# Palatini Gauss-Bonnet Theory

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The theory does possess local degrees of freedom when $n > 1$, but identifying them turns out to be an intricate question, as for pure Chern-Simons theory in $2n+1$ dimensions with $n \geq 2$.

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Based on M. Bañados and M.H, arXiv :2511.20903 [hep-th] (to appear in JHEP) and in preparation

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where $\epsilon_{\gamma_1\gamma_2\cdots\gamma_n\beta_1\beta_2\cdots\beta_n}$ is the $GL(2n)$ numerically invariant tensor density of weight minus one. The combination $\sqrt{|g|}\,\epsilon^{\alpha_1\alpha_2\cdots\alpha_n}{}_{\beta_1\beta_2\cdots\beta_n}$, which is the only place where the metric enters, is a $GL(2n)$ tensor acting as a multilinear form.

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If the metric was fixed to be the Minkowski metric, restricting the group to the homogeneous Lorentz group, the action would be the Gauss-Bonnet topological invariant. However, because the metric is varied in the action principle, the dynamics is non-trivial.

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Exhibiting the missing gauge symmetries is a hard task in spacetime dimensions $\geq 4$.

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$$\psi \equiv B^{\alpha}_{\ \alpha} = 0\,, \qquad \omega \equiv B^{\alpha}_{\ \beta} B^{\beta}_{\ \alpha} - 2k^2\eta = 0 \quad (\eta = \frac{g}{|g|})\,.$$

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$$\mathcal{G}^\alpha_{\ \beta} = -\nabla_1 B^\alpha_{\ \beta}.$$

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which is a **new** (= unlisted before) gauge symmetry.

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However, in the presence of boundaries, there are non trivial boundary degrees of freedom, as explained later in the talk.

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To understand the dynamical structure, the safest way to proceed is to go to the Hamiltonian formulation.

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with

$$\mathcal{G}^{\alpha}_{\ \beta} = -\nabla_i p^{i\alpha}_{\ \beta} + 2\pi^{\alpha\gamma}g_{\gamma\beta}, \quad \phi^{i\nu}_{\ \ \mu} = \phi^{i\beta}_{\ \ \alpha} = p^{i\beta}_{\ \alpha} - \sqrt{g}\epsilon^{\beta\gamma}_{\ \ \alpha\delta}\, R^{\delta}_{\ \gamma jk}\epsilon^{ijk}.$$

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By following the standard (Dirac) procedure, one can rewrite the action in Hamiltonian form,

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($p^{i\alpha}_{\ \ \beta}$ are the conjugate momenta to the spatial components $\Gamma^{\beta}_{\ \alpha i}$; $\pi^{\alpha\beta}$ are the conjugate momenta to $g_{\alpha\beta}$; $\Gamma^{\beta}_{\ \alpha 0}$, $X^{\alpha}_{\ \beta i}$ and $Y_{\alpha\beta}$ are Lagrange multipliers for the constraints $\mathcal{G}^{\alpha}_{\ \beta} \approx 0$, $\phi^{i\beta}_{\ \ \alpha} \approx 0$ and $\pi^{\alpha\beta} \approx 0$)

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The constraints are not independent because $\nabla_i\phi^{i\alpha}_{\ \alpha} + \mathcal{G}^{\alpha}_{\ \alpha} - 2\pi^{\alpha}_{\ \alpha} = 0$. There are $16 + 3 \times 16 + 10 - 1 = 73$ independent constraints.

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\begin{aligned}
[\mathcal{G}^{\alpha}{}_{\beta}, \mathcal{G}^{\gamma}{}_{\delta}] &= \delta^{\gamma}{}_{\beta} \mathcal{G}^{\alpha}_{\delta} - \delta^{\alpha}_{\delta} \mathcal{G}^{\gamma}{}_{\beta} \\
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[\mathcal{G}^{\alpha}{}_{\beta}, \pi^{\gamma\delta}] &= \pi^{\alpha\delta} \delta^{\gamma}_{\beta} + \pi^{\gamma\alpha} \delta^{\delta}_{\beta} \\
[\phi^{i\alpha}{}_{\beta}, \phi^{i\gamma}{}_{\delta}] &= \Omega^{\kappa\lambda\alpha\gamma}{}_{\beta\delta} \epsilon^{ijk} \nabla_{k} g_{\kappa\lambda} \\
[\phi^{i\alpha}{}_{\beta}, \pi^{\kappa\lambda}] &= \Omega^{\kappa\lambda\alpha\gamma}{}_{\beta\delta} \epsilon^{ijk} R^{\delta}{}_{\gamma jk} \\
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\Omega^{\kappa\lambda\alpha\gamma}{}_{\beta\delta} = \frac{1}{2}\sqrt{g}\left(g^{\kappa\lambda}\epsilon^{\alpha\gamma}{}_{\beta\delta} - g^{\alpha\sigma}\epsilon^{\rho\gamma}{}_{\beta\delta} - g^{\alpha\rho}\epsilon^{\sigma\gamma}{}_{\beta\delta} - g^{\gamma\rho}\epsilon^{\alpha\sigma}{}_{\beta\delta} - g^{\gamma\sigma}\epsilon^{\alpha\rho}{}_{\beta\delta}\right).
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Among these $16 + 3 \times 16 + 10 - 1 = 73$ constraints, some are first class and generate the gauge symmetries while some are second class.

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Just as in those references, we focus our attention on the generic situation where the rank of the matrix of the Poisson brackets takes the maximum possible value compatible with the constraints,
which is the generic case.

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We have thus identified $16+1+3+3-1=22$ independent first class constraints, which account for all gauge symmetries listed above.

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These symmetries are hard to write analytically, in covariant form... completely unexpected!

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| Fields | $\{p^{i\alpha}_{\beta}, A^{\gamma}_{\delta j}\}$, $\{g_{\alpha\beta}, \pi^{\gamma\delta}\}$ | $2\times3\times16 + 2\times10 = 116$ |
| Constraints | $\mathcal{G}^{\alpha}_{\beta}, \phi^{i\gamma}_{\delta}, \pi^{\kappa\lambda}$ | $15+3\times16+10 = 73$ |
| Symmetries | gl(4), diff, projective, Weyl, **X** | $16+3+3+1-1+\mathbf{3} = 25$ |
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This theory has therefore $\frac{116-2\times25-48}{2} = 9$ physical (local) degrees of freedom.

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We thus consider the theory on the interval,

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where $b$ does not depend on $r$ (but can of course depend on the unwritten time $t$).

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This is achieved here by solving the Gauss constraint inside the action and eliminating the corresponding Lagrange multiplier ("inverse Lagrange multiplier method").

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where $\gamma^{\alpha\beta}$ is the inverse to the anti-de Sitter metric,

$$ds^2 = \gamma_{\alpha\beta} dq^\alpha dq^\beta = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\varphi^2$$

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for a free particle following geodesics in $AdS_3$.

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Note that neither $h$ nor $g$ depends on time ; these are global symmetries.

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In our derivation, one copy comes from the left boundary, the other comes from the right boundary.

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It is negative but satisfies the Breitenlhoner-Freedman bound with Euclidean signature, provided $k$ is small enough.

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THANK YOU !