \left[1 + \lambda - \sqrt{1 + \lambda(\lambda - 1)} \right], 0 \right), \left(\frac{1}{3} \left[1 + \lambda + \sqrt{1 + \lambda(\lambda - 1)} \right], 0 \right) \right\}$$

with critical values

$$S = \{T_i = \log \mathcal{B}(x_i, y_i; \lambda)\} = \{T_1 = \log t_1, T_2 = \log t_2\}$$

► We construct the double fibration

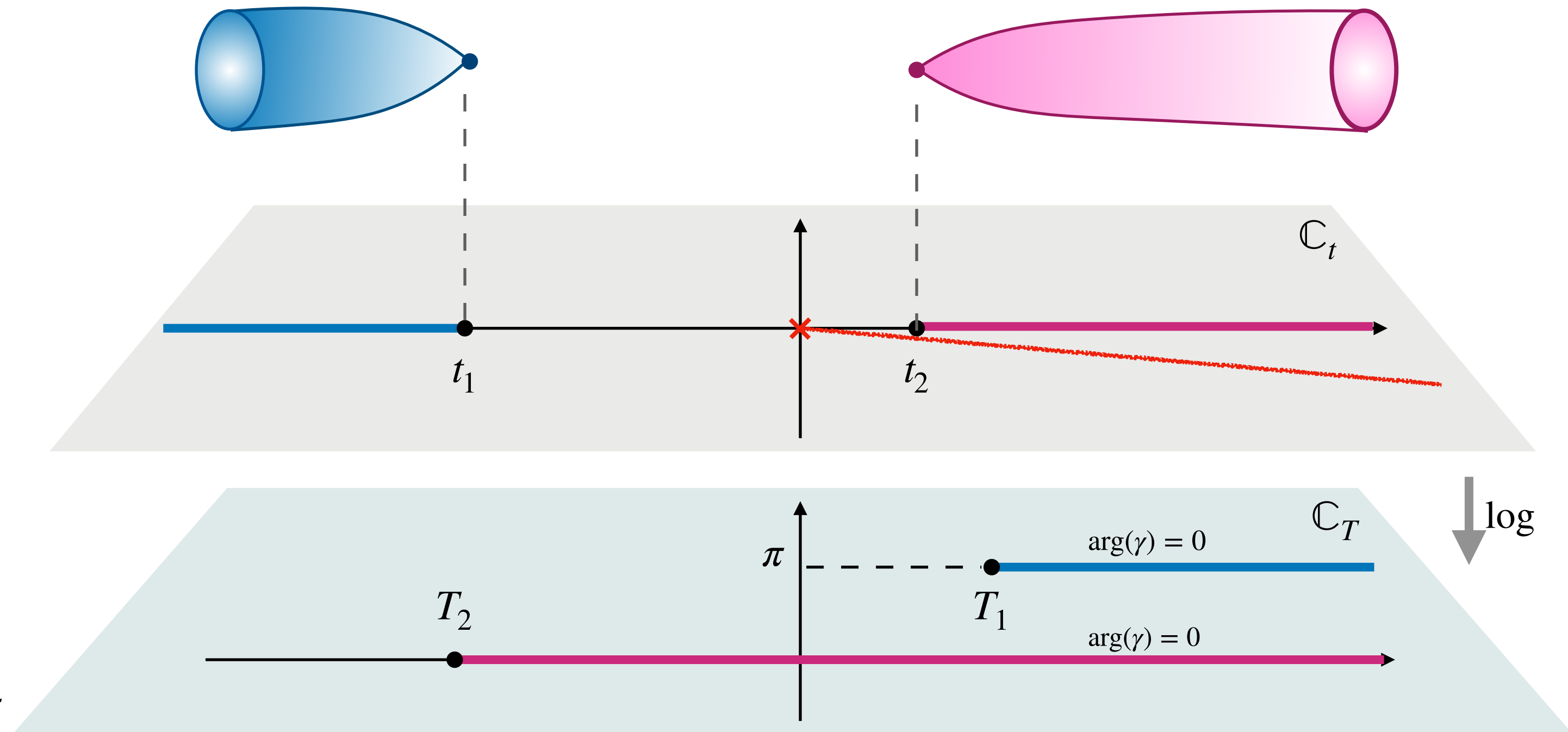
► We determine the vanishing cycles using Picard-Lefschetz theory

► For fixed γ we construct the thimbles tracing the vanishing cycles along the direction $\theta = \arg \gamma$

► Stokes' jumps

$$\theta_1 = \arg(\gamma_1) = \arctan \left(\frac{\pi}{\log |t_1|} + \log |t_2| \right) \sim 0.470933\pi$$

$$\theta_2 = \arg(\gamma_2) = \arctan \left(\frac{\pi}{\log |t_1|} + \log |t_2| \right) + \pi \sim 1.470933\pi$$



Regime $\lambda = 0$

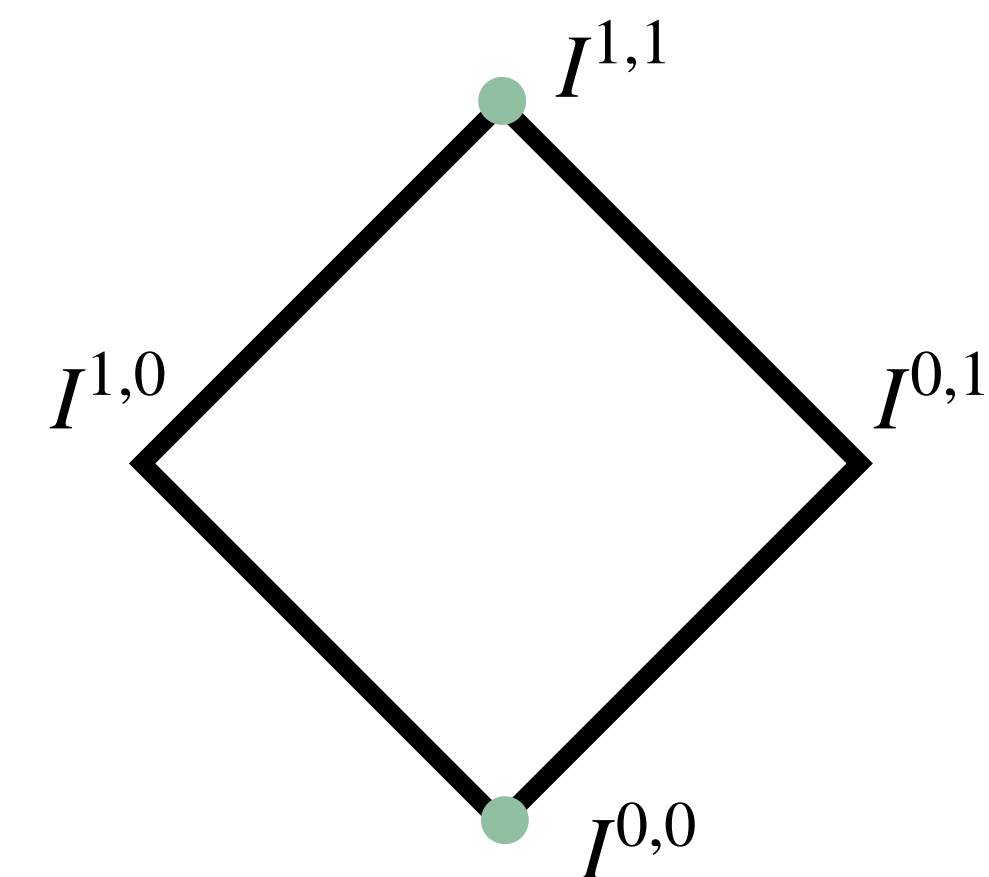
(Unpublished)

- The curve $\mathcal{E}_0 : y^2 - x^2(x - 1) = 0$ in the family $\mathcal{E}_\lambda$ of Legendre curves is singular.
- To understand the type of singularity we can study the monodromy in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ around $\lambda = 0$

$$M_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{---} \text{---} \text{---} \text{---} \text{---} \rightarrow N_0 = \log M_0 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

This contains the information about the Hodge-Deligne splitting that replaces the Hodge decomposition as we approach the locus $\lambda = 0$

$$H^1(\mathcal{E}_0) = \underbrace{H^{1,0}(\mathcal{E}_0)}_{\dim H^{1,0} = 1} \oplus \underbrace{H^{0,1}(\mathcal{E}_0)}_{\dim H^{0,1} = 1} \xrightarrow{\lambda \mapsto 0}$$



Type II

Regime $\lambda = 0$

(Unpublished)

► If we continue using the twist $\alpha = d \log \mathcal{B}(x, y; 0)$, we obtain a single non-degenerate critical point at

$$(x_1, y_1) = \left(\frac{2}{3}, 0\right)$$

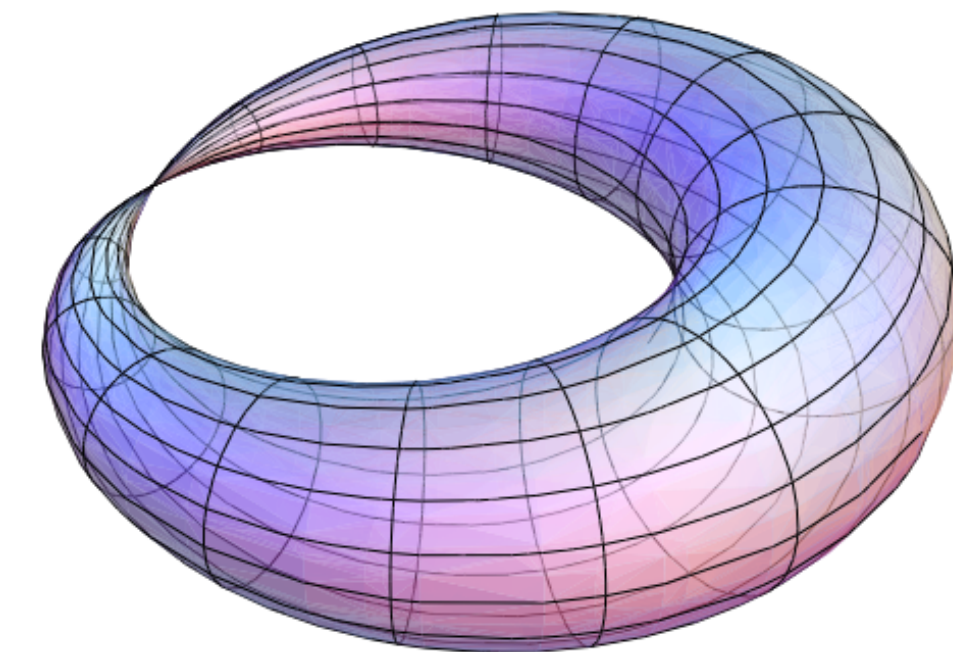
Rank reduction : $2 \mapsto 1$?

➡ **One single thimble:**

No Stokes' phenomena ?

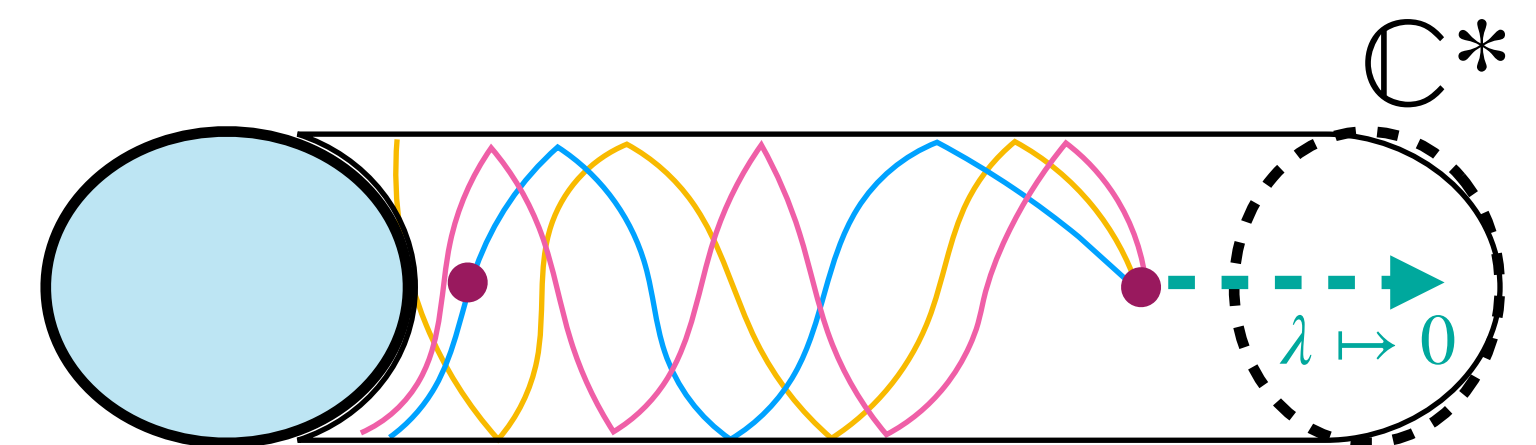
► The main problem is that the locus $\mathcal{B} = 0$, when $\lambda = 0$, is singular and now the form $\alpha = d \log \mathcal{B}$ is **not globally well defined** in $\bar{X}$.

We obtain an indeterminate form at $(0,0)$.



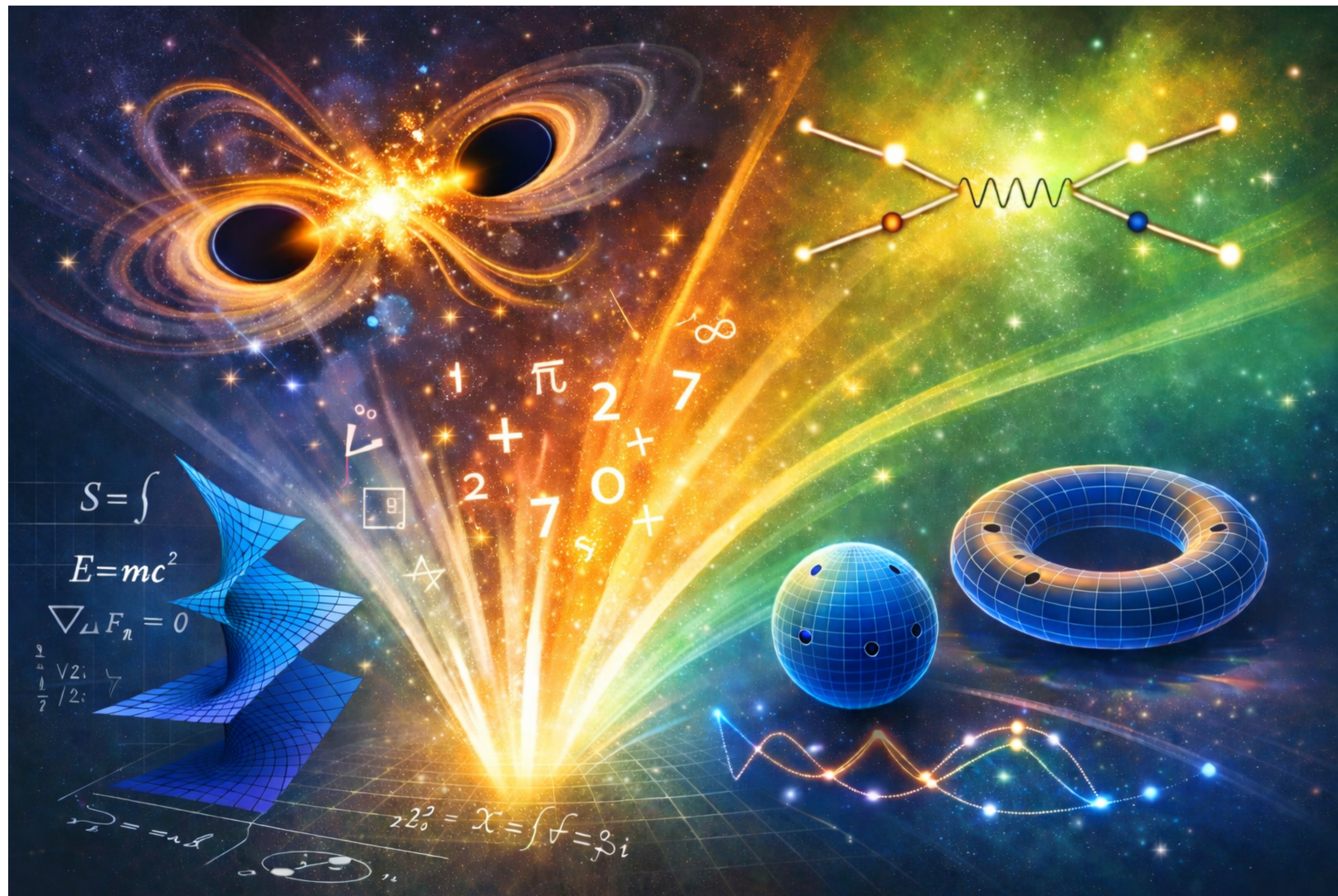
- *Blow up*: No new critical points!

- *Approaching parametrically the limit $\lambda \mapsto 0$* : Accumulation of Stokes lines leading to thimble degeneration.



Conclusions and future directions

- **Exponential integrals** provide a well defined pairing between **twisted de Rham** co-cycles and **Betti cycles** over complex manifolds that allow to accomodate in the same framework a **wide range of physically relevant integrals**.
- The **WCS** analysis allows for an **analytic continuation** of the MIs decomposition in the parameter γ to study Stokes' phenomena to assure a **sharp counting** of the co-homology dimension.
- ➔ To test the analysis in Feynman integrals in different representations;
- ➔ To study in detail the dependence of the pairing on the kinematic parameters ($\sim$ complex structure moduli for vanishing cycles);
- ➔ To performe explicit computations for families of higher *CYs* (beyond elliptic curves): $K3, CY_3, CY_4, \dots$;
- ➔ To use the method to analyze string amplitudes;
- ➔ ...



20-22 of April 2026

Organizers:

Roberta Angius

(University of Hamburg)

Anthony Massidda

(University of Insubria)

Matteo Pegorin

(University of Padova)

Registrations will open **soon** in:

<https://www.mitp.uni-mainz.de/mitp-events-2026/>

Loops & Geometry:

Hidden Structures in Multiloop Amplitudes



Stay tuned!

Thank you for your attention!