# Finite $N$ precursors of the free cumulants

Sylvain Lacroix – LPTHE – Sorbonne Université

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Based on 2508.21483 with Jean-Bernard Zuber

# Introduction

# Cumulants in ordinary and free probability

- Cumulants $c_d$: related to moments by combinatorics of partitions

$$c_1 = m_1, \qquad c_2 = m_2 - m_1^2, \qquad c_3 = m_3 - 3m_2 m_1 + 2m_1^3$$

$$c_4 = m_4 - 4m_3 m_1 - 3m_2^2 + 12 m_2 m_1^2 - 6m_1^4\,, \qquad \dots$$

- Additivity property:

$$a \text{ and } b \text{ independent} \quad \Longrightarrow \quad c_d(a+b) = c_d(a) + c_d(b)\,, \;\; \forall\, d$$

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- Free probability: non-commutative random variables [Voiculescu '83]
- Free cumulants $\kappa_d$ (from non-crossing partitions) [Speicher '94]:

$$\kappa_4 = m_4 - 4m_3 m_1 - 2m_2^2 + 10 m_2 m_1^2 - 5 m_1^4\,, \qquad \dots$$

$$a \text{ and } b \text{ free} \quad \Longrightarrow \quad \kappa_d(a+b) = \kappa_d(a) + \kappa_d(b)\,, \ \ \forall\, d$$

# Free cumulants for matrices

- Freeness arises in large size limit of random matrices with unitary invariant distributions [Voiculescu '93]: algebraic point of view
- Moments and free cumulants: invariant polynomials on $N \times N$ matrices

$$m_k(A) = \frac{1}{N}\mathrm{Tr}(A^k)\,, \qquad \kappa_d(A) = P_d\big(m_1(A), \dots, m_d(A)\big)$$

- $A$, $B$ deterministic and $U \in \mathrm{U}(N)$ random (Haar-distributed)
  $\rightarrow$ $A$ and $UBU^{-1}$ free as $N \rightarrow \infty$
- Additivity property (in expectation value):

$$\int_{\mathrm{U}(N)} DU\, \kappa_d(A + UBU^{-1}) \approx \kappa_d(A) + \kappa_d(B) \qquad \text{as } N \rightarrow \infty$$

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- **Question:** polynomials satisfying this property at finite $N$?

# Finite $N$ precursors of free cumulants

- Finite precursors of free cumulants [Capitaine Casalis '06]: $N$-dependent invariant polynomials $K_d$ ($d \leq N$), coinciding with $\kappa_d$ when $N \to \infty$
- Addivity property at finite $N$:

$$\int_{\mathrm{U}(N)} DU\, K_d(A + UBU^{-1}) = K_d(A) + K_d(B)$$

- Application: $a, b$ independent random matrices with $\mathrm{U}(N)$-inv dist

$$\mathbb{E}\big[K_d(a+b)\big] = \mathbb{E}\big[K_d(a)\big] + \mathbb{E}\big[K_d(b)\big]$$

# Content of this talk

- [SL Zuber '25]: alternative route to $K_d$ from HCIZ integral
  $\rightarrow$ recover known results but also discover new ones

- Plan of this talk:
  1. Definition and additivity of $K_d$ from HCIZ
  2. Explicit expressions at finite $N$
  3. Large $N$ limit and relation to Hurwitz numbers
  4. Generating function of $K_d$
  5. Averages over sums of conjugacy orbits for general polynomials

# Definition and additivity of the precursors from HCIZ integral

# HCIZ integral and precursors

- HCIZ integral [Harish-Chandra '57, Itzykson Zuber '80]:

$$Z(A,B\,;x) := \int_{\mathrm{U}(N)} DU\, e^{Nx\,\mathrm{Tr}(AUBU^{-1})}$$

- Converging power series expansion in $x$

  Coefficient of $x^d$: invariant polynomial of degree $d$ in both $A$ and $B$

- Free cumulant precursor: "single trace coefficient" of degree $d \leq N$

$$K_d(A) := \frac{d}{N}\left[x^d\,\mathrm{Tr}(B^d)\right] Z(A,B\,;x)$$

  Invariant polynomial in $A$ of degree $d$

# First few precursors

- First few precursors:

$$
\begin{aligned}
K_1(A) &= \frac{1}{N}\,\mathrm{Tr}(A) \\
K_2(A) &= \frac{1}{N^2-1}\big(N\,\mathrm{Tr}(A^2) - \mathrm{Tr}^2(A)\big) \\
K_3(A) &= \frac{N}{(N^2-1)(N^2-4)}\big(N^2\,\mathrm{Tr}(A^3) - 3N\,\mathrm{Tr}(A)\,\mathrm{Tr}(A^2) + 2\mathrm{Tr}^3(A)\big) \\
K_4(A) &= \frac{N^2}{(N^2-1)(N^2-4)(N^2-9)}\big(N(N^2+1)\,\mathrm{Tr}(A^4) - 4(N^2+1)\mathrm{Tr}(A)\,\mathrm{Tr}(A^3) \\
&\qquad -(2N^2-3)\mathrm{Tr}^2(A^2) + 10N\,\mathrm{Tr}^2(A)\,\mathrm{Tr}(A^2) - 5\mathrm{Tr}^4(A)\big)
\end{aligned}
$$

- See later for more general formulas

# Additivity of the precursors

$$Z(A, C\,;x) := \int_{\mathrm{U}(N)} DU\, e^{Nx\,\mathrm{Tr}(AUCU^{-1})}$$

- Multiplicativity of the HCIZ integral:

$$\int_{\mathrm{U}(N)} DU\, Z(A + UBU^{-1}, C\,;x) = Z(A, C\,;x)\, Z(B, C\,;x)$$

# Additivity of the precursors

$$Z(A, C\,;x) := \int_{\mathrm{U}(N)} DU\, e^{Nx\,\mathrm{Tr}(AUCU^{-1})} = 1 + \frac{N}{d} K_d(A)\,\mathrm{Tr}(C^d)\,x^d + \cdots$$

- Multiplicativity of the HCIZ integral:

$$\int_{\mathrm{U}(N)} DU\, Z(A + UBU^{-1}, C\,;x) = Z(A, C\,;x)\, Z(B, C\,;x)$$

- Extract single trace coefficients: additivity of $K_d$

$$\int_{\mathrm{U}(N)} DU\, K_d(A + UBU^{-1}) = K_d(A) + K_d(B)$$

# Generalised precursors

- Generalised precursor $K_\alpha$ (of degree $d \leq N$):

$$K_\alpha(A) := \frac{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!}{N^\ell} \left[x^d \operatorname{Tr}(B^{\alpha_1}) \cdots \operatorname{Tr}(B^{\alpha_\ell})\right] Z(A, B\,; x)$$

- Labelled by partition $\alpha = (\alpha_1, \ldots, \alpha_\ell) \vdash d \qquad (\alpha_1 \geq \cdots \geq \alpha_\ell \geq 1)$

$$\alpha_1 + \cdots + \alpha_\ell = d$$

  Degree $d(\alpha) = d$, length $\ell(\alpha) = \ell$

- Multiplicity $\hat{\alpha}_k$: number of times $k$ appears in $\alpha$

- For length 1 partition $\alpha = (d)$: precursor $K_{(d)}(A) = K_d(A)$

# Generalised precursors

$$K_\alpha(A) := \frac{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!}{N^\ell} \left[ x^d \operatorname{Tr}(B^{\alpha_1}) \cdots \operatorname{Tr}(B^{\alpha_\ell}) \right] Z(A, B\,; x)$$

- $\{K_\alpha\}_{\alpha \vdash d}$ basis of invariant polynomials of degree $d \leq N$
- Examples for degree 2:

$$K_{(2)}(A) = \frac{1}{N^2 - 1} \left( N \operatorname{Tr}(A^2) - \operatorname{Tr}^2(A) \right)$$

$$K_{(1,1)}(A) = -\frac{1}{N(N^2 - 1)} \left( \operatorname{Tr}(A^2) - N \operatorname{Tr}^2(A) \right)$$

# Explicit expressions of the (generalised) precursors

# Representation theory of symmetric and unitary groups

- Permutation $\sigma$ in symmetric group $\mathrm{S}_d$ with cycle type $[\sigma] \vdash d$:

$$[\sigma] = (\alpha_1, \ldots, \alpha_\ell) \quad \Longleftrightarrow \quad \sigma \text{ has } \ell \text{ cycles of sizes } \alpha_i$$

- Characterises conjugacy classes:

$$\mathsf{Cl}_\alpha = \{\sigma \in \mathrm{S}_d \text{ with cycle type } [\sigma] = \alpha\}\,, \qquad |\mathsf{Cl}_\alpha| = \frac{d!}{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!}$$

- Irreducible reps $\mathsf{V}_\lambda^{\mathrm{S}_d}$ $(\lambda \vdash d)$, with characters $\chi_\lambda(\alpha)$

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- Irreducible reps $\mathrm{V}_\lambda^{\mathrm{S}_d}$ $(\lambda \vdash d)$, with characters $\chi_\lambda(\alpha)$

- Irreducible polynomial reps $\mathrm{V}_\lambda^{\mathrm{U}(N)}$ $(\ell(\lambda) \leq N)$ of unitary group $\mathrm{U}(N)$
- Character: Schur polynomial $s_\lambda(A)$, of degree $d(\lambda)$
  $\rightarrow$ form a basis of invariant polynomials on $N \times N$ matrices

# Generalised precursors in terms of Newton polynomials

- Newton polynomials $p_\alpha(A) := \mathrm{Tr}(A^{\alpha_1}) \cdots \mathrm{Tr}(A^{\alpha_\ell})$
- Generalised precursor $K_{[\sigma]}$ $(\sigma \in \mathrm{S}_d)$ in Newton basis:

$$K_{[\sigma]}(A) = N^{d-\ell([\sigma])} \sum_{\tau \in \mathrm{S}_d} \mathrm{Wg}([\sigma\tau^{-1}])\, p_{[\tau]}(A)$$

- Weingarten coefficient of $\alpha \vdash d$ ($N$-dependent):

$$\mathrm{Wg}(\alpha) := \frac{1}{d!} \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq N}} \frac{(\dim \mathrm{V}_\lambda^{\mathrm{S}_d})^2}{\dim \mathrm{V}_\lambda^{\mathrm{U}(N)}} \chi_\lambda(\alpha)$$

[Weingarten '78, Samuel '80, Collins '02]

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[Weingarten '78, Samuel '80, Collins '02]

- $K_{[\sigma]}(A)$: convolution of $\sigma \mapsto \mathrm{Wg}([\sigma])$ and $\sigma \mapsto p_{[\sigma]}(A)$ in $\mathbb{C}[\mathrm{S}_d]$
  $\rightarrow$ recovers matrix cumulants of [Casalis Capitaine '06 '08]

# Generalised precursors in terms of Schur polynomials

- Frobenius-Schur relation: Newton $\leftrightarrow$ Schur polynomials

$$p_\alpha(A) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq N}} \chi_\lambda(\alpha)\, s_\lambda(A)$$

- Generalised precursor $K_\alpha$ ($\alpha \vdash d$) in Schur basis:

$$K_\alpha(A) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq N}} D_\lambda\, \chi_\lambda(\alpha)\, s_\lambda(A)\,, \qquad D_\lambda := \frac{\dim \mathrm{V}_\lambda^{\mathrm{S}_d}}{d!\, \dim \mathrm{V}_\lambda^{\mathrm{U}(N)}}$$

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- Precursor $K_d$ from hook partitions $\lambda_{d,t} = (d-t, 1, \ldots, 1)$:

$$K_d(A) = \sum_{t=0}^{d-1} (-1)^t\, D_{\lambda_{d,t}}\, s_{\lambda_{d,t}}(A)\,, \qquad D_{\lambda_{d,t}} = \frac{N^{d-1}(N-t-1)!}{(N+d-t-1)!}$$

# Large $N$ limit and Hurwitz numbers

# Moments and large $N$ limit

- Moment (normalised trace):

$$m_k(A) := \frac{1}{N}\mathrm{Tr}(A^k)$$

- For $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ a partition:

$$m_\alpha(A) := m_{\alpha_1}(A) \cdots m_{\alpha_\ell}(A) = \frac{1}{N^{\ell(\alpha)}} p_\alpha(A)$$

- Sequence $(A_N)_{N \in Z_{\geq 1}}$ with limiting eigenvalue distribution as $N \to \infty$
  $\to$ moments $m_k(A_N)$ stay finite
- In this talk, large $N$ limit of invariant polynomial $f(A)$:
  - rewrite $f(A)$ in terms of $m_k(A)$, with $N$-dependent coefficients
  - take the limit $N \to \infty$ while keeping $m_k(A)$ finite

# Precursors in terms of moments

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\begin{aligned}
K_1 &= m_1 \\
K_2 &= \frac{N^2}{N^2-1}(m_2 - m_1^2) \\
K_3 &= \frac{N^4}{(N^2-1)(N^2-4)}(m_3 - 3m_2 m_1 + 2m_1^3) \\
K_4 &= \frac{N^4}{(N^2-1)(N^2-4)(N^2-9)} \\
&\quad \left((N^2+1)(m_4 - 4m_3 m_1 - 2m_2^2 + 10 m_2 m_1^2 - 5m_1^4) + 5(m_2 - m_1^2)^2\right) \\
K_5 &= \cdots
\end{aligned}
$$

# Large $N$ limits of precursors and free cumulants

- Large $N$ limits of first precursors:

$$\lim_{N\to\infty} K_1 = \kappa_1 := m_1$$
$$\lim_{N\to\infty} K_2 = \kappa_2 := m_2 - m_1^2$$
$$\lim_{N\to\infty} K_3 = \kappa_3 := m_3 - 3m_2m_1 + 2m_1^3\,,$$
$$\lim_{N\to\infty} K_4 = \kappa_4 := m_4 - 4m_3m_1 - 2m_2^2 + 10m_2m_1^2 - 5m_1^4$$

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\lim_{N\to\infty} K_4 &= \kappa_4 := m_4 - 4m_3 m_1 - 2m_2^2 + 10 m_2 m_1^2 - 5m_1^4
\end{aligned}$$

- More generally, from HCIZ [Itzykson Zuber '80]:

$$\lim_{N\to\infty} K_d = \kappa_d := \sum_{\beta \vdash d} (-1)^{1+\ell(\beta)} \frac{(d+\ell(\beta)-2)!}{(d-1)! \prod_{k=1}^{n} \hat{\beta}_k!} \, m_\beta$$

$\kappa_d$ free cumulant of degree $d$

# Large $N$ limits of generalised precursors

- Large $N$ limits of generalised precursors:

$$\lim_{N\to\infty} K_\alpha = \kappa_\alpha := \prod_{i=1}^{\ell(\alpha)} \kappa_{\alpha_i}$$

- Proof using finiteness of $\lim\limits_{N\to\infty} \dfrac{1}{N^2} \log Z(A, B\,; x)$

- Warning: $K_\alpha$ is not factorised before large $N$ limit

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$\dfrac{1}{N}$–corrections to the large $N$ limit of (generalised) precursors?

# Topological expansion of generalised precursors

- "Topological" expansion of the generalised precursors as $N \to \infty$:

$$K_\alpha = \frac{1}{|\mathrm{Cl}_\alpha|} \sum_{g \geq 1-\ell(\alpha)} N^{2(1-\ell(\alpha)-g)} \left( \sum_{\beta \vdash n} (-1)^{\ell(\alpha)+\ell(\beta)} H_g^{\bullet,\leq}(\alpha,\beta)\, m_\beta \right)$$

- $H_g^{\bullet,\leq}(\alpha,\beta)$: disconnected weakly monotone double Hurwitz numbers of type $(\alpha,\beta)$ and genus $g$ (count certain covers $\Sigma_g \to \mathbb{CP}^1$ of degree $d$)

- HCIZ: generating func of $H_g^{\bullet,\leq}(\alpha,\beta)$ [Goulden Guay-Paquet Novak '11]

# Topological expansion of precursors

- Topological expansion of the precursors:

$$K_d = \frac{1}{(d-1)!} \sum_{g \geq 0} N^{-2g} \left( \sum_{\beta \vdash n} (-1)^{1+\ell(\beta)} H_g^{\bullet,\leq}((d),\beta)\, m_\beta \right)$$

- Large $N$ limit gives back the free cumulant:

$$\lim_{N\to\infty} K_d = \kappa_d = \frac{1}{(d-1)!} \left( \sum_{\beta \vdash n} (-1)^{1+\ell(\beta)} H_0^{\bullet,\leq}((d),\beta)\, m_\beta \right)$$

- Recovers expression of genus 0 Hurwitz number [Novak '14]:

$$H_0^{\bullet,\leq}((d),\beta) = \frac{(d+\ell(\beta)-2)!}{\prod_k \hat{\beta}_k!}$$

# Moments-cumulant relation in free probability

- Inverted relation between moments and generalised precursors:

$$m_d = \sum_{g \geq 0} N^{-2g} \sum_{\alpha \vdash d} \mathrm{P}_g(\alpha)\, K_\alpha$$

$\mathrm{P}_g(\alpha) := |\,\text{permutations } \sigma \in \mathrm{S}_d \text{ of cycle type } [\sigma] = \alpha \text{ and genus } g\,|$

- Large $N$ limit recovers moment-cumulant relation of free probability:

$$m_d = \sum_{\alpha \vdash d} \mathrm{P}_0(\alpha)\, \kappa_\alpha$$

$\mathrm{P}_0(\alpha) = |\,\text{non-crossing partitions of } \{1, \dots, d\} \text{ by subsets of size } \alpha_i\,|$

- Counting permutations/partitions by genus and type [Kreweras '72, Cori '75, Harer Zagier '86, Cori Hetyei '13, Coquereaux Zuber '23, Hock '23]

# Generating function of the precursors

# Generating function of the precursors

- Generating function of $K_d$:

$$\mathcal{K}(A;x) \;:=\; \frac{1}{N}\int_{\mathrm{U}(N)} DU\, e^{N\mathrm{Tr}(AU^{-1})}\,\mathrm{Tr}\left(\frac{1}{1-xU}\right)$$

$$=\; 1+\sum_{d=1}^{N} x^d K_d(A) + \mathrm{O}(x^{N+1})$$

- Proof: extract $K_d$ from HCIZ by orthogonality relation

$$\int_{\mathrm{U}(N)} DU\, p_\alpha(U^{-1})\, p_\beta(U) = \frac{d!}{|\mathrm{Cl}_\alpha|}\delta_{\alpha\beta}$$

- Consequence: $\dfrac{\mathcal{K}(A;x)-1}{x} \xrightarrow{N\to\infty}$ Voiculescu $\mathcal{R}$-transform

# Invariant polynomials and sums of conjugacy orbits

# Algebra of invariant polynomials

- $\mathcal{A}$: graded algebra of invariant polynomials on $N \times N$ matrices:

$$\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d \,, \qquad \mathcal{A}_d \cdot \mathcal{A}_{d'} \subset \mathcal{A}_{d+d'}$$

  $\mathcal{A}_d$: degree $d$ polynomials

- Netwon basis $\{p_\alpha\}_{\alpha \in \mathcal{P}}$: $\qquad (\mathcal{P}_0 = \{\emptyset\}, p_\emptyset = 1)$

$$\mathcal{P} = \bigsqcup_{d \geq 0} \mathcal{P}_d \,, \qquad \mathcal{P}_d = \{\alpha \vdash d \,|\, \alpha_i \leq N\}$$

- Structure constants:

$$p_\alpha \, p_\beta = \sum_{\gamma \in \mathcal{P}} \Pi_{\alpha\beta}{}^{\gamma} \, p_\gamma$$

# Orbit coproduct

- Average of $f \in \mathcal{A}_d$ over sum of $\mathrm{U}(N)$-conjugacy orbits:

$$\Delta f(A,B) := \iint_{\mathrm{U}(N)^2} DU\, DV\, f(UAU^{-1}+VBV^{-1}) = \int_{\mathrm{U}(N)} DU\, f(A+UBU^{-1})$$

Sum of products of invariant polynomials in $A$ and $B$

- Defines orbit coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$\Delta f = \sum_i g_i \otimes h_i \quad \Longleftrightarrow \quad \Delta f(A,B) = \sum_i g_i(A)\, h_i(B)$$

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- Application in random matrix theory: $a, b$ independent random matrices with $\mathrm{U}(N)$-inv dist

$$\mathbb{E}[f(a+b)] = \mathbb{E}[\Delta f(a,b)] = \sum_i \mathbb{E}[g_i(a)]\, \mathbb{E}[h_i(b)]$$

# Properties of the orbit coproduct

- Orbit coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$:

$$\Delta f(A,B) := \iint_{\mathrm{U}(N)^2} DU\, DV\, f(UAU^{-1} + VBV^{-1}) = \int_{\mathrm{U}(N)} DU\, f(A + UBU^{-1})$$

- Properties:
  - $(\mathcal{A}, \cdot, \Delta)$ not a bialgebra
  - Graded: $\Delta(\mathcal{A}_d) = \bigoplus_{k=0}^{d} \mathcal{A}_k \otimes \mathcal{A}_{d-k}$
  - Cocommutative: $\Delta f(A,B) = \Delta f(B,A)$
  - Coassociative: $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$

- Precursors are primitive elements: $\Delta K_d = K_d \otimes 1 + 1 \otimes K_d$

# Orbit coproduct in dual Newton basis

- Dual Newton polynomials $\{p_\star^\alpha\}_{\alpha\in\mathcal{P}}$ from HCIZ integral:

$$Z(A,B\,;x) = \sum_{\alpha\in\mathcal{P}} x^{d(\alpha)} p_\star^\alpha(A)\, p_\alpha(B)$$

- For degree $d(\alpha) \leq N$, proportional to generalised precursor:

$$p_\star^\alpha = \frac{N^{\ell(\alpha)}|\mathrm{Cl}_\alpha|}{d(\alpha)!} K_\alpha$$

- **Theorem:** orbit coproduct in dual Newton basis

$$\Delta p_\star^\gamma = \sum_{\alpha,\beta\in\mathcal{P}} \Pi_{\alpha\beta}{}^\gamma\, p_\star^\alpha \otimes p_\star^\beta$$

Proof: $\displaystyle\int_{\mathrm{U}(N)} DU\, Z(A+UBU^{-1},C\,;x) = Z(A,C\,;x)\, Z(B,C\,;x)$

# Algebra-coalgebra duality

- Symmetric coefficients $\eta_{\alpha\beta}$ ($N$-dependent):

$$Z(A,B\,;x) = \sum_{\alpha,\beta\in\mathcal{P}} x^{d(\alpha)}\,\eta_{\alpha\beta}\,p_\star^\alpha(A)\,p_\star^\beta(B)$$

- Scalar product:

$$\eta\big(p_\alpha, p_\beta\big) := \eta_{\alpha\beta}$$

$\{p_\alpha\}_{\alpha\in\mathcal{P}}$ and $\{p_\star^\alpha\}_{\alpha\in\mathcal{P}}$ dual bases

- **Corollary:** $\eta(\cdot,\cdot)$ induces algebra isomorphism $(\mathcal{A},\cdot) \longrightarrow (\mathcal{A}^\star, \Delta^\star)$

$$\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A} \qquad \Longleftrightarrow \qquad \Delta^\star : \mathcal{A}^\star \otimes \mathcal{A}^\star \to \mathcal{A}^\star$$

# Conclusion

# Conclusion: finite $N$ precursors of the free cumulants

- Finite $N$ precursors of the free cumulants from HCIZ integral:
  - additive with respect to average over sums of $\mathrm{U}(N)$-orbits
  - natural expansions in terms of Newton and Schur polynomials
  - tend to free cumulants at large $N$
  - $\frac{1}{N^2}$-corrections in terms of monotone Hurwitz numbers
- Other points discussed in [SL Zuber '25]:
  - Average over GUE, relation to matrix integrals, Horn problem ...
  - Probablistic aspects and relation with [Collins Gurau Lionni '24]
  - Differences with [Arizmendi Perales '16]: no finite $N$ convolution à la [Marcus Spielman Srivastava '15] associated with $K_d$
- Future perspectives:
  - Other classical groups than $\mathrm{U}(N)$?
  - $q$- and $\beta$-deformations?
  - Relation with [Kunisky Moore Wein '24]?

# Conclusion: orbit coproduct

- Orbit coproduct: behaviour of general invariant polynomials with respect to averaging over sums of $\mathrm{U}(N)$-orbits

- Perspective: more general orbit coproduct for $V$ rep of $G$ (compact)
- $\Delta_V : P(V)^G \to P(V)^G \otimes P(V)^G$ defined by

$$\Delta_V f(a,b) := \int_G DU\, f(a+U.b)\,, \qquad a,b \in V$$

- Study $\Delta_V$ by replacing HCIZ integral by

$$Z_V(a,b\,;x) := \int_G DU\, e^{z\,\langle a, U.b\rangle}$$

$$\int_G DU\, Z_V(a+U.b, c\,;x) = Z_V(a,c\,;x)\, Z_V(b,c\,;x)$$

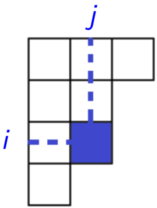
Thank you for your attention!

# Generalised precursors in terms of Schur polynomials

- Generalised precursor $K_\alpha$ ($\alpha \vdash d$) in Schur basis:

$$K_\alpha(A) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq N}} D_\lambda \, \chi_\lambda(\alpha) \, s_\lambda(A)$$

$$D_\lambda := \frac{\dim \mathrm{V}_\lambda^{\mathrm{S}_d}}{d! \, \dim \mathrm{V}_\lambda^{\mathrm{U}(N)}} = \prod_{(i,j) \in \lambda} (N + j - i)^{-1}$$



# Topological expansion of generalised precursors

- "Topological" expansion of the generalised precursors as $N \to \infty$:

$$K_\alpha = \frac{1}{|\mathrm{Cl}_\alpha|} \sum_{g \geq 1-\ell(\alpha)} N^{2(1-\ell(\alpha)-g)} \left( \sum_{\beta \vdash n} (-1)^{\ell(\alpha)+\ell(\beta)} H_g^{\bullet,\leq}(\alpha,\beta)\, m_\beta \right)$$

- $H_g^{\bullet,\leq}(\alpha,\beta)$: disconnected weakly monotone double Hurwitz number of type $(\alpha,\beta)$ and genus $g$
- Counts covers $\Sigma_g \to \mathbb{CP}^1$ of degree $d$ with:
  - $\Sigma_g$ surface of genus $g$ (potentially disconnected)
  - 2 branch points with ramification profiles $\alpha$ and $\beta$
  - $r = 2g - 2 - \ell(\alpha) - \ell(\beta)$ simple branch points with monodromies $(a_i b_i)$ such that $a_i < b_i$ and $b_1 \leq \cdots \leq b_r$
- HCIZ generating func of $H_g^{\bullet,\leq}(\alpha,\beta)$ [Goulden Guay-Paquet Novak '11]

# Average of generalised precursors over Gaussian weight

- Gaussian Unitary Ensemble:

$$\mathbb{E}_{A\sim \mathrm{GUE}(N,\sigma)}\big(f(A)\big) := \frac{1}{Z_{\mathrm{GUE}}} \int DA\, f(A)\, e^{-\frac{1}{2\sigma^2} N\,\mathrm{Tr}(A^2)}$$

- Wick-like relation for generalised precursors:

$$\mathbb{E}_{A\sim \mathrm{GUE}(N,\sigma)}\big(K_\alpha(A)\big) = \delta_{\alpha,[2^d]}\, \sigma^{2d}$$

Similar to ordinary/free cumulants with normal/semi-circle law

- Proof: compute $\mathbb{E}_A\big(Z(A,B\,;x)\big)$ (Fourier transform of Gaussian dist)

# Orbit coproduct in Schur basis

- Schur basis of $\mathcal{A}$: $\{s_\lambda\}_{\ell(\lambda)\leq N}$
- Self-dual up to content coefficient $C_\lambda$: $\eta(s_\lambda, s_\mu) = C_\lambda \, \delta_{\lambda\mu}$

$$Z(A,B\,;x) = \sum_{\ell(\lambda)\leq N} \frac{x^{d(\lambda)}}{C_\lambda} s_\lambda(A)\, s_\lambda(B)\,, \qquad C_\lambda := \frac{d!}{N^{\ell(\alpha)}} \frac{\dim \mathrm{V}_\lambda^{\mathrm{U}(N)}}{\dim \mathrm{V}_\lambda^{\mathrm{S}_d}}$$

- **Corollary:** orbit coproduct in Schur basis

$$\Delta s_\lambda = \sum_{\ell(\mu),\ell(\nu)\leq N} \frac{C_\lambda}{C_\mu \, C_\nu} \, c_{\mu\nu}{}^\lambda \, s_\mu \otimes s_\nu$$

Littlewood-Richardson coefficients: $s_\mu \, s_\nu = \sum_{\ell(\lambda)\leq N} c_{\mu\nu}{}^\lambda \, s_\lambda$