Proper elements and effective stability in Celestial Mechanics

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Outline

1. Introduction
2. Proper elements and space debris dynamics
3. Effective stability and satellite dynamics
4. Effective estimates in rotational dynamics

Outline

Problems and Methods of Celestial Mechanics

• PROBLEMS OF CELESTIAL MECHANICS
• Combination of orbital and rotational motions around a (not spherical body, not rigid) central body
• Different stability times: tens of years (artificial satellites) to millions of years (planets)
• Several degrees of freedom with variables varying on different time scales (Earth: rotates in 1 day, orbits around the Sun in 1 year, Earth's precession occurs on thousands years)
• Conservative and dissipative problems: if present, typically the dissipation is small $\rightarrow$ weakly dissipative systems

• MATHEMATICAL METHODS
• Perturbation theory: proper elements, which are quasi-integrals characterizing the dynamics, application to the space debris problem.
• Nekhoroshev's theorem: effective stability estimates, application to satellite and rotational dynamics.

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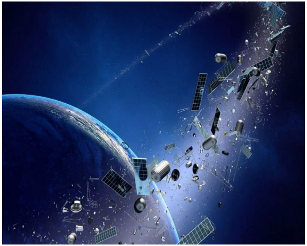
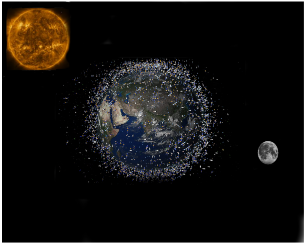
Space debris





- Satellite ESA numbers:

▷ 6340 rocket launches since 1957

▷ 14710 satellites placed into Earth orbit

▷ 9780 satellites still in space

- Space debris ESA numbers:

▷ 54000 > 10 cm

▷ 1.2 millions 1-10 cm

▷ 140 millions 1 mm - 1 cm.

Mathematical tools:

1. Perturbative methods: to compute approximate solutions.
2. Proper elements: which are quantities (nearly) constant over time.
3. Statistical techniques: to analyze the results.
4. Machine Learning: to cluster and classify the fragments.

Results mainly in collaboration with Tudor Vartolomei, Univ. Iasi (Romania)

1. dynamics and stability of the fragments generated by a break-up event;
2. cluster and classify space debris through proper elements (Hirayama 1918 for asteroid families and later Brouwer, Kozai, Milani, Knezevich, etc.).

Hamiltonian model

• Conservative geolunisolar model above 2000 km of altitude (no atmosphere!), including Earth (with a non-spherical shape), Sun and Moon (treated as third-body perturbations), SRP.

• Action–angle Delaunay variables:

▷ Actions L, G, H, related to the orbital elements a, e, i by

$$L = \sqrt{\mathcal{G}\, m_E a}\ , \qquad G = L\sqrt{1-e^2}\ , \qquad H = G\cos i\ .$$

▷ Angles M, ω, Ω = mean anomaly, argument of perigee, longitude of the ascending node.

• Hamiltonian:

$$\mathcal{H} = -\frac{(\mathcal{G}\, m_E)^2}{2L^2} + \mathcal{H}_{Earth}(L, G, H, M, \omega, \Omega, \theta) + \mathcal{H}_{Moon}(L, G, H, M, \omega, \Omega, \Lambda_M)$$
$$+\mathcal{H}_{Sun}(L, G, H, M, \omega, \Omega, \Lambda_S) + \mathcal{H}_{SRP}(L, G, H, M, \omega, \Omega, \Lambda_S)$$

with θ = sidereal time, Λ_M, Λ_S = orbital elements of Moon and Sun.

Classification of the arguments in the disturbing functions

• The Fourier expansions of $\mathcal{H}_{Earth}$, $\mathcal{H}_{Sun/SRP}$, $\mathcal{H}_{Moon}$ contain an infinite number of terms of the form (k_j integers):

$$\mathcal{A}^{Earth}_{k_1 k_2 k_3 k_4}(L, G, H)\cos(k_1 M + k_2\theta + k_3\omega + k_4\Omega),$$

$$\mathcal{A}^{Sun/SRP}_{k_1 k_2 k_3 k_4 k_5 k_6}(L, G, H, L_S, G_S, H_S)\cos(k_1 M + k_2 M_S + k_3\omega + k_4\omega_S + k_5\Omega + k_6\Omega_S),$$

$$\mathcal{A}^{Moon}_{k_1 k_2 k_3 k_4 k_5 k_6}(L, G, H, L_M, G_M, H_M)\cos(k_1 M + k_2 M_M + k_3\omega + k_4\omega_M + k_5\Omega + k_6\Omega_M)$$

• The angles involved in the above combinations may be classified as follows:
– fast angles: M and θ (periods of days);
– semi–fast angles: M_M and M_S (periods of 1 month and 1 year);
– slow angles: ω, Ω, ω_M, Ω_M, ω_S, Ω_S (periods of years).

Terms and resonances

- Classification of the terms of the expansions:

– *short periodic terms*: involving the fast angles (M, θ) with $\dot{M}$ and $\dot{\theta}$ not commensurable;
– *resonant (tesseral) terms*: involving the fast angles (M, θ) with a commensurability between $\dot{M}$ and $\dot{\theta}$ (say, $k_1\dot{M} - k_2\dot{\theta} = 0$);
– *semi–secular terms*: depending on the semi-fast angles M_M, M_S;
– *secular terms*: depending on the slow angles ω, Ω, $\omega_{M/S}$, $\Omega_{M/S}$ $\Rightarrow$ long-term variations of e and i over tens (or hundreds) of years.

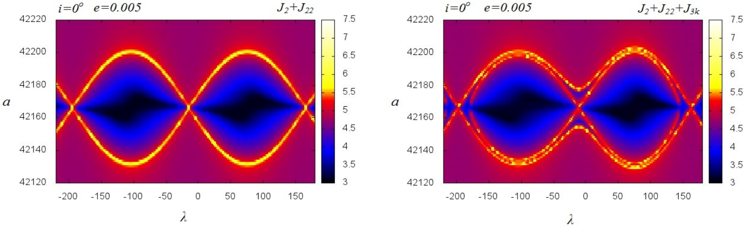


Figure: Around the geosynchronous resonance $\dot{\lambda} = \dot{M} - \dot{\theta} = 0$.

Proper elements

• Initial Hamiltonian:

$$\mathcal{H}_1(\underline{J},\underline{\varphi}) = h_1(\underline{J}) + \varepsilon R_1(\underline{J},\underline{\varphi}) \ .$$

• Under a non-resonance assumption on the frequency, implement a canonical change of variables through a Lie series transformation to normalize the Hamiltonian at order Q:

$$\mathcal{H}_Q(\underline{J}',\underline{\varphi}') \ = \ h_Q(\underline{J}') + \varepsilon^Q R_Q(\underline{J}',\underline{\varphi}') \ .$$

• Neglecting $O(\varepsilon^Q)$, the actions are integrals for

$$\mathcal{H}_Q(\underline{J}',\underline{\varphi}') = h_Q(\underline{J}') \quad \Rightarrow \quad \underline{\dot{J}}' = -\frac{\partial h_Q(\underline{J}')}{\partial \underline{\varphi}'} = \underline{0} \ ,$$

quasi-integrals for $\mathcal{H}_Q = h_Q + \varepsilon^K R_Q$.
• Proper elements (quasi-integrals) are back-transformed to the original variables through the generating function.

Osculating/Mean/Proper elements

Definition

Osculating orbital elements are obtained integrating the full Hamiltonian or Cartesian equations of motion.

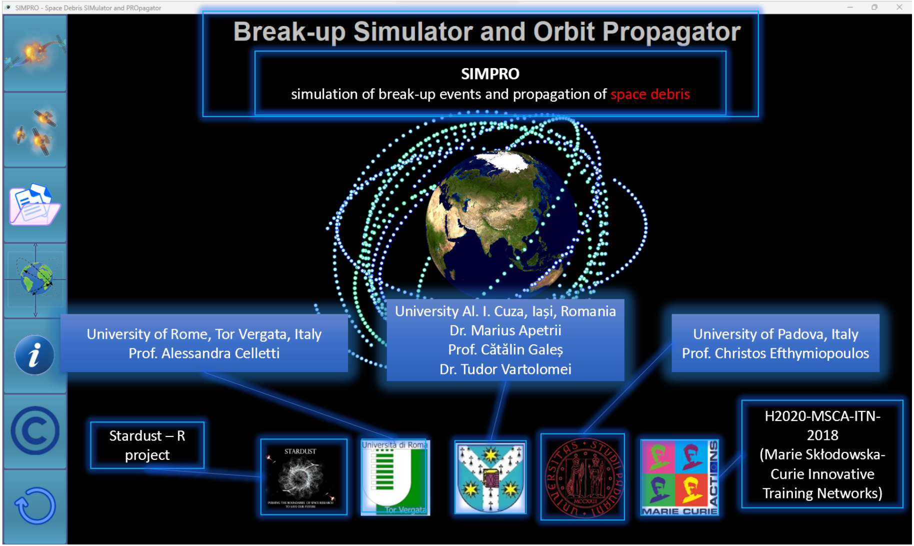
Definition

Mean orbital elements are the orbital elements obtained after averaging the Hamiltonian w.r.t. the short-period variables (mean anomaly of the debris and sidereal time).

Definition

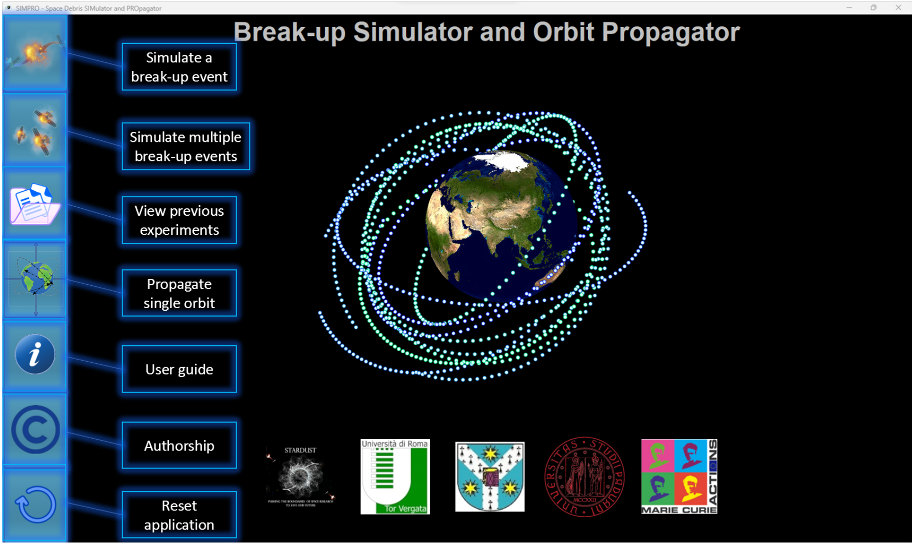
Proper elements are obtained averaging the Hamiltonian function w.r.t. fast, semi-fast and long-period variables to filter out short-period oscillations and isolate the secular (long-term) behaviour.

SIMPRO



SIMPRO, based on NASA EVOLVE 4.0, is freely available on
https://github.com/simproproject/simpro_app

SIMPRO



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Proper elements: three groups clustering

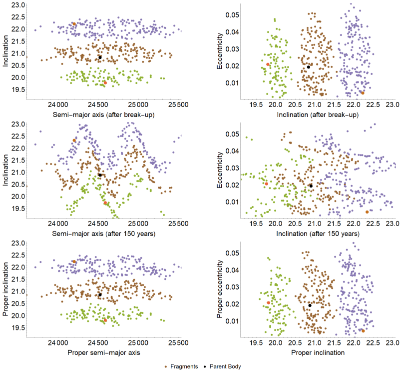
• Proper elements are of fundamental importance for clustering space debris: find a map from datapoints to categories, based on the distribution of points in space.

• Clusterization methods: `KMeans` (unsupervised ML method), `DBSCAN` (Density-based spatial clustering of applications with noise), `GaussianMixture` (models the probability density with a mixture of Gaussian distributions).

proper elements at the initial time

mean elements after 150 years

proper elements after 150 years



• 3 satellites, $a - i$ (left), $e - i$ (right) at $a = 24600$ km, $i = 20^o, 21^o, 22^o$.

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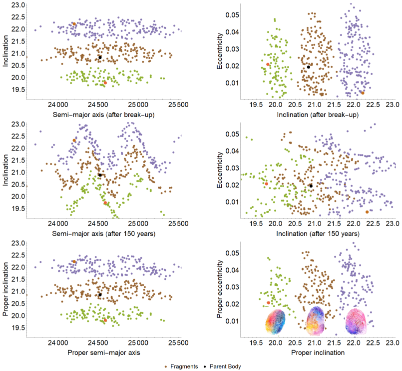

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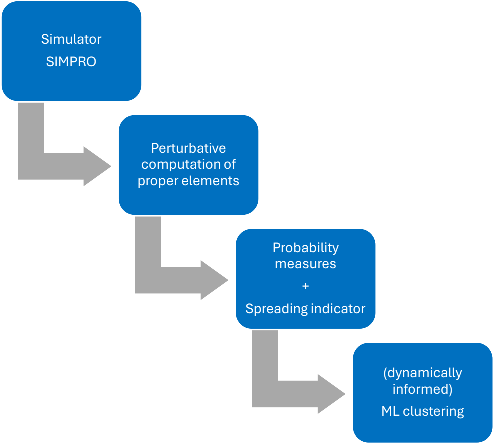
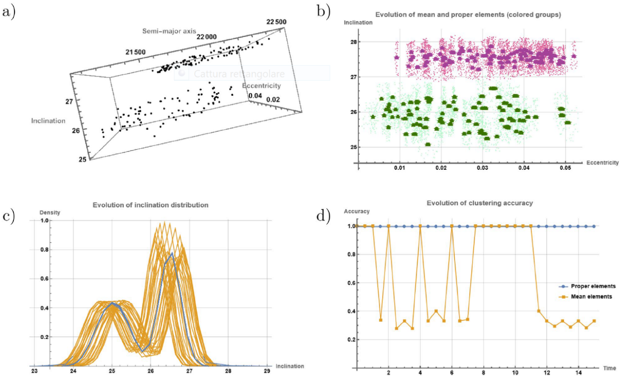
Work-flow

- Celletti A., Vartolomei T., "A dynamics based procedure for clustering and classifying space debris", Scientific Reports (2025)



PDFs and spreading indicator

▷ Spreading indicator introduced to measure the variation of mean and proper elements: information on the applicability of the clusterization methods.
▷ Compute the evolution over 40 years of the probability density functions (PDFs) of the (varying) mean (yellow) and (overlapping) proper (blue) inclinations. Proper PDFs: same shape over time ⇒ mixture of two normal distributions.



Dissipative proper elements [A.C.,Gales,Lhotka,Vartolomei]

• For objects below 2000 km of altitude, one needs to consider the dissipative effect due to the atmospheric drag:

$$\dot{a} = -\frac{1}{2\pi}\int_0^{2\pi} B\,\rho\,v\frac{a}{1-e^2}\left[1+e^2+2e\cos f-\omega_E\cos i\sqrt{\frac{a^3(1-e^2)^3}{\mu_E}}\right]dM$$

$$\dot{e} = -\frac{1}{2\pi}\int_0^{2\pi} B\,\rho\,v\left[e+\cos f-\frac{r^2\omega_E\cos i}{2\sqrt{\mu_E a(1-e)^2}}\Big(2(e+\cos f)-e\sin^2 f\Big)\right]dM$$

▷ First (direct) method: build a canonical transformation in terms of Lie series, averaging the original Hamiltonian over the angle variables.
▷ Second (patching) method: partition the total integration time into N subintervals and compute a distinct generating function for each sub-interval via the direct transformation.
▷ Third (Brouwer-Hori) method: based on the transformation on the Hamiltonian part and also on the dissipative term.
▷ Fourth (Kamel) method: transform the dissipative system into a Hamiltonian one by doubling the phase space variables and compute the proper elements by eliminating its periodic terms.

Brouwer-Hori method

Proposition

Consider the equations of motion:

$$\dot{\varphi}_i = \frac{\partial \mathcal{H}}{\partial J_i} + F_{\varphi,i} \ , \qquad \dot{J}_i = -\frac{\partial \mathcal{H}}{\partial \varphi_i} - F_{J,i} \ , \quad i = 1,2 \ ,$$

for a Hamiltonian function $\mathcal{H}$ and non–gravitational contributions $(F_{\varphi,i}, F_{J,i})$. Under a canonical change of variables $(\underline{\varphi}, \underline{J}) \to (\underline{\varphi}', \underline{J}')$ the transformed equations are

$$\dot{\varphi}'_i = \frac{\partial \mathcal{H}'}{\partial J'_i} + F'_{\varphi,i} \ , \qquad \dot{J}'_i = -\frac{\partial \mathcal{H}'}{\partial \varphi_i} - F'_{J,i} \ , \quad i = 1,2 \ ,$$

where

$$\begin{aligned} F'_{\varphi,i} &\equiv \underline{F}_\varphi \cdot \frac{\partial \underline{J}}{\partial J'_i} + \underline{F}_J \cdot \frac{\partial \underline{\varphi}}{\partial J'_i} \\ F'_{J,i} &\equiv \underline{F}_\varphi \cdot \frac{\partial \underline{J}}{\partial \varphi'_i} + \underline{F}_J \cdot \frac{\partial \underline{\varphi}}{\partial \varphi'_i} \ , \qquad i = 1,2 \ . \end{aligned}$$

Kamel method

• Non-Hamiltonian vector field: $\dot{\underline{y}} = \underline{g}(\underline{y}, t)$, introduce an *adjoint* vector $\underline{Y} \in \mathbb{R}^n$ and a Hamiltonian K, defined as the projection of $\underline{g}$ on $\underline{Y}$:

$$K(\underline{y}, \underline{Y}, t) \equiv \underline{Y} \cdot \underline{g}(\underline{y}, t) \ . \tag{1}$$

Hamilton's equations associated to K are given by

$$\begin{aligned} \dot{\underline{y}} &= \frac{\partial K}{\partial \underline{Y}} = \underline{g}(\underline{y}, t) \\ \dot{\underline{Y}} &= -\frac{\partial K}{\partial \underline{y}} = -\underline{Y} \cdot \frac{\partial \underline{g}}{\partial \underline{y}} \ . \end{aligned}$$

• This *Hamiltonianization* procedure requires to double the order of the system, since the Hamiltonian K is $2n$-dimensional with an explicit time dependence. Once obtained the Hamiltonian K, we implement the standard normalization procedure.

Comparison between Brouwer-Hori, direct, patching

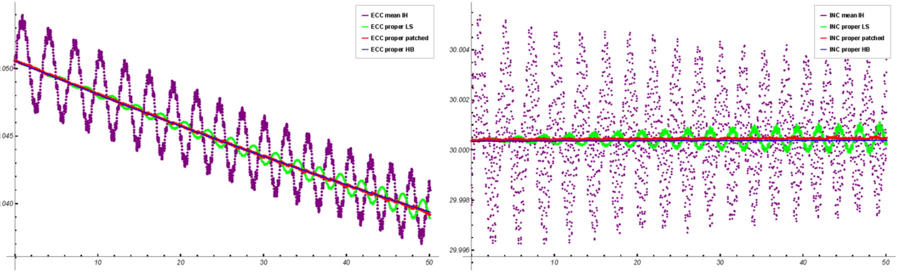


Figure: Comparison between the Brouwer-Hori (blue), direct (green) and patching (red) methods for $a = 7500$ km, $e = 0.05$, $i = 30^o$, $M = 0^o$, $\omega = 0^o$, $\Omega = 0^o$, $A/m = 1\text{m}^2/\text{kg}$ from [Celletti Gales Lhotka Vartolomei, 2025].

Outline

Nekhoroshev theorem (simplified)

Theorem (non-resonant formulation)

Let $\mathcal{H}(\underline{J},\underline{\varphi}) = h(\underline{J}) + \varepsilon R(\underline{J},\underline{\varphi})$ *on a domain* $D \times \mathbb{T}^n$*, such that it can be extended to a complex domain* $D_{r_0} \times \mathbb{T}^n_{s_0}$ *for* $r_0, s_0 > 0$*.*
Assume that the frequency $\underline{\omega} = \frac{\partial h}{\partial \underline{J}}$ *is non-resonant for some* $\alpha, K > 0$*:*

$$|\underline{\omega} \cdot \underline{k}| \geq \alpha \qquad \underline{k} \in \mathbb{Z}^n \backslash \{0\} \ , \quad |\underline{k}| \leq K \ .$$

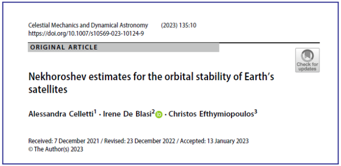
Assume that $\|\varepsilon R\| < E_0$ *is small on the complex domain.*

Then, there exists constants C_1*,* C_2*,* ε_**,* γ_1*,* γ_2*, such that*

$$\|\underline{J}(t) - \underline{J}(0)\| \leq C_1 (\frac{\varepsilon}{\varepsilon_*})^{\gamma_1} \ , \qquad |t| < C_2 \ exp((\frac{\varepsilon_*}{\varepsilon})^{\gamma_2}) \ .$$

See [Pöschel 1993], also in the resonant case for quasi-convex Hamiltonians.

The satellite geolunisolar model


Celestial Mechanics and Dynamical Astronomy (2023) 135:10
https://doi.org/10.1007/s10569-023-10124-9

ORIGINAL ARTICLE

Check for updates

Nekhoroshev estimates for the orbital stability of Earth's satellites

Alessandra Celletti[1] · Irene De Blasi[2] · Christos Efthymiopoulos[3]

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• Geolunisolar model: Keplerian part + J_2 (non-spherical Earth) + gravitational effects of Sun and Moon (on the ecliptic plane):

$$\mathcal{H}_{gls} = \mathcal{H}_{kep} + \mathcal{H}_{J_2} + \mathcal{H}_S + \mathcal{H}_M \ .$$

• Studied in [De Blasi, Celletti, Efthymiopoulos 2021] to give bounds on the stability of the Lidov-Kozai integral.

Nekhoroshev assumption

▷ $h(\underline{J})$ is *convex* if $(\partial^2 h(\underline{J}))\ \underline{u}\cdot\underline{u} = 0 \Leftrightarrow \underline{u} = \underline{0}$, $\forall \underline{u} \in \mathbb{R}^n$;
▷ $h(\underline{J})$ is *quasi-convex*, if $\underline{\omega}(\underline{J}) \neq 0$ and

$$\forall \underline{u} \in \mathbb{R}^n \qquad \begin{cases} \underline{\omega}(\underline{J}) \cdot \underline{u} = 0 \\ (\partial^2 h(\underline{J}))\ \underline{u}\cdot\underline{u} = 0 \end{cases} \Leftrightarrow \underline{u} = \underline{0};$$

▷ $h(\underline{J})$ is *three-jet non degenerate*, if $\underline{\omega}(\underline{J}) \neq 0$ and

$$\forall \underline{u} \in \mathbb{R}^n \qquad \begin{cases} \underline{\omega}(\underline{J}) \cdot \underline{u} = 0 \\ (\partial^2 h(\underline{J}))\ \underline{u}\cdot\underline{u} = 0 \\ \sum_{i,j,k=1}^n \frac{\partial^3 h}{\partial J_i \partial J_j \partial J_k}(\underline{J}) u_i u_j u_k = 0 \end{cases} \Leftrightarrow \underline{u} = \underline{0}.$$

• Convexity ⇒ Quasi-convexity ⇒ Three-jet non-degeneracy.

• The J_2 model (without Sun and Moon) is three-jet, but not quasi-convex.
• The geolunisolar model is quasi-convex ⇒ the lunisolar perturbation removes the degeneracy from the J_2 model!

Exponential stability estimates for a satellite

• Prepare the Hamiltonian by considering the secular Hamiltonian (averaged over all fast variables, expanded around reference values to order $N = 12$, normalized to order $Q = 6$), which has the form

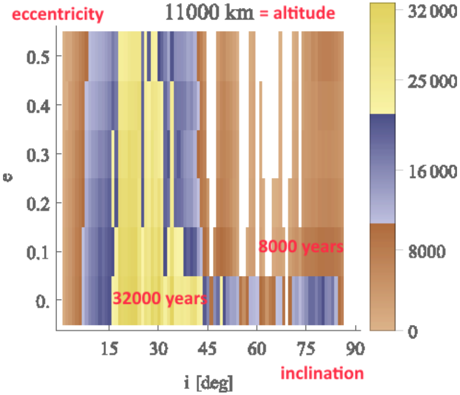
$$\mathcal{H}_Q(\underline{J}', \underline{\varphi}') = h_Q(\underline{J}') + \varepsilon^Q \; R_Q(\underline{J}', \underline{\varphi}')$$

• Implement Pöschel version of Nekhoroshev's theorem for quasi-convex Hamiltonians in non-resonant domains

$\rightsquigarrow$ stability estimates for different a, e, i.

Exponential stability estimates for a satellite

• Stability time (in years) for (i, e) in the domain of applicability of the Theorem.



• Stability times have been obtained through Nekhoroshev's theorem also for the equilateral Lagrangian equilibrium points.

Outline

Set up (joint work with A. Dogkas and A. Guido)

- Let $\mathcal{H}(\underline{J},\underline{\varphi}) = h(\underline{J}) + \varepsilon R(\underline{J},\underline{\varphi})$, $(\underline{J},\underline{\varphi}) \in D \subseteq \mathbb{R}^n \times \mathbb{T}^n$ open domain.
  - Define $\Pi D \subseteq \mathbb{R}^n$ the projection of D in the action space.
  - Consider a complex extension of the domain: $V_{r_0}\Pi D \times W_{s_0}\mathbb{T}^n$ for $r_0, s_0 \in \mathbb{R}_+$.
  - Introduce the norm:

$$\|R\|_{r_0,s_0} = \sup_{\underline{J}\in V_{r_0}\Pi D} \sum_{\underline{k}\in\mathbb{Z}^n_K} |R_{\underline{k}}(\underline{J})| \, \exp\left(\|\underline{k}\|_1 s_0\right) \ ,$$

  where $\mathbb{Z}^n_K = \{\underline{k} \in \mathbb{Z}^n : \|\underline{k}\|_1 \leq K\}$.
  - Let E and M be upper bounds on:

$$\left\|\varepsilon R(\underline{J},\underline{\varphi})\right\|_{r_0,s_0} \leq E \ , \qquad \sup_{\underline{J}\in V_{r_0}\Pi D} \left\|\partial^2_{\underline{J}} h(\underline{J})\right\|_2 \leq M \ .$$

Some definitions

Definition 1: resonant frequency

The frequency vector $\underline{\omega}(\underline{J})$ satisfies a resonance of order $\underline{k} \in \mathbb{Z}^n \backslash \{\underline{0}\}$, if

$$\underline{k} \cdot \underline{\omega}(\underline{J}) = 0 \ .$$

Definition 2: α, K non- resonant condition

A domain $D \subseteq \mathbb{R}^n \times \mathbb{T}^n$ is called α, K non-resonant modulo Λ, for $\Lambda \subseteq \mathbb{Z}^n_K \backslash \{\underline{0}\}$, if

$$|\underline{k} \cdot \underline{\omega}(\underline{J})| \geq \alpha \quad \forall \underline{k} \in \mathbb{Z}^n_K \backslash \{\Lambda \cup \underline{0}\} \ , \quad (\underline{J}, \underline{\varphi}) \in D$$

for some $\alpha \in \mathbb{R}_+$. If $\Lambda = \emptyset$, D is a completely α, K non-resonant domain.

Definition 3: Diophantine condition

A frequency vector $\underline{\omega}(\underline{J})$ is said to satisfy the Diophantine condition if

$$|\underline{k} \cdot \underline{\omega}(\underline{J})| \geq \frac{C}{\|k\|_1^\tau} \quad \forall \ \underline{k} \in \mathbb{Z}^n \backslash \{\underline{0}\} \ , \quad C \in \mathbb{R}_+ \ , \quad \tau \geq n-1 \ .$$

Non Resonant Stability Estimate

Theorem (Poeschel 1993)

Given $\mathcal{H} = h + \varepsilon R$ in D, let $r_0, s_0 > 0$. Suppose the domain D is completely α, K-non-resonant. Let $\ell > 0$ and let r be such that

$$r \le \min\left(\frac{\alpha}{M\,K}\left(1 - \frac{1}{\ell}\right), r_0\right) .$$

Assume E satisfies

$$\|\varepsilon R\|_{r_0,s_0} \le E \le \frac{1}{2^7\ell}\ \frac{\alpha\, r}{K} .$$

Then, for any initial condition $(\underline{J}_0, \underline{\varphi}_0) \in D$, the actions are bounded as

$$\|\underline{J}(t) - \underline{J}_0\| \le r \qquad \text{for all} \quad |t| \le T \equiv \frac{s_0 r}{5E}\ \exp\left(\frac{K s_0}{6}\right) .$$

$\Rightarrow$ explicit estimates on parameters ensuring stability, under applicability conditions on the parameters $(r_0, s_0, M, K, \alpha, E, \ell)$. Since these conditions allow some freedom in parameter choice, we propose an optimization algorithm to maximize the stability time.

Optimization of Non Resonant Stability Estimates

For a given initial condition $(\underline{J}_0, \underline{\varphi}_0)$, choose the parameters that maximize the stability time under the conditions:

$$
\begin{aligned}
E_{min} &\equiv \|\varepsilon R(\underline{J}_0, \underline{\varphi}_0)\|_{r_0, s_0} \le E \le \frac{1}{2^7 \ell} \, \frac{\alpha \, r}{K} \\
\alpha &\le \alpha_{max} \equiv \inf_{\underline{k} \in \mathbb{Z}^n_K} |\underline{k} \cdot \underline{\omega}(\underline{J}_0)| \\
M &\ge M_{min} \equiv \sup_{\|\underline{J} - \underline{J}_0\| \le r_0} \left(\|\partial^2_{\underline{J}} h(\underline{J})\|_2 \right) \\
r &\le r_{max} \equiv \min \left(\frac{\alpha}{MK} \left(1 - \frac{1}{\ell} \right), r_0 \right) \\
K &\ge \sup_{\underline{k} \in \Lambda_R} (\|\underline{k}\|_1) = K_{min}
\end{aligned}
$$

Choice of arbitrary parameters

$$(r_0, s_0, M, K, \alpha, E, \ell)$$

$$\downarrow$$

$$E = E_{min} \text{ and } r = r_{max},$$

$$\downarrow$$

$$M = M_{min} \text{ and } \alpha = \alpha_{max}$$

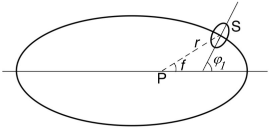
$$\downarrow$$

$$(r_0, s_0, K, \ell)$$

such that $T \equiv \frac{s_0 r}{5E} \, \exp\left(\frac{K s_0}{6}\right)$ is maximized.

Conservative Spin-orbit model

- Assumptions:
  - Triaxal satellite S
  - Spin axis $\perp$ to the orbital plane
  - S moves on a Keplerian orbit around a central planet P.



- 1D time dependent Hamiltonian:

$$\mathcal{H}(J_1, \varphi_1, t) = \frac{J_1^2}{2} - \frac{\varepsilon}{2} \frac{a^3}{r(t)^3} \cos(2\varphi_1 - 2f(t))$$

with $\varepsilon = \frac{3(I_2 - I_1)}{2I_3}$; frequency: $\underline{\omega} = (\omega_1, 1) = (J_1, 1)$.

Spin orbit resonance of type $k_2 : k_1$

• A spin-orbit resonance of type $k_2 : k_1$ (order $(k_1, -k_2)$) occurs if (Definition 1) $\underline{\omega} \cdot \underline{k} = 0$ with $\underline{\omega} = (\omega_1, 1)$ and $\underline{k} = (k_1, -k_2)$, i.e.

$$k_1 \omega_1 - k_2 = 0 \ ,$$

which implies that the satellite makes k_1 rotations within k_2 orbital revolutions.

• We want to approach a specific resonance without intersecting any other resonances:

Sequences of Diophantine frequencies $\rightarrow$ Resonant frequency $\frac{k_2}{k_1}$.

Sequences of Diophantine frequencies

- Non-autonomous 1D system: $\underline{\omega}(\underline{J}) = (\omega_1(J_1), 1)$:

Close to a resonance of order $(k_1, -k_2)$, we seek sequences of irrational frequencies that approximate the resonant frequency $\omega_1 = k_2/k_1$:

$$\Gamma_{z,s}^{(k_1,k_2)} = \frac{k_2}{k_1} - \frac{s}{z+\gamma} \xrightarrow{z\to\infty} (\frac{k_2}{k_1})^- \ , \quad \Delta_{z,s}^{(k_1,k_2)} = \frac{k_2}{k_1} + \frac{s}{z+\gamma} \xrightarrow{z\to\infty} (\frac{k_2}{k_1})^+$$

for $z \in \mathbb{Z}\backslash\{0\}$, $s \in \mathbb{Q}$, $\gamma = \frac{\sqrt{5}-1}{2}$ golden ratio.

Proposition

The numbers $\Gamma_{z,s}^{(k_1,k_2)}$, $\Delta_{z,s}^{(k_1,k_2)}$ satisfy the inequality

$$|k_1\omega_1 - k_2| \geq \frac{C}{|k_1|}, \quad \forall (k_1, k_2) \in \mathbb{Z}^2\backslash\{\underline{0}\} \ .$$

Consequence of Liouville theorem: Let ω be an algebraic number of degree n; then ω satisfies the Diophantine condition for some positive constant C and for $\tau = n-1$.

Implementation of perturbation theory

• From $\mathcal{H}(\underline{J},\underline{\varphi}) = h(\underline{J}) + \varepsilon R(\underline{J},\underline{\varphi})$, implement a Lie canonical transformation:

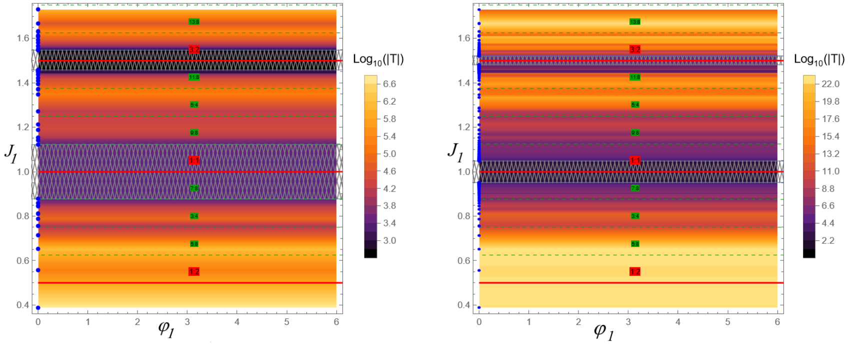
$$(\underline{J},\underline{\varphi}) \longrightarrow (\underline{J}',\underline{\varphi}')$$

defined by a generating function $\chi = \chi(\underline{J}',\underline{\varphi}')$, such that the new Hamiltonian is

$$\begin{aligned} \mathcal{H}'(\underline{J}',\underline{\varphi}') &= exp(L_{\chi(\underline{J}',\underline{\varphi}')})\mathcal{H}(\underline{J}',\underline{\varphi}') \\ &= h'(\underline{J}') + \varepsilon^2 R'(\underline{J}',\underline{\varphi}') \ . \end{aligned}$$

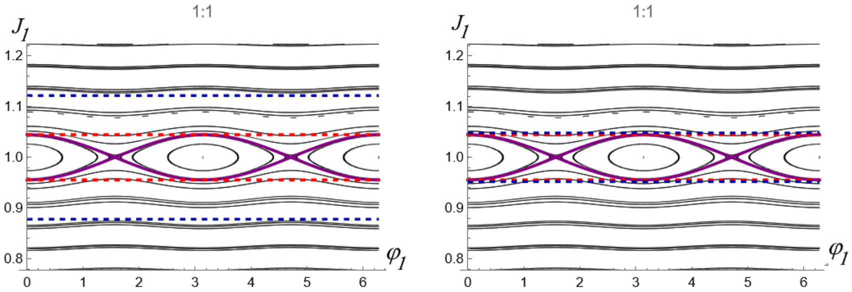
• This procedure can be iterated up to some optimal order: $\mathcal{H}' = h' + \varepsilon^2 R'$, $\mathcal{H}'' = h'' + \varepsilon^3 R''$, $\mathcal{H}''' = h''' + \varepsilon^4 R'''$, etc, before the accumulation of small divisors starts increasing the norm of the remainder.

- Consider initial conditions such that $\underline{\omega}(J_1(0))_z = (\Gamma_{z,s}^{(k_1,k_2)}, 1)$ or $\underline{\omega}(J_1(0))_z = (\Delta_{z,s}^{(k_1,k_2)}, 1)$.



a) b)

Stability time for $z = 2, \ldots, 100$ with $s = 1.6$ (1 : 1 resonance) and $s = 0.6$ (3 : 2 resonance), $\varepsilon = 10^{-3}$ (a) after two perturbative steps, (b) after three perturbative steps.

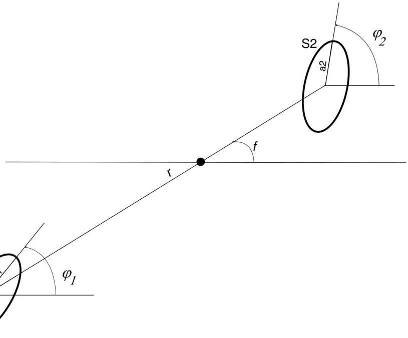


a) b)

Comparison between the last value of J_1 for which the algorithm provides results (blue line) and a resonant normal form approximation of the width of the 1:1 resonance (red line) for $\varepsilon = 10^{-3}$ (a) after two perturbative steps, (b) after three perturbative steps.

Conservative Spin-spin-orbit model

- Assumptions:
  - Triaxal satellites S_1, S_2
  - Respective spin axes $\perp$ to the orbital plane
  - S_1, S_2 move around each other on Keplerian orbits around their barycenter (the relative position is given by $(r(t), f(t))$).



2D time dependent Hamiltonian

$$\mathcal{H}(J_1, J_2, \varphi_1, \varphi_2, t) = J_1^2 + J_2^2 + V_1(r, f, \varphi_1, \varphi_2) + V_2(r, f, \varphi_1, \varphi_2)$$

$$V_1 = -\frac{m}{r^3(t)} \sum_{k=1,2} \frac{\varepsilon_k I_c^{(k)}}{2M_{S_k}} \cos(\underline{2\varphi_k - 2f(t)})$$

$$V_2 = -\frac{m}{r(t)^5} \sum_{k=1,2} \left(\frac{\varepsilon_k I_c^{(k)}}{M_{S_k}}\right)^2 \left(\frac{5}{112} + \frac{25}{48}\cos(4\varphi_k - 4f(t))\right)$$
$$- \frac{m}{r(t)^5}\left(\frac{\varepsilon_1 I_c^{(1)}}{M_{S_1}}\right)\left(\frac{\varepsilon_2 I_c^{(2)}}{M_{S_2}}\right)\left(\frac{35}{24}\cos(\underline{4f(t) - 2\varphi_1 - 2\varphi_2}) + \frac{1}{8}\cos(\underline{2\varphi_1 - 2\varphi_2})\right)$$

with $\varepsilon_k = \frac{3(I_2^{(k)} - I_1^{(k)})}{2I_3^{(k)}}$ and $m = GM_{S_1}M_{S_2}$.

Spin-spin-orbit problem: previous results

- Boué, Maciejewski, Misquero, Scheeres, etc.

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The Spin–Spin Problem in Celestial Mechanics

Alessandra Celletti[1] · Joan Gimeno[2] · Mauricio Misquero[3]

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- Full model (coupling between orbital and rotational motions) and Keplerian model (centers of mass moving on coplanar Keplerian ellipses), existence of periodic and quasi-periodic solutions, linear stability, interaction between rotational and orbital motions.

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Research paper

The dynamics of the spin–spin problem in Celestial Mechanics☆

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- (1:1,1:1), (3:2,3:2), (1:1,3:2) resonances: analysis of the linear stability of the equilibria in the conservative and dissipative settings, proving that the (linear) stability depends on the eccentricity; analysis of higher order resonant islands, which appear in the Keplerian case, but are destroyed in the full problem.

Sequences of Diophantine frequencies

- Non-autonomous 2D system: $\underline{\omega}(\underline{J}) = (\omega_1(\underline{J}), \omega_2(\underline{J}), 1)$.

- In the general n-dim case, define the vector $\underline{\omega} = (\omega_1, \ldots, \omega_{n-1}, 1) \in \mathbb{R}^n$ such that:

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} b_1 & & & \\ \vdots & & \mathcal{A} & \\ b_{n-1} & & & \\ 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{n-1} \end{pmatrix} , \qquad (2)$$

where $(b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1}$, the $(n-1) \times (n-1)$ dimensional matrix $\mathcal{A}$ has rational coefficients a_j with $\det \mathcal{A} \neq 0$, and α is a real algebraic number of degree n. The vectors defined by (2) satisfy the Diophantine condition.

- For $n = 3$ (2D non-autonomous system) choose α as the solution of

$$\alpha^3 + \alpha^2 - 1 = 0 \ .$$

The number $1/\alpha$ is the smallest Pisot-Vijayaraghavan (PV) number of degree 3.

Sequence of Diophantine frequencies

- A spin-spin-orbit resonance of type $k_3 : k_1 : k_2$ occurs when:

$$\underline{\omega} \cdot \underline{k} = 0 \qquad \text{with} \quad \underline{\omega} = (\omega_1, \omega_2, 1) \quad \text{and} \quad \underline{k} = (k_1, k_2, -k_3)$$

Definition

- An intersection between the spin-spin-orbit resonances of types $k_{3,1} : k_1 : 0$ and $k_{3,2} : 0 : k_2$ is a spin-orbit/spin-orbit resonance of type $(k_{3,1} : k_1)_{S_1}$, $(k_{3,2} : k_2)_{S_2}$ defined by the set of equations:

$$k_1\omega_1 - k_{3,1} \ = \ 0 \ , \qquad k_2\omega_2 - k_{3,2} = 0 \ .$$

- Diophantine frequencies for 2D time-dependent Hamiltonian

Given a spin-orbit/spin-orbit resonance of type $(k_{3,1} : k_1)_{S_1}, (k_{3,2} : k_2)_{S_2}$:

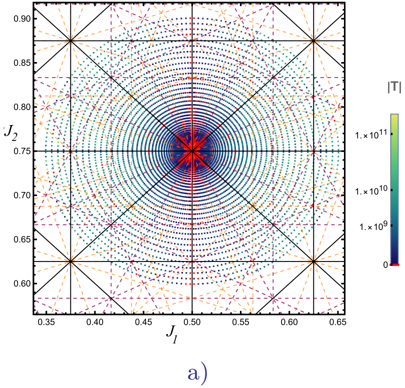
$$\omega_{1,z}^{(k_1,k_{3,1})} = \frac{k_{3,1}}{k_1} \pm \left(\frac{\tilde{a}_1}{z}\alpha + \frac{\tilde{a}_2}{z}\alpha^2\right) \xrightarrow{z\to\infty} \left(\frac{k_{3,1}}{k_1}\right)^{\pm},$$

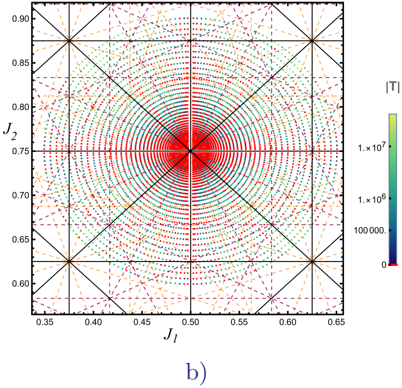
$$\omega_{2,z}^{(k_2,k_{3,2})} = \frac{k_{3,2}}{k_2} \pm \left(\frac{\tilde{a}_3}{z}\alpha + \frac{\tilde{a}_4}{z}\alpha^2\right) \xrightarrow{z\to\infty} \left(\frac{k_{3,2}}{k_2}\right)^{\pm}$$

with $\tilde{a}_j$ rationals and z integer.

Results

- We consider as initial conditions $(\underline{J}(0), \underline{\varphi}(0)) = (J_1(0), J_2(0), 0, 0)$ such that $\underline{\omega}(J_1(0), J_2(0))_z = (\omega_{1,z}^{(k_1,k_{3,1})}, \omega_{2,z}^{(k_2,k_{3,2})}, 1)$





Stability time around the spin-spin-orbit resonance $(1:1)_{S_1}, (3:2)_{S_2}$, (a) for $\varepsilon_1 = \varepsilon_2 = 10^{-5}$ after two perturbative steps and (b) for $\varepsilon_1 = 3 \cdot 10^{-5}$ and $\varepsilon_2 = 10^{-4}$ after two perturbative steps.

What next?

• Two main directions of research:

(1) natural debris: rings systems using an ellipsoidal model or a topographic feature model, adopt epicyclic variables, study corotation and Lindblad resonances through perturbation and bifurcation theory (in collaboration with I. De Blasi, S. Di Ruzza).

(2) include noise: start from the Sharma-Parathasarathy stochastic two-body problem, include dissipative effects (generalized Stokes drag) and find a balance between noise and dissipation, so that angular momentum and energy are *weak* integrals, i.e. the expectation at any time is the same as that at the initial time (in collaboration with C. Lhotka).

Main references

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▷ Nekhoroshev theorem and satellite dynamics:
• Celletti A., De Blasi I., Efthymiopoulos C., "Nekhoroshev estimates for the orbital stability of Earth's satellites", CM&DA (2023)

▷ Effective estimates in rotational dynamics:
• Celletti A., Dogkas A., Guido A., "Effective stability estimates close to resonances with applications to rotational dynamics", submitted to Nonlinearity (2025-2026)

CELMEC IX

- **CELMEC IX:**
14-18 September 2026, San Martino al Cimino, Viterbo, Italy
https://www.mat.uniroma2.it/~celmec/celmec9/

- **CELMEC prizes:** Topical issues in Cel. Mech. Dyn. Astron.:

▷ Analytical and semi-analytical results in Celestial Mechanics
▷ Pioneering computational techniques in Dynamical Astronomy

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