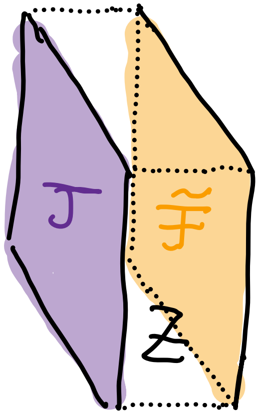
Fully Local Unitary Topological Quantum Field Theories

Lukas Müller, LMU Munich
Global Categorical Symmetries, June 26 2026
based on joint work with Giovanni Ferrer, Brett Hungar, Theo Johnson-Freyd, Cameron Krulewski, Nivedita, David Penneys, David Reutter, Claudia Scheimbauer, Luuk Stehouwer, Chetan Vuppulury (2403.01651)
Giovanni Ferrer, David Penneys, and Luuk Stehouwer (2606.11334), as well as work in progress with Theo Johnson-Freyd, Cameron Krulewski and Luuk Stehouwer.

Motivation:



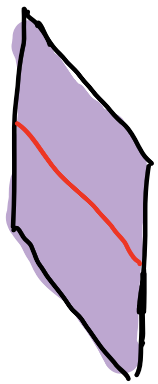
$\cong \mathcal{F}$

SymTFT

Quiche: $(\mathcal{J}, \mathcal{Z})$ "abstract symmetry structure"

Physical boundary: $\tilde{\mathcal{F}}$ "representation on $\mathcal{F}$"

Topological defects in $(\mathcal{J}, \mathcal{Z})$ "generalized group elements"

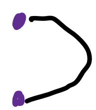
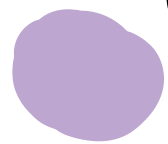


Extended Operator States

$\longleftrightarrow$

correspondence

[Kapustin '10]

$\mathcal{Z}\Big($  $\Big) =$ Category of lines on 

$\Downarrow$

Extended TQFTs

Q: What is unitary representation theory of generalized symmetries?

Ask $\mathcal{Z}, \mathcal{J}, \mathcal{F}$ to be unitary $\Rightarrow$ What is a unitary extended TQFT? $\Uparrow$

Mathematics motivation: TQFTs are representation theory for manifolds. What is unitary rep. theory?

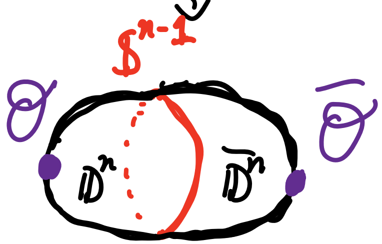
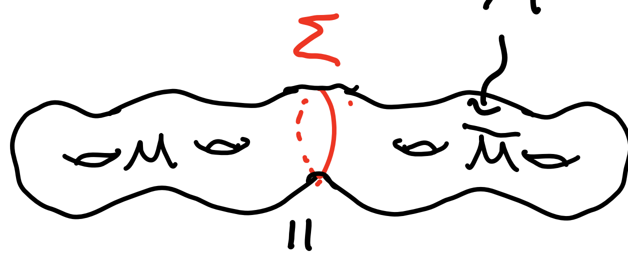
Reflection positivity [Freed Hopkins '16]

$$\langle \mathcal{O}\, R\mathcal{O} \rangle > 0 \longleftarrow \text{in a QFT}$$

$$= \langle \mathcal{O} | \mathcal{O} \rangle$$

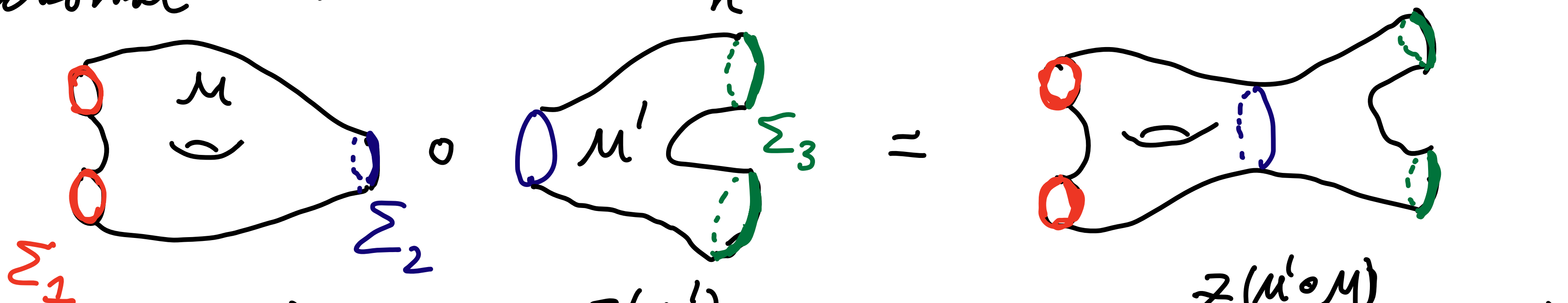
$$\text{"}\int \mathcal{O} e^{-S} = |\mathcal{O}\rangle\text{"} \in \mathcal{H}$$

Compactify

$Z\Big($  $\Big) \geq 0 \rightsquigarrow Z\Big($  $\Big) \geq 0$

$$= \langle Z(M) \,|\, Z(M) \rangle_{Z(\Sigma)}$$

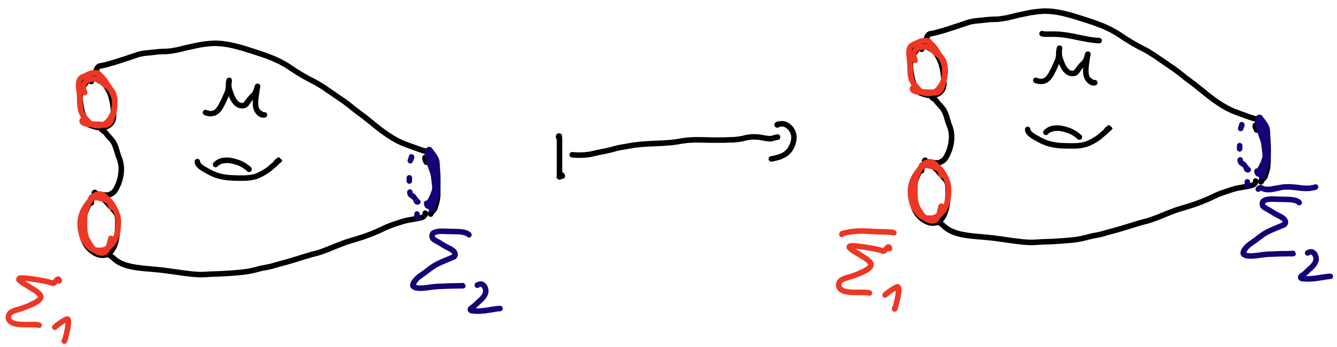
Functorial TQFT: $Z : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}_{\mathbb{C}}$



$$Z(\Sigma_1) \xrightarrow{Z(M)} Z(\Sigma_2) \xrightarrow{Z(M')} Z(\Sigma_3) = Z(\Sigma_1) \xrightarrow{Z(M' \circ M)} Z(\Sigma_3)$$

Reflection positivity

$\langle \Psi | \varphi \rangle$ is complex anti-linear in Ψ!



"Reflection"

A reflection theory is a $\mathbb{Z}_2$-equivariant sym monoidal functor $Z : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}_{\mathbb{C}}$. Data: $Z(\bar{\Sigma}) \cong \overline{Z(\Sigma)}$

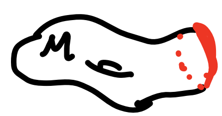
$$\begin{array}{ccc} \mathrm{Bord}_n & & \mathrm{Vect}_{\mathbb{C}} \\ \circlearrowleft & & \circlearrowleft \\ \overline{(\;)} & & \overline{(\;)} \end{array}$$

conditions:

$$\begin{tikzcd} Z(\Sigma) = Z(\bar{\bar{\Sigma}}) \arrow[r, "\sim"] \arrow[rr, bend right, "\mathrm{id}_{Z(\Sigma)}"'] & \overline{Z(\bar{\Sigma})} \arrow[r, "\sim"] & \overline{\overline{Z(\Sigma)}} = Z(\Sigma) \end{tikzcd}$$

$$\begin{tikzcd} Z(\bar{\Sigma}) \arrow[r] \arrow[d, "Z(\bar{M})"'] & \overline{Z(\Sigma)} \arrow[d, "\overline{Z(M)}"] \\ Z(\bar{\Sigma}') \arrow[r] & \overline{Z(\Sigma')} \end{tikzcd}$$

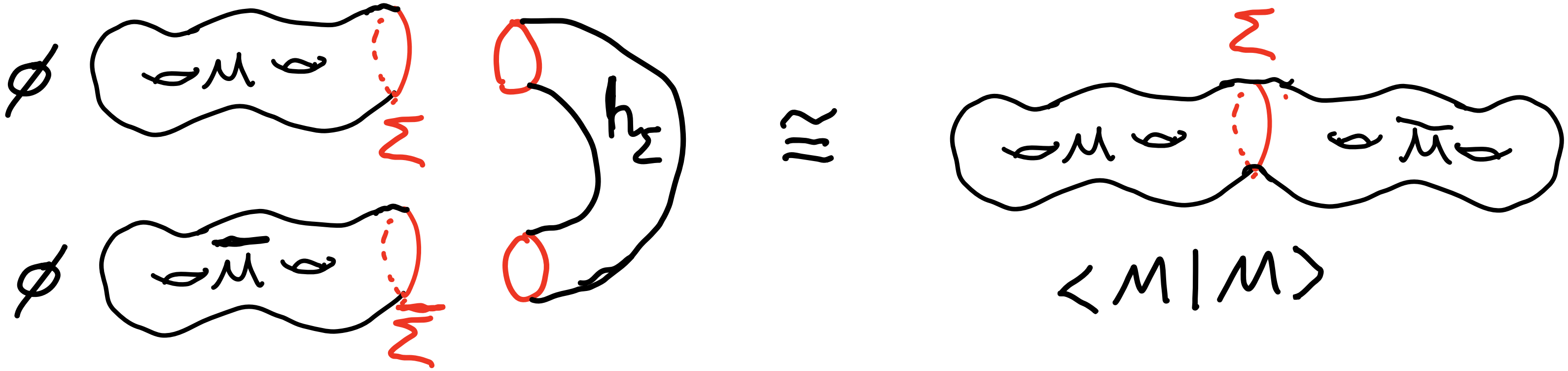
e.g.



$$\begin{tikzcd} Z(\bar{\emptyset}) \cong \mathbb{C} \arrow[r, phantom, "="] \arrow[d, "Z(\bar{M})"'] & \mathbb{C} \cong Z(\emptyset) \arrow[d, "\overline{Z(M)}"] \\ Z(\bar{\Sigma}) \arrow[r, "\sim"] & \overline{Z(\Sigma)} \end{tikzcd}$$

<u>Reflection positivity</u>

$|M\rangle \in Z(\Sigma)$



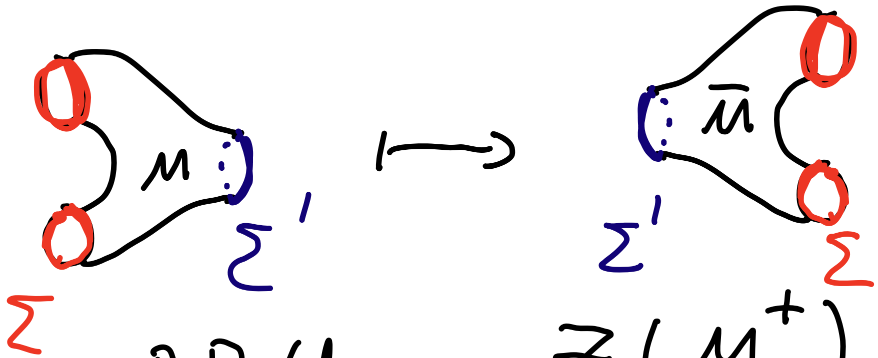
$\overline{|M\rangle} \in \overline{Z(\Sigma)} \cong Z(\overline{\Sigma}) \ni |\overline{M}\rangle$

Proposition: In every reflection TQFT Z
$\langle\, , \,\rangle : \overline{Z(\Sigma)} \otimes Z(\Sigma) \cong Z(\overline{\Sigma} \otimes \Sigma) \xrightarrow{Z(h_\Sigma)} \mathbb{C}$ is a hermitian inner product.

Def. Z is <u>reflection positive</u> if all the pairings $\langle\, , \,\rangle$ are positive.

Remark: By the operator state correspondence $Z(\Sigma)$ is the vector space of operators in the "interiour of Σ".

Dagger categories



$M:\Sigma\to\Sigma' \longmapsto M^\dagger:\Sigma'\to\Sigma$

In a RP theory $Z(M^\dagger) = Z(M)^\dagger$ ($\dagger$-functor)

Def. A *dagger category* is a category $\mathcal{C}$ equipped with an identity on objects functor $\dagger:\mathcal{C}\to\mathcal{C}^{op}$ such that $\dagger^2=\mathrm{id}$

Thm RP topological QFTs $\Leftrightarrow$ symmetric monoidal $\dagger$-functors
$Z:\mathrm{Bord}_n\longrightarrow\mathrm{Hilb}$

Classification in 2D: Hilbert space A with a commutative FA structure ($\eta:\mathbb{C}\to A$, $\mu:A\otimes A\to A$, $\Delta:A\to A\otimes A$, $\varepsilon:A\to\mathbb{C}$) such that $\mu^\dagger=\Delta$ and $\eta^\dagger=\varepsilon$.

DW: $A=\mathrm{Cl}(G)$ $\varepsilon(f)=f(e)$

$\langle f|g\rangle = \frac{1}{|G|}\sum_{h\in G}\overline{f(h)}\,g(h)$ $\quad\Delta(f)(g_1,g_2)=f(g_1\cdot g_2)$

• TV for a unitary fusion category

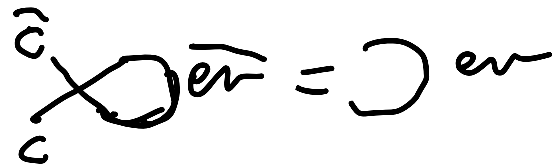
Why is the FH definition evil?

$$(\mathcal{C}, \otimes, (\ )^\vee) \circlearrowleft_{\overline{(\ )}} \mathbb{Z}_2 \rightsquigarrow \mathcal{C}^\times \circlearrowleft_{\overline{(\ )}^\vee} \mathbb{Z}_2$$

CH $\mathbb{Z}_2$-action

$$f : c \to c' \mapsto \bar{c}^\vee \xrightarrow{\bar{f}^{\vee\,-1}} \bar{c}'^\vee$$

Def. A <u>hermitian object</u> in $\mathcal{C}$ is a fixed point for the $\mathbb{Z}_2$-action $\overline{(\ )}^\vee$ on $\mathcal{C}^\times$:

$\overline{(c)}^\vee \xrightarrow{\sim} c$ + coh. $\iff$ ev: $\bar{c} \otimes c \to 1$, 

Example • Hermitian objects in Vect are hermitian vector spaces.

•  is a hermitian object in Bord_n.

$\mathbb{Z}_2$-equivariant functors preserve hermitian objects

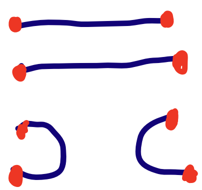
RP: reflection + condition that h is mapped to a Hilbert space pairing!

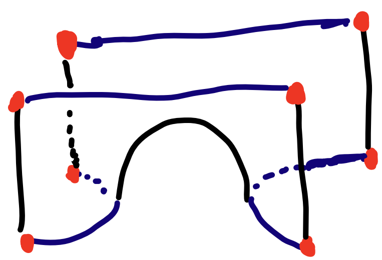
Extended TQFTs

$Z(M) \in \mathbb{C}$, $Z(\Sigma) \in Vect$, $Z(S) \in Cat$, ...

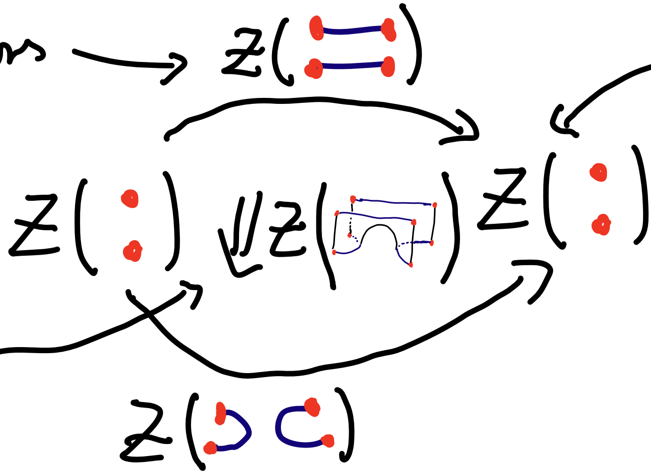
$\rightsquigarrow$ $Bord_n$ is an n-category. (We will focus on $Bord_2$)

Obj 1-Mor 2-Mor ...... diffeomorphisms and isotopies.





linear functors → linear category semi-simple with finitely many isomorphism classes of simple objects



natural transformation

2-vector space

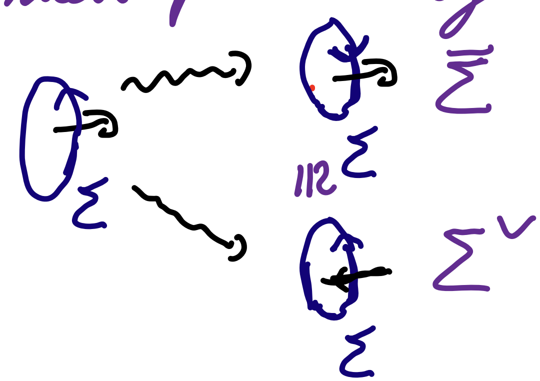
Extended reflection TQFTs:

$$\begin{array}{ccc} Z : \mathrm{Bord}_2 & \longrightarrow & 2\mathrm{Vect}_{\mathbb{C}} \\ \overline{(\,)} \circlearrowleft & & \circlearrowleft \overline{(\,)} \\ \mathbb{Z}_2 & & \mathbb{Z}_2 \end{array}$$

$\mathbb{Z}_2$-equivariant sym monoidal 2-functor.

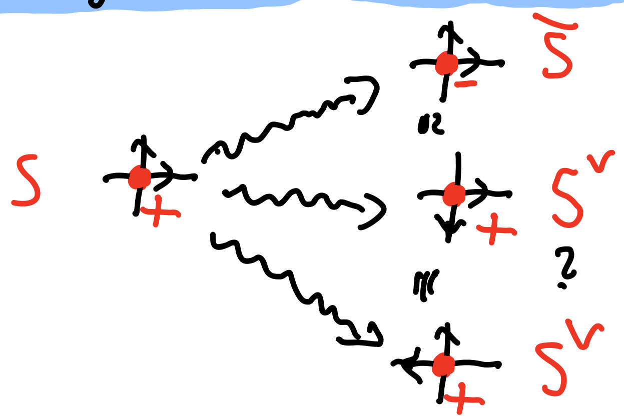
Positivity: ?

unextended positivity was about hermitian structures. $\overline{\Sigma} \cong \Sigma^\vee$



O(1)-many choices for $\to$

In higher codimension:



We have O(2) many choices for ↑→. Category theory picks directions. Physics shouldn't!

$\Rightarrow$ O(2)-hermitian structures

$O(2)$-hermitian structures

$\mathcal{B}$ a sym. mono. bicategory where all objects are fully dualizable. $\underset{CH}{\Longrightarrow}$ $\underset{(\;)^\vee}{O(2) \curvearrowright \mathcal{B}^\times}$. $\underset{\overline{(\;)}}{\mathbb{Z}_2 \curvearrowright \mathcal{B}^\times}$ (more generally O)

$\underset{\overline{(\;)}^\vee}{O(2) \curvearrowright \mathcal{B}^\times}$ using the map $\det : O(2) \longrightarrow \mathbb{Z}_2$

Def. An <u>$O(2)$-hermitian</u> object is a fixed point for this $O(2)$-action.

Example. Every object of the oriented bordism 2-category has a cannonical $O(2)$-hermitian structure.

Not true in Bord_2^{fr}!

O(2)-hermitian 2-vector spaces.

Thm (G. Ferrer, CM, D Penneys, L. Stehouwer '26)

Baez 2-Hilbert spaces are O(2)-hermitian 2-vector spaces.

2-vector space $\mathcal{H}$ equipped with a $\dagger$-structure $\dagger: \mathcal{H} \to \overline{\mathcal{H}}^{op}$ and a trace $\lambda_a: \mathrm{End}_{\mathcal{H}}(a) \to \mathbb{C}$ s.t. $\lambda_a(f \circ g) = \lambda_b(g \circ f)$ for $a \xrightarrow{g} b \xrightarrow{f} a$ and $\langle f_1, f_2 \rangle := \lambda(f_1^\dagger \circ f_2)$ defines a Hilbert space pairing on $\mathcal{H}(a,b)$.

Example • Let G be a finite group. $u\mathrm{Rep}(G)$ is a 2-Hilbert space with $\dagger$ the adjoint and $\lambda = \mathrm{tr}$.

• Every unitary fusion category is a 2 Hilbert space.

Back to positivity: Extended reflection theories preserve $O(2)$-hermitian objects.

- Require to map the cannonical $O(2)$-hermitian objects of Bord_2 to Baez 2-Hilbert spaces.

$$\begin{tikzcd} & 2\mathrm{Hilb} \arrow[d] \\ * \arrow[ur, dashed, bend left, "\text{data}"] \arrow[r, "Z(S)"] & (2\mathrm{Vect}^{\times})^{O(2)} \end{tikzcd}$$

neither injective nor surjective.

- Conditions on 1-morphisms!

Thm (LM, L Stehouwer ??)

2-dimensional RP fully local TQFTs are classified by the Baez 2-Hilbert space they assign to $\bullet^{+}$.

Example: u $\mathrm{Rep}(G) \rightsquigarrow$ DW theory

Generalizations

- Works for general stable tangential structures $BG \to BO$
- Requires extending $\overline{(\,)}$ to an action of O [Krulewski LM Stehouwer '25] "higher spin action"
- Target s2Vect

Thm (LM, L Stehouwer '22)

- Unitary fully local spin TQFTs are classified by super Baez 2-Hilbert spaces.

Ex: $(G, c \in Z(G)) \rightsquigarrow$ usRep(G, c)

- For tangential structures associated to an internal fermionic symmetry group K unitary TQFTs are classified by unitary 2-representations of K.

Ex: Invertible uTQFT: $\mathbb{Z}_2$-choice: Action on sHilb or Cl_1-Mod $H^0(K, \mathbb{Z}_2)$

- For each $k \in K$ we can act by $\mathbb{C} \otimes (-)$ or $\Pi\mathbb{C} \otimes (-) \rightsquigarrow H^1(K, \mathbb{Z}_2)$
- For pairs $k, k' \in K$ coherence isomorphisms $\rightsquigarrow H^2(K, U(1))$

$\rightsquigarrow$ Atiyah-Hirzebruch spectral sequence

Higher dimension (wip)

- $\mathcal{C}$ be a fully dualizable n-category $\overset{CH}{\rightsquigarrow} O(n) \overset{(\,)^\vee}{\circlearrowright} \mathcal{C}^\times$
- $O(n)$-hermitian objects fixed points for $\overline{(\,)}^\vee$ ← $O \overset{\overline{(\,)}}{\circlearrowright} \mathcal{C}$
- For $X \to BO$ a stable tangential structure $pt \in \mathrm{Bord}_n^{X_n}$ has a cannonical $O(n)$-hermitian structure.

Thm (T Johnson-Freyd, C Krulewski, LM, L. Stehouwer)
Unitary n-dim fully local TQFT classified by fixed points for an action of X on the unitary objects of the target via the map $X \to BO \to \mathrm{Aut}(u\mathcal{C}^\times)$

Invertible TQFTs

$$u\mathrm{TQFT}^\times(\Sigma^{n+1} I\mathbb{Z}(1)) \simeq \mathrm{Map}(\$, \Sigma^{n+1} I\mathbb{Z}(1)^X)$$
$$\simeq \mathrm{Map}(\$_X, \Sigma^{n+1} I\mathbb{Z}(1)) \simeq \mathrm{Map}(MTX, \Sigma^{n+1} I\mathbb{Z}(1))$$

Higher dimension (wip)

- $\mathcal{C}$ be a fully dualizable n-category $\overset{CH}{\rightsquigarrow}$ $O(n) \overset{()^\vee}{\circlearrowright} \mathcal{C}^\times$
- $O(n)$-hermitian objects fixed points for $\overline{()}^\vee$ ← $O \overset{\overline{()}}{\circlearrowright} \mathcal{C}$
- For $X \to BO$ a stable tangential structure $pt \in \mathrm{Bord}_n^{X_n}$ has a cannonical $O(n)$-hermitian structure.

Thm (T Johnson-Freyd, C Krulewski, LM, L. Stehouwer)
Unitary n-dim fully local TQFT classified by fixed points for an action of X on the unitary objects of the target via the map $X \to BO \to \mathrm{Aut}(u\mathcal{C}^\times)$

Invertible TQFTs

$$uTQFT^\times(\Sigma^{n+1} I\mathbb{Z}(1)) \simeq \mathrm{Map}(\$, \Sigma^{n+1} I\mathbb{Z}(1)^X)$$
$$\simeq \mathrm{Map}(\$_X, \Sigma^{n+1} I\mathbb{Z}(1)) \simeq \mathrm{Map}(MTX, \Sigma^{n+1} I\mathbb{Z}(1))$$

Thank you for your attention!