On Some of the Math for Continuous Symmetries

GCS '26, IHP Paris

Pelle Steffens

Technical University of Munich

June 26th 2026

Finite G-symmetries

We're pretty familiar with the following picture:

Finite G-symmetries

We're pretty familiar with the following picture:

- G a finite group.

Finite G-symmetries

We're pretty familiar with the following picture:

- G a finite group.
- Consider $(n+1)$-dimensional *Dijkgraaf-Witten theory* σ_{BG}^{n+1} with its right regular boundary ρ, the *quiche*.

Finite G-symmetries

We're pretty familiar with the following picture:

- G a finite group.
- Consider $(n+1)$-dimensional *Dijkgraaf-Witten theory* σ_{BG}^{n+1} with its right regular boundary ρ, the *quiche*.
- n-Dimensional theories with G-symmetry live on the left boundary of the *sandwich*.

Constructing G-gauge theory

For a finite group G, there are two ways to construct the theory σ_{BG}^{n+1}.

Constructing G-gauge theory

For a finite group G, there are two ways to construct the theory σ^{n+1}_{BG}.

- Via path integral quantization: since G is finite, the path integral is well-defined. Fully extended version via higher categorical machinery of *ambidexterity* and *higher semiadditivity*.

Constructing G-gauge theory

For a finite group G, there are two ways to construct the theory σ_{BG}^{n+1}.

- Via path integral quantization: since G is finite, the path integral is well-defined. Fully extended version via higher categorical machinery of *ambidexterity* and *higher semiadditivity*.
- Via the cobordism hypothesis: the sym. monoidal ∞-category $\mathrm{Rep}_G^{\otimes}$ is fully dualizable in the Morita category $\mathrm{Mor}_n(\mathrm{Cat}_{\infty,k})$ of k-linear ∞-categories for any n.

Constructing G-gauge theory

For a finite group G, there are two ways to construct the theory σ^{n+1}_{BG}.

- Via path integral quantization: since G is finite, the path integral is well-defined. Fully extended version via higher categorical machinery of *ambidexterity* and *higher semiadditivity*.
- Via the cobordism hypothesis: the sym. monoidal ∞-category $\mathrm{Rep}_G^{\otimes}$ is fully dualizable in the Morita category $\mathrm{Mor}_n(\mathrm{Cat}_{\infty,k})$ of k-linear ∞-categories for any n.

Recall the higher Morita $(\infty, n+2)$-category $\mathrm{Mor}_n(\mathrm{Cat}_{\infty,k})$:

(O) Objects are $\mathbb{E}_n$-monoidal k-linear ∞-categories.

(M_1) 1-Morphisms between two such A, B are $\mathbb{E}_{n-1}$-monoidal A-B-bimodule categories.

$(M_{\dots})$ ...

(M_n) n-Morphisms are bimodules in $\mathbb{E}_1$-monoidal bimodules in $\mathbb{E}_2$-monoidal bimodules in...

(M_{n+1}) $(n+1)$-Morphisms are k-linear bimodule functors.

(M_{n+2}) $(n+2)$-Morphisms are natural transformations.

Continuous G-symmetries

What if G is a continuous (better: *smooth*, that is, Lie) group? There is a very rich theory of *infinite dimensional* G-representations. We might want to label defects by such.

Continuous G-symmetries

What if G is a continuous (better: *smooth*, that is, Lie) group? There is a very rich theory of *infinite dimensional* G-representations. We might want to label defects by such.

Remark

In infinite dimensions, representations of Lie groups must be *continuous* (better: *smooth*) for a topological vector space structure $\Rightarrow$ consider not $\mathbb{R}$-linear categories, but categories enriched in topological vector spaces (*or something of that flavour*).

Continuous G-symmetries

What if G is a continuous (better: *smooth*, that is, Lie) group? There is a very rich theory of *infinite dimensional* G-representations. We might want to label defects by such.

Remark

In infinite dimensions, representations of Lie groups must be *continuous* (better: *smooth*) for a topological vector space structure $\Rightarrow$ consider not $\mathbb{R}$-linear categories, but categories enriched in topological vector spaces (*or something of that flavour*).

Goals

(1) Set up convenient target categories of topological (*or something of that flavour*) n-vector spaces (like the higher Morita category).

(2) Construct an analogue of σ_{BG} for G a *compact* Lie group

  (1) via the cobordism hypothesis, verifying the dualizability of a (topologically enriched version of) $\mathrm{Rep}_G^{\otimes}$ in $(n+1)\mathrm{TopVect}$.

  (2) directly, as a functor

$$\mathrm{Bord}_n^{\mathrm{fr}\otimes} \longrightarrow (n+1)\mathrm{TopVect}^{\otimes},$$

Continuous G-symmetries

What if G is a continuous (better: *smooth*, that is, Lie) group? There is a very rich theory of *infinite dimensional* G-representations. We might want to label defects by such.

Remark

In infinite dimensions, representations of Lie groups must be *continuous* (better: *smooth*) for a topological vector space structure $\Rightarrow$ consider not $\mathbb{R}$-linear categories, but categories enriched in topological vector spaces (*or something of that flavour*).

Goals

(1) Set up convenient target categories of topological (*or something of that flavour*) n-vector spaces (like the higher Morita category).

(2) Construct an analogue of σ_{BG} for G a *compact* Lie group

  (1) via the cobordism hypothesis, verifying the dualizability of a (topologically enriched version of) $\mathrm{Rep}_G^{\otimes}$ in $(n+1)\mathrm{TopVect}$.

  (2) directly, as a functor

$$\mathrm{Bord}_n^{\mathrm{fr}\otimes} \longrightarrow (n+1)\mathrm{TopVect}^{\otimes},$$

Remark

Hilbert spaces are infinite dimensional: we cannot expect to do better than a once-categorified $(n+1)$-dimensional theory.

Broader context

Continuous symmetries in the physics literature:

- *A symTFT for continuous symmetries* [Brennan-Sun, 24].
- *SymTFTs for Continuous non-Abelian Symmetries* [Bonetti-Del Zotto-Minasian, 24].
- ...

Broader context

Continuous symmetries in the physics literature:

- *A symTFT for continuous symmetries* [Brennan-Sun, 24].
- *SymTFTs for Continuous non-Abelian Symmetries* [Bonetti-Del Zotto-Minasian, 24].
- ...

The TFT is quantum BF-theory.

Broader context

Continuous symmetries in the physics literature:

- *A symTFT for continuous symmetries* [Brennan-Sun, 24].
- *SymTFTs for Continuous non-Abelian Symmetries* [Bonetti-Del Zotto-Minasian, 24].
- ...

The TFT is quantum BF-theory.

Remark

Classical BF in n-dimensions is an AKSZ sigma model with target $T^\vee[n-1]BG$ (formal nbhd at id: $\mathfrak{g}[-1] \oplus \mathfrak{g}^\vee[2-n]$). For $n = 3$, we have

$$T^\vee[2]BG \simeq B(T^\vee G).$$

BF in 3 dimensions is Chern-Simons for $T^\vee G \simeq G \ltimes \mathfrak{g}^\vee$.

Broader context

Continuous symmetries in the physics literature:

- *A symTFT for continuous symmetries* [Brennan-Sun, 24].
- *SymTFTs for Continuous non-Abelian Symmetries* [Bonetti-Del Zotto-Minasian, 24].
- ...

The TFT is quantum BF-theory.

Remark

Classical BF in n-dimensions is an AKSZ sigma model with target $T^\vee[n-1]BG$ (formal nbhd at id: $\mathfrak{g}[-1] \oplus \mathfrak{g}^\vee[2-n]$). For $n = 3$, we have

$$T^\vee[2]BG \simeq B(T^\vee G).$$

BF in 3 dimensions is Chern-Simons for $T^\vee G \simeq G \ltimes \mathfrak{g}^\vee$.

Today's talk may be regarded as an elaboration of parts of Dan Freed's talk last November (Lie group quiche).

Broader context

Continuous symmetries in the physics literature:

- *A symTFT for continuous symmetries* [Brennan-Sun, 24].
- *SymTFTs for Continuous non-Abelian Symmetries* [Bonetti-Del Zotto-Minasian, 24].
- ...

The TFT is quantum BF-theory.

Remark

Classical BF in n-dimensions is an AKSZ sigma model with target $T^\vee[n-1]BG$ (formal nbhd at id: $\mathfrak{g}[-1] \oplus \mathfrak{g}^\vee[2-n]$). For $n = 3$, we have

$$T^\vee[2]BG \simeq B(T^\vee G).$$

BF in 3 dimensions is Chern-Simons for $T^\vee G \simeq G \ltimes \mathfrak{g}^\vee$.

Today's talk may be regarded as an elaboration of parts of Dan Freed's talk last November (Lie group quiche). See also:

- *Geometric categories for continuous gauging* [Stockall-Yu, 25]
- *Classical SymTFTs for Continuous Symmetries via Higher Symplectic Geometry* [Xu, 26]

Strategy

In algebro-geometric representation theory: for G an algebraic group

$$\mathrm{Rep}_G := \mathrm{QCoh}(BG).$$

It follows from the geometry of BG and good properties of QCoh that Rep_G is highly dualizable.

Strategy

In algebro-geometric representation theory: for G an algebraic group

$$\mathrm{Rep}_G := \mathrm{QCoh}(BG).$$

It follows from the geometry of BG and good properties of QCoh that Rep_G is highly dualizable.

Remark

These are *large* categories of representations. For maximum flexibility and control, work with *presentable* ∞*-categories*

$$\mathrm{Pr}^{\mathrm{L}} \subset \widehat{\mathrm{Cat}}_\infty,$$

the place to do categorical algebra.

Strategy

In algebro-geometric representation theory: for G an algebraic group

$$\mathrm{Rep}_G := \mathrm{QCoh}(BG).$$

It follows from the geometry of BG and good properties of QCoh that Rep_G is highly dualizable.

Remark

These are *large* categories of representations. For maximum flexibility and control, work with *presentable ∞-categories*

$$\mathrm{Pr}^{\mathrm{L}} \subset \widehat{\mathrm{Cat}}_\infty,$$

the place to do categorical algebra.

Idea

Develop a good theory of *quasi-coherent sheaves* on manifolds. For M a manifold, we should roughly have

$$\mathrm{QCoh}(M) = \{\text{Derived } \infty\text{-category of modules for the ring } C^\infty(M) \text{ in TopVect}\}$$

enriched in TopVect. Then set

$$\mathrm{Rep}_G := \mathrm{QCoh}(BG)$$

for G a Lie group. Note that $BG \in \mathrm{Shv}(\mathrm{Mfd})$, a *smooth stack*.

Outline

(1) What is TopVect? We need a good functional analytic setting for homological/homotopical algebra.

Outline

(1) What is TopVect? We need a good functional analytic setting for homological/homotopical algebra.

(2) Construct BF for G a compact Lie group via the cobordism hypothesis.

Outline

(1) What is TopVect? We need a good functional analytic setting for homological/homotopical algebra.

(2) Construct BF for G a compact Lie group via the cobordism hypothesis.

(3) Construct BF for G a compact Lie group by quantizing the classical AKSZ theory.

Acknowledgements

Various parts of this story are joint with subsets of collaboration members and affiliates {Claudia Scheimbauer, Will Stewart, Jackson van Dyke}.

Acknowledgements

Various parts of this story are joint with subsets of collaboration members and affiliates {Claudia Scheimbauer, Will Stewart, Jackson van Dyke}. Shoutout to these folks for many quality conversations.

Convenient categories for functional analysis

What should we mean by TopVect?

Convenient categories for functional analysis

What should we mean by TopVect? From your first functional analysis lecture, you might expect a reasonable choice to be

$$\{\text{Hausdorff, complete, locally convex topological vector spaces}\}\,.$$

Convenient categories for functional analysis

What should we mean by TopVect? From your first functional analysis lecture, you might expect a reasonable choice to be

$$\{\text{Hausdorff, complete, locally convex topological vector spaces}\}\,.$$

These are built up from Banach spaces. Let E be Hausdorff complete locally convex, then the topology of E is induced by a system $\{p_\alpha : E \to \mathbb{R}_{\geq 0}\}_\alpha$ of seminorms, and there is an equivalence

$$E \simeq \lim \left\{ \widehat{E/\ker(p_\alpha)} \right\}_\alpha$$

for $\widehat{E/\ker(p_\alpha)}$ the completion of the normed space $E/\ker(p_\alpha)$.

Convenient categories for functional analysis

What should we mean by TopVect? From your first functional analysis lecture, you might expect a reasonable choice to be

$$\{\text{Hausdorff, complete, locally convex topological vector spaces}\}\,.$$

These are built up from Banach spaces. Let E be Hausdorff complete locally convex, then the topology of E is induced by a system $\{p_\alpha : E \to \mathbb{R}_{\geq 0}\}_\alpha$ of seminorms, and there is an equivalence

$$E \simeq \lim \left\{ E/\widehat{\ker(p_\alpha)} \right\}_\alpha$$

for $E/\widehat{\ker(p_\alpha)}$ the completion of the normed space $E/\ker(p_\alpha)$. So the category of Hausdorff complete LCTVS is essentially Pro(Ban), formal inverse limits of Banach spaces and continuous maps.

Convenient categories for functional analysis

What should we mean by TopVect? From your first functional analysis lecture, you might expect a reasonable choice to be

$$\{\text{Hausdorff, complete, locally convex topological vector spaces}\}.$$

These are built up from Banach spaces. Let E be Hausdorff complete locally convex, then the topology of E is induced by a system $\{p_\alpha : E \to \mathbb{R}_{\geq 0}\}_\alpha$ of seminorms, and there is an equivalence

$$E \simeq \lim \left\{ E / \widehat{\ker(p_\alpha)} \right\}_\alpha$$

for $E / \widehat{\ker(p_\alpha)}$ the completion of the normed space $E/\ker(p_\alpha)$. So the category of Hausdorff complete LCTVS is essentially Pro(Ban), formal inverse limits of Banach spaces and continuous maps. E.g.

$$C^\infty(\mathbb{R}^n) \longrightarrow \ldots \longrightarrow C^{k+1}(\mathbb{R}^n) \longrightarrow C^k(\mathbb{R}^n) \longrightarrow \ldots .$$

Convenient categories for functional analysis

What should we mean by TopVect? From your first functional analysis lecture, you might expect a reasonable choice to be

$$\{\text{Hausdorff, complete, locally convex topological vector spaces}\}.$$

These are built up from Banach spaces. Let E be Hausdorff complete locally convex, then the topology of E is induced by a system $\{p_\alpha : E \to \mathbb{R}_{\geq 0}\}_\alpha$ of seminorms, and there is an equivalence

$$E \simeq \lim \left\{ \widehat{E/\ker(p_\alpha)} \right\}_\alpha$$

for $\widehat{E/\ker(p_\alpha)}$ the completion of the normed space $E/\ker(p_\alpha)$. So the category of Hausdorff complete LCTVS is essentially Pro(Ban), formal inverse limits of Banach spaces and continuous maps. E.g.

$$C^\infty(\mathbb{R}^n) \longrightarrow \ldots \longrightarrow C^{k+1}(\mathbb{R}^n) \longrightarrow C^k(\mathbb{R}^n) \longrightarrow \ldots.$$

From the perspective of a category theorist, this is not so good: Pro(Ban) is not *presentable*. We want something of the form

$$\mathrm{Ind}(\_),$$

formal *direct* limits.

Convenient categories for functional analysis

Two obvious superior candidates for TopVect stare us in the face:

Convenient categories for functional analysis

Two obvious superior candidates for TopVect stare us in the face:

(1) Ind(Ban) (essentially (complete) *bornological vector spaces*, axiomatize bounded sets instead of open sets).

Convenient categories for functional analysis

Two obvious superior candidates for TopVect stare us in the face:

(1) Ind(Ban) (essentially (complete) *bornological vector spaces*, axiomatize bounded sets instead of open sets).

(2) $\mathrm{Pro}(\mathrm{Ban})^{op} \simeq \mathrm{Ind}(\mathrm{Ban}^{op})$ (less classical objects, usually described via *stereotype duality* $\mathrm{Ban}^{op} \simeq \mathrm{Smith}$, the *Smith spaces*).

Convenient categories for functional analysis

Two obvious superior candidates for TopVect stare us in the face:

(1) Ind(Ban) (essentially (complete) *bornological vector spaces*, axiomatize bounded sets instead of open sets).

(2) $\mathrm{Pro(Ban)}^{op} \simeq \mathrm{Ind(Ban}^{op})$ (less classical objects, usually described via *stereotype duality* $\mathrm{Ban}^{op} \simeq \mathrm{Smith}$, the *Smith spaces*).

Remark

Ind(Ban) and Ind(Smith) are dual to each other in Pr^{L}.

Convenient categories for functional analysis

Two obvious superior candidates for TopVect stare us in the face:

(1) Ind(Ban) (essentially (complete) *bornological vector spaces*, axiomatize bounded sets instead of open sets).

(2) $\mathrm{Pro}(\mathrm{Ban})^{op} \simeq \mathrm{Ind}(\mathrm{Ban}^{op})$ (less classical objects, usually described via *stereotype duality* $\mathrm{Ban}^{op} \simeq \mathrm{Smith}$, the *Smith spaces*).

Remark

Ind(Ban) and Ind(Smith) are dual to each other in Pr^{L}.

Neither of these categories are abelian. Abelian completion yields the two modern approaches to functional analysis (over $\mathbb{R}$)

Convenient categories for functional analysis

Two obvious superior candidates for TopVect stare us in the face:

(1) Ind(Ban) (essentially (complete) *bornological vector spaces*, axiomatize bounded sets instead of open sets).

(2) $\mathrm{Pro}(\mathrm{Ban})^{op} \simeq \mathrm{Ind}(\mathrm{Ban}^{op})$ (less classical objects, usually described via *stereotype duality* $\mathrm{Ban}^{op} \simeq \mathrm{Smith}$, the *Smith spaces*).

Remark

Ind(Ban) and Ind(Smith) are dual to each other in Pr^{L}.

Neither of these categories are abelian. Abelian completion yields the two modern approaches to functional analysis (over $\mathbb{R}$)

(1) Ben-Bassat-Kelly-Kremnizer take the smallest (universal) abelian completion of Ind(Ban).

Convenient categories for functional analysis

Two obvious superior candidates for TopVect stare us in the face:

(1) Ind(Ban) (essentially (complete) *bornological vector spaces*, axiomatize bounded sets instead of open sets).

(2) $\mathrm{Pro}(\mathrm{Ban})^{op} \simeq \mathrm{Ind}(\mathrm{Ban}^{op})$ (less classical objects, usually described via *stereotype duality* $\mathrm{Ban}^{op} \simeq \mathrm{Smith}$, the *Smith spaces*).

Remark

Ind(Ban) and Ind(Smith) are dual to each other in Pr^{L}.

Neither of these categories are abelian. Abelian completion yields the two modern approaches to functional analysis (over $\mathbb{R}$)

(1) Ben-Bassat-Kelly-Kremnizer take the smallest (universal) abelian completion of Ind(Ban).

(2) Clausen-Scholze take Ind(Smith) and do a more complicated abelian completion that introduces non-locally convex objects $\Rightarrow$ *liquid condensed vector spaces*.

Convenient categories for functional analysis

Two obvious superior candidates for TopVect stare us in the face:

(1) Ind(Ban) (essentially (complete) *bornological vector spaces*, axiomatize bounded sets instead of open sets).

(2) $\mathrm{Pro}(\mathrm{Ban})^{op} \simeq \mathrm{Ind}(\mathrm{Ban}^{op})$ (less classical objects, usually described via *stereotype duality* $\mathrm{Ban}^{op} \simeq \mathrm{Smith}$, the *Smith spaces*).

Remark

Ind(Ban) and Ind(Smith) are dual to each other in Pr^{L}.

Neither of these categories are abelian. Abelian completion yields the two modern approaches to functional analysis (over $\mathbb{R}$)

(1) Ben-Bassat-Kelly-Kremnizer take the smallest (universal) abelian completion of Ind(Ban).

(2) Clausen-Scholze take Ind(Smith) and do a more complicated abelian completion that introduces non-locally convex objects $\Rightarrow$ *liquid condensed vector spaces*.

I will follow (1) and take

$$\mathrm{TopVect} := \{\text{Abelian envelope of } \mathrm{Ind}(\mathrm{Ban})\}\,.$$

Easier and has some technical advantages (afforded by local convexity).

What is this abelian envelope?

Right modules for endomorphisms of a generating projective.

What is this abelian envelope?

Right modules for endomorphisms of a generating projective.

Fact (Köthe)

Let Ban_κ *be the category of Banach spaces that contain a dense set of cardinality* $< \kappa$*, then* Ban_κ *has enough projectives, and* $B \in \mathrm{Ban}_\kappa$ *is projective if and only if* $B \cong l^1(\lambda)$ *for* $\lambda < \kappa$*.*

What is this abelian envelope?

Right modules for endomorphisms of a generating projective.

Fact (Köthe)

Let Ban_κ *be the category of Banach spaces that contain a dense set of cardinality* $< \kappa$*, then* Ban_κ *has enough projectives, and* $B \in \mathrm{Ban}_\kappa$ *is projective if and only if* $B \cong l^1(\lambda)$ *for* $\lambda < \kappa$*.*

It turns out that

$$\{\text{Abelian envelope of } \mathrm{Ind}(\mathrm{Ban}_{\omega_1})\} = \mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))},$$

What is this abelian envelope?

Right modules for endomorphisms of a generating projective.

Fact (Köthe)

Let Ban_κ *be the category of Banach spaces that contain a dense set of cardinality* $< \kappa$*, then* Ban_κ *has enough projectives, and* $B \in \mathrm{Ban}_\kappa$ *is projective if and only if* $B \cong l^1(\lambda)$ *for* $\lambda < \kappa$*.*

It turns out that

$$\{\text{Abelian envelope of } \mathrm{Ind}(\mathrm{Ban}_{\omega_1})\} = \mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))},$$

so

$$\mathrm{TopVect} := \{\text{Right modules for a countably infinite matrix algebra}\}$$

What is this abelian envelope?

Right modules for endomorphisms of a generating projective.

Fact (Köthe)

Let Ban_κ *be the category of Banach spaces that contain a dense set of cardinality* $< \kappa$*, then* Ban_κ *has enough projectives, and* $B \in \mathrm{Ban}_\kappa$ *is projective if and only if* $B \cong l^1(\lambda)$ *for* $\lambda < \kappa$*.*

It turns out that

$$\{\text{Abelian envelope of } \mathrm{Ind}(\mathrm{Ban}_{\omega_1})\} = \mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))},$$

so

$$\mathrm{TopVect} := \{\text{Right modules for a countably infinite matrix algebra}\}$$

$\mathrm{End}(l^1(\mathbb{N}))$ are (countably infinite) $l^{1,\infty}$-*matrices*:

- columns are in $l^1(\mathbb{N})$.
- the row of absolute sums is bounded (in $l^\infty(\mathbb{N}) = l^1(\mathbb{N})^\vee$).

What is this abelian envelope?

Right modules for endomorphisms of a generating projective.

Fact (Köthe)

Let Ban_κ *be the category of Banach spaces that contain a dense set of cardinality* $< \kappa$*, then* Ban_κ *has enough projectives, and* $B \in \mathrm{Ban}_\kappa$ *is projective if and only if* $B \cong l^1(\lambda)$ *for* $\lambda < \kappa$*.*

It turns out that

$$\{\text{Abelian envelope of } \mathrm{Ind}(\mathrm{Ban}_{\omega_1})\} = \mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))},$$

so

$$\mathrm{TopVect} := \{\text{Right modules for a countably infinite matrix algebra}\}$$

$\mathrm{End}(l^1(\mathbb{N}))$ are (countably infinite) $l^{1,\infty}$*-matrices*:

- columns are in $l^1(\mathbb{N})$.
- the row of absolute sums is bounded (in $l^\infty(\mathbb{N}) = l^1(\mathbb{N})^\vee$).

This is old stuff, goes back to Waelbroeck ('60s).

Aggregated vector spaces

Think of an element of $\mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))}$ as a pair

$$(V, l^\infty(V))$$

of a vector space together with a formal set of 'bounded sequences', which can be paired with $l^{1,\infty}$-matrices to produce new bounded sequences.

Aggregated vector spaces

Think of an element of $\mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))}$ as a pair

$$(V, l^\infty(V))$$

of a vector space together with a formal set of ‘bounded sequences’, which can be paired with $l^{1,\infty}$-matrices to produce new bounded sequences. I will write

$$\mathcal{A}\mathrm{Vect} := \mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))},$$

aggregated vector spaces.

Aggregated vector spaces

Think of an element of $\mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))}$ as a pair

$$(V, l^\infty(V))$$

of a vector space together with a formal set of ‘bounded sequences’, which can be paired with $l^{1,\infty}$-matrices to produce new bounded sequences. I will write

$$\mathcal{A}\mathrm{Vect} := \mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))},$$

aggregated vector spaces. Obvious derived ∞-category:

$$\mathcal{A}\mathrm{Mod} = \mathrm{RMod}_{\mathrm{End}(l^1(\mathbb{N}))}(\mathcal{S}\mathrm{p}).$$

Aggregated vector spaces

Think of an element of $\mathsf{RMod}_{\mathsf{End}(l^1(\mathbb{N}))}$ as a pair

$$(V, l^\infty(V))$$

of a vector space together with a formal set of 'bounded sequences', which can be paired with $l^{1,\infty}$-matrices to produce new bounded sequences. I will write

$$\mathcal{A}\mathsf{Vect} := \mathsf{RMod}_{\mathsf{End}(l^1(\mathbb{N}))},$$

aggregated vector spaces. Obvious derived ∞-category:

$$\mathcal{A}\mathsf{Mod} = \mathsf{RMod}_{\mathsf{End}(l^1(\mathbb{N}))}(\mathcal{S}\mathrm{p}).$$

Fact

The functor

$$\mathsf{LCTVS} \longrightarrow \mathcal{A}\mathsf{Vect}, \qquad V \longmapsto \mathrm{Hom}_{\mathsf{LCTVS}}(l^1(\mathbb{N}), V)$$

is fully faithful on an enormous class of locally convex TVS (the ultrabornological *ones).*

Aggregated vector spaces

Think of an element of $\mathsf{RMod}_{\mathsf{End}(l^1(\mathbb{N}))}$ as a pair

$$(V, l^\infty(V))$$

of a vector space together with a formal set of 'bounded sequences', which can be paired with $l^{1,\infty}$-matrices to produce new bounded sequences. I will write

$$\mathcal{A}\mathsf{Vect} := \mathsf{RMod}_{\mathsf{End}(l^1(\mathbb{N}))},$$

aggregated vector spaces. Obvious derived ∞-category:

$$\mathcal{A}\mathsf{Mod} = \mathsf{RMod}_{\mathsf{End}(l^1(\mathbb{N}))}(\mathcal{S}\mathrm{p}).$$

Fact

The functor

$$\mathsf{LCTVS} \longrightarrow \mathcal{A}\mathsf{Vect}, \qquad V \longmapsto \mathrm{Hom}_{\mathsf{LCTVS}}(l^1(\mathbb{N}), V)$$

is fully faithful on an enormous class of locally convex TVS (the ultrabornological *ones).*

Fact

$\mathcal{A}\mathsf{Mod}^{\widehat{\otimes}} \in \mathsf{Calg}(\mathrm{Pr}^{\mathrm{L}})$ *is symmetric monoidal with a completed tensor product* $\widehat{\otimes}$ *preserving colimits in both variables, defined by*

$$l^1(S)\widehat{\otimes} l^1(T) \cong l^1(S \times T).$$

Aggregated n-vector spaces/categories

The symmetric monoidal $(\infty, 2)$-category of presentable $\mathcal{A}$Mod-enriched ∞-categories:

$$\mathsf{Pr}^{\mathrm{L}}_{\mathcal{A}\mathsf{Mod}}.$$

Aggregated n-vector spaces/categories

The symmetric monoidal $(\infty, 2)$-category of presentable $\mathcal{A}\mathsf{Mod}$-enriched ∞-categories:

$$\mathsf{Pr}^{\mathrm{L}}_{\mathcal{A}\mathsf{Mod}}.$$

Feed into the higher Morita category $\Rightarrow$ sym. monoidal $(\infty, n+2)$-category

$$\mathsf{Mor}_n(\mathsf{Pr}^{\mathrm{L}}_{\mathcal{A}\mathsf{Mod}}) = \{\text{Presentably } \mathbb{E}_n\text{-monoidal } \mathcal{A}\mathsf{Mod}\text{-enriched categories}\}.$$

Goal

Let G be a compact Lie group. Find an $(n+1)$-dualizable object

$$\mathcal{A}\mathrm{Rep}_G \in \mathrm{Mor}_n(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

of *aggregated (smooth) G-representations*, containing *at least* all smooth G-representations on Fréchet spaces, like

$$C^\infty(G),$$

the regular representation, but also

$$\mathcal{D}(G),$$

distributions on G.

Dualizbility in higher Morita categories

Theorem (Scheimbauer-S-Stewart)

$\mathcal{C} \in \mathrm{Mor}_n(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$ *is* $(n+1)$*-dualizable* $\Leftrightarrow$ $\mathcal{C}$ *is dualizable as a module category for*

$$\int_{S^k} \mathcal{C}$$

for $-1 \leq k \leq n-1$.

Dualizbility in higher Morita categories

Theorem (Scheimbauer-S-Stewart)

$\mathcal{C} \in \mathrm{Mor}_n(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$ *is* $(n+1)$*-dualizable* $\Leftrightarrow$ $\mathcal{C}$ *is dualizable as a module category for*

$$\int_{S^k} \mathcal{C}$$

for $-1 \leq k \leq n-1$*.*

Corollary (Generalizing (Brochier-Jordan-Snyder, '18))

For an object of $\mathrm{Mor}_n(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$ *to be* $(n+1)$*-dualizable, it suffices to be* Gaitsgory rigid*.*

Dualizbility in higher Morita categories

Theorem (Scheimbauer-S-Stewart)

$\mathcal{C} \in \mathrm{Mor}_n(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$ is $(n+1)$-dualizable $\Leftrightarrow$ $\mathcal{C}$ is dualizable as a module category for

$$\int_{S^k} \mathcal{C}$$

for $-1 \leq k \leq n-1$.

Corollary (Generalizing (Brochier-Jordan-Snyder, '18))

For an object of $\mathrm{Mor}_n(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$ to be $(n+1)$-dualizable, it suffices to be Gaitsgory rigid*.*

Rigidity is very strong form of self-duality: if an $\mathbb{E}_n$-monoidal $\mathcal{C}$ is rigid, then $\mathcal{C}$ is also rigid $\Rightarrow$ dualizable and self-dual as a module category for $\int_{S^k} \mathcal{C}$ for $-1 \leq k \leq n-1$.

Gaitsgory rigidity

Definition

Let $\mathcal{V}^{\otimes} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. A monoidal $\mathcal{V}$-enriched ∞-category $\mathcal{B}$ is *Gaitsgory rigid* if:

(1) The associativity square

$$\begin{tikzcd} \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[d] \\ \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[r] & \mathcal{B} \end{tikzcd}$$

is vertically right adjointable (internal to $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$).

(2) The map $\mathcal{V} \to \mathcal{B} \to \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B}$ is the coevaluation of a duality.

Gaitsgory rigidity

Definition

Let $\mathcal{V}^{\otimes} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. A monoidal $\mathcal{V}$-enriched ∞-category $\mathcal{B}$ is *Gaitsgory rigid* if:

(1) The associativity square

$$\begin{tikzcd} \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[d] \\ \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[r] & \mathcal{B} \end{tikzcd}$$

is vertically right adjointable (internal to $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$).

(2) The map $\mathcal{V} \to \mathcal{B} \to \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B}$ is the coevaluation of a duality.

A Gaitsgory rigid $\mathcal{B}^{\otimes}$ is dualizable, self-dual and a Frobenius algebra object of $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$.

Gaitsgory rigidity

Definition

Let $\mathcal{V}^{\otimes} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. A monoidal $\mathcal{V}$-enriched ∞-category $\mathcal{B}$ is *Gaitsgory rigid* if:

(1) The associativity square

$$\begin{tikzcd} \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[d] \\ \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[r] & \mathcal{B} \end{tikzcd}$$

is vertically right adjointable (internal to $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$).

(2) The map $\mathcal{V} \to \mathcal{B} \to \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B}$ is the coevaluation of a duality.

A Gaitsgory rigid $\mathcal{B}^{\otimes}$ is dualizable, self-dual and a Frobenius algebra object of $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$.

Remark

If $\mathcal{B}$ is compactly generated and $\mathcal{V} = \mathcal{S}\mathrm{p}$, then $\mathcal{B}$ Gaitsgory rigid $\Leftrightarrow$ every compact object has a dual.

Gaitsgory rigidity

Definition

Let $\mathcal{V}^{\otimes} \in \mathsf{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. A monoidal $\mathcal{V}$-enriched ∞-category $\mathcal{B}$ is *Gaitsgory rigid* if:

(1) The associativity square

$$\begin{tikzcd} \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[d] \\ \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \arrow[r] & \mathcal{B} \end{tikzcd}$$

is vertically right adjointable (internal to $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$).

(2) The map $\mathcal{V} \to \mathcal{B} \to \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B}$ is the coevaluation of a duality.

A Gaitsgory rigid $\mathcal{B}^{\otimes}$ is dualizable, self-dual and a Frobenius algebra object of $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$.

Remark

If $\mathcal{B}$ is compactly generated and $\mathcal{V} = \mathcal{S}\mathrm{p}$, then $\mathcal{B}$ Gaitsgory rigid $\Leftrightarrow$ every compact object has a dual.

The ∞-category $\mathcal{A}\mathsf{Rep}_G$ to come is rigid in $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathsf{Mod}}$ but *not* compactly generated!

Quasi-coherent sheaves in differential geometry

We want to say $\mathcal{A}\mathrm{Rep}_G = \mathrm{QCoh}(BG)$.

Quasi-coherent sheaves in differential geometry

We want to say $\mathcal{A}\mathrm{Rep}_G = \mathrm{QCoh}(BG)$.

Definition

Let M be a manifold, then C^∞_M is a sheaf of aggregated commutative algebras on M.

$$\mathrm{QCoh}(M) := \{\text{Sheaves of aggregated } C^\infty_M\text{-modules.}\}$$

Quasi-coherent sheaves in differential geometry

We want to say $\mathcal{A}\mathrm{Rep}_G = \mathrm{QCoh}(BG)$.

Definition

Let M be a manifold, then C^∞_M is a sheaf of aggregated commutative algebras on M.

$$\mathrm{QCoh}(M) := \{\text{Sheaves of aggregated } C^\infty_M\text{-modules.}\}$$

QCoh is a sheaf

$$\mathrm{Mfd}^{op} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

of presentably symmetric monoidal $\mathcal{A}$Mod-enriched ∞-categories.

Quasi-coherent sheaves in differential geometry

We want to say $\mathcal{A}\mathrm{Rep}_G = \mathrm{QCoh}(BG)$.

Definition

Let M be a manifold, then C^∞_M is a sheaf of aggregated commutative algebras on M.

$$\mathrm{QCoh}(M) := \{\text{Sheaves of aggregated } C^\infty_M\text{-modules.}\}$$

QCoh is a sheaf

$$\mathrm{Mfd}^{op} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

of presentably symmetric monoidal $\mathcal{A}$Mod-enriched ∞-categories.

$$(f : M \to N) \longmapsto (f^* : \mathrm{QCoh}(M) \leftarrow \mathrm{QCoh}(N))$$

Quasi-coherent sheaves in differential geometry

We want to say $\mathcal{A}\mathrm{Rep}_G = \mathrm{QCoh}(BG)$.

Definition

Let M be a manifold, then C^∞_M is a sheaf of aggregated commutative algebras on M.

$$\mathrm{QCoh}(M) := \{\text{Sheaves of aggregated } C^\infty_M\text{-modules.}\}$$

QCoh is a sheaf

$$\mathrm{Mfd}^{op} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

of presentably symmetric monoidal $\mathcal{A}$Mod-enriched ∞-categories.

$$(f : M \to N) \longmapsto (f^* : \mathrm{QCoh}(M) \leftarrow \mathrm{QCoh}(N))$$

Extend to smooth stacks Shv(Mfd) by right Kan extension:

$$\mathrm{QCoh}(X) = \lim_{M \to X} \mathrm{QCoh}(X)$$

Quasi-coherent sheaves in differential geometry

We want to say $\mathcal{A}\mathrm{Rep}_G = \mathrm{QCoh}(BG)$.

Definition

Let M be a manifold, then C^∞_M is a sheaf of aggregated commutative algebras on M.

$$\mathrm{QCoh}(M) := \{\text{Sheaves of aggregated } C^\infty_M\text{-modules.}\}$$

QCoh is a sheaf

$$\mathrm{Mfd}^{op} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

of presentably symmetric monoidal $\mathcal{A}$Mod-enriched ∞-categories.

$$(f : M \to N) \longmapsto (f^* : \mathrm{QCoh}(M) \leftarrow \mathrm{QCoh}(N))$$

Extend to smooth stacks Shv(Mfd) by right Kan extension:

$$\mathrm{QCoh}(X) = \lim_{M \to X} \mathrm{QCoh}(X)$$

Proper base change holds: given a proper pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}, \quad \text{then} \quad \begin{tikzcd} \mathrm{QCoh}(W) \arrow[r] \arrow[d] & \mathrm{QCoh}(Y) \arrow[d] \\ \mathrm{QCoh}(Z) \arrow[r] & \mathrm{QCoh}(X) \end{tikzcd}$$

is right adjointable internal to $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$.

Smooth representations

Definition

Let $G = \; G_1 \rightrightarrows G_0 \;$ be a Lie groupoid, then the presentable symmetric monoidal $\mathcal{A}$Mod-enriched ∞-category $\mathcal{A}\mathrm{Rep}_G$ of aggregated G-representations is

$$\mathrm{QCoh}([G_0/G_1]).$$

The limit of the diagram

$$\mathrm{QCoh}([G_0/G_1]) = \lim \left\{ \; \mathrm{QCoh}(G_0) \rightrightarrows \mathrm{QCoh}(G_1) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathrm{QCoh}(G_1 \times_{G_0} G_1) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots \; \right\}$$

of ∞-categories.

Smooth representations

Definition

Let $G = \begin{tikzcd} G_1 \arrow[r, shift left] \arrow[r, shift right] & G_0 \end{tikzcd}$ be a Lie groupoid, then the presentable symmetric monoidal $\mathcal{A}\mathrm{Mod}$-enriched ∞-category $\mathcal{A}\mathrm{Rep}_G$ of aggregated G-representations is

$$\mathrm{QCoh}([G_0/G_1]).$$

The limit of the diagram

$$\mathrm{QCoh}([G_0/G_1]) = \lim \left\{ \begin{tikzcd} \mathrm{QCoh}(G_0) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{QCoh}(G_1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{QCoh}(G_1 \times_{G_0} G_1) \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \end{tikzcd} \right\}$$

of ∞-categories. Let G be a Lie group, then $C^\infty(G)$ is a commutative Hopf algebra object of $\mathcal{A}\mathrm{Mod}$. By comonadic descent:

$$\mathcal{A}\mathrm{Rep}_G \simeq \mathrm{coMod}_{C^\infty(G)}(\mathcal{A}\mathrm{Mod}).$$

Smooth representations

Definition

Let $G = \;\; G_1 \rightrightarrows G_0$ be a Lie groupoid, then the presentable symmetric monoidal $\mathcal{A}$Mod-enriched ∞-category $\mathcal{A}\mathrm{Rep}_G$ of aggregated G-representations is

$$\mathrm{QCoh}([G_0/G_1]).$$

The limit of the diagram

$$\mathrm{QCoh}([G_0/G_1]) = \lim \left\{ \mathrm{QCoh}(G_0) \rightrightarrows \mathrm{QCoh}(G_1) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathrm{QCoh}(G_1 \times_{G_0} G_1) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots \right\}$$

of ∞-categories. Let G be a Lie group, then $C^\infty(G)$ is a commutative Hopf algebra object of $\mathcal{A}$Mod. By comonadic descent:

$$\mathcal{A}\mathrm{Rep}_G \simeq \mathrm{coMod}_{C^\infty(G)}(\mathcal{A}\mathrm{Mod}).$$

For V a Fréchet space, a $C^\infty(G)$-comodule structure is a map

$$V \longrightarrow C^\infty(G)\widehat{\otimes} V \cong C^\infty(G, V), \quad \text{adjoint to} \quad G \times V \longrightarrow V,$$

a smooth G-representation.

Smooth representations

Definition

Let $G = G_1 \rightrightarrows G_0$ be a Lie groupoid, then the presentable symmetric monoidal $\mathcal{A}$Mod-enriched ∞-category $\mathcal{A}\mathrm{Rep}_G$ of aggregated G-representations is

$$\mathrm{QCoh}([G_0/G_1]).$$

The limit of the diagram

$$\mathrm{QCoh}([G_0/G_1]) = \lim \left\{ \mathrm{QCoh}(G_0) \rightrightarrows \mathrm{QCoh}(G_1) \substack{\textstyle\rightarrow \\[-1ex] \textstyle\rightarrow \\[-1ex] \textstyle\rightarrow} \mathrm{QCoh}(G_1 \times_{G_0} G_1) \substack{\textstyle\rightarrow \\[-1ex] \textstyle\rightarrow \\[-1ex] \textstyle\rightarrow \\[-1ex] \textstyle\rightarrow} \cdots \right\}$$

of ∞-categories. Let G be a Lie group, then $C^\infty(G)$ is a commutative Hopf algebra object of $\mathcal{A}$Mod. By comonadic descent:

$$\mathcal{A}\mathrm{Rep}_G \simeq \mathrm{coMod}_{C^\infty(G)}(\mathcal{A}\mathrm{Mod}).$$

For V a Fréchet space, a $C^\infty(G)$-comodule structure is a map

$$V \longrightarrow C^\infty(G)\widehat{\otimes} V \cong C^\infty(G, V), \quad \text{adjoint to} \quad G \times V \longrightarrow V,$$

a smooth G-representation. For G a Lie group, there is a fiber functor equivalence

$$\mathcal{A}\mathrm{Rep}_G \simeq \mathcal{A}\mathrm{RMod}^{\mathrm{firm}}_{C_c^\infty(G)}$$

for $C_c^\infty(G)$ the (nonunital firm) *smooth convolution algebra* of G.

Rigidity for $\mathcal{A}\mathrm{Rep}_G$

Theorem (S.)

Let X be a proper *Lie groupoid quotient (ie $X = BG$ for G compact), then*

$$\mathrm{QCoh}(X) \otimes_{\mathcal{A}\mathrm{Mod}} \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(X \times X)$$

is an equivalence.

Rigidity for $\mathcal{A}\mathrm{Rep}_G$

Theorem (S.)

Let X be a proper *Lie groupoid quotient (ie $X = BG$ for G compact), then*

$$\mathrm{QCoh}(X) \otimes_{\mathcal{A}\mathrm{Mod}} \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(X \times X)$$

is an equivalence.

This is quite nontrivial, depends on the existence of *normalized* Haar measures on compact Lie groups.

Rigidity for $\mathcal{A}\mathrm{Rep}_G$

Theorem (S.)

Let X be a proper *Lie groupoid quotient (ie $X = BG$ for G compact), then*

$$\mathrm{QCoh}(X) \otimes_{\mathcal{A}\mathrm{Mod}} \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(X \times X)$$

is an equivalence.

This is quite nontrivial, depends on the existence of *normalized* Haar measures on compact Lie groups. This would fail dramatically without the 'topological' enrichment (compare $C^\infty(M) \otimes C^\infty(N) \ncong C^\infty(M \times N)$).

Rigidity for $\mathcal{A}\mathrm{Rep}_G$

Theorem (S.)

Let X be a proper *Lie groupoid quotient (ie $X = BG$ for G compact), then*

$$\mathrm{QCoh}(X) \otimes_{\mathcal{A}\mathrm{Mod}} \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(X \times X)$$

is an equivalence.

This is quite nontrivial, depends on the existence of *normalized* Haar measures on compact Lie groups. This would fail dramatically without the 'topological' enrichment (compare $C^\infty(M) \otimes C^\infty(N) \ncong C^\infty(M \times N)$).

Corollary

$\mathrm{QCoh}(X)$ *is Gaitsgory rigid in* $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$.

Rigidity for $\mathcal{A}\mathrm{Rep}_G$

Theorem (S.)

Let X be a proper *Lie groupoid quotient (ie $X = BG$ for G compact), then*

$$\mathrm{QCoh}(X) \otimes_{\mathcal{A}\mathrm{Mod}} \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(X \times X)$$

is an equivalence.

This is quite nontrivial, depends on the existence of *normalized* Haar measures on compact Lie groups. This would fail dramatically without the 'topological' enrichment (compare $C^\infty(M) \otimes C^\infty(N) \ncong C^\infty(M \times N)$).

Corollary

$\mathrm{QCoh}(X)$ *is Gaitsgory rigid in* $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$.

Proof: apply proper base change; in the pullback diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"] & X \times X \arrow[d, "\mathrm{id} \times \Delta"] \\ X \times X \arrow[r, "\Delta \times \mathrm{id}"] & X \times X \times X, \end{tikzcd}$$

of smooth stacks, all maps are proper.

Corollary

Let G a compact Lie group (or a proper Lie groupoid). For any n, the symmetric monoidal ∞-category $\mathcal{A}\mathrm{Rep}_G^{\widehat{\otimes}}$ determines a once-categorified theory $(n+1)$-dimensional theory

$$\sigma^{n+1}_{BF(G)} : \mathrm{Bord}_n^{\mathrm{fr}\otimes} \longrightarrow \mathrm{Mor}_{n-1}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

.

Theorem (Scheimbauer)

G a compact Lie groupoid (or a proper Lie groupoid). The composite

$$\mathrm{Bord}_{n-1}^{\mathrm{fr}\otimes} \hookrightarrow \mathrm{Bord}_{n}^{\mathrm{fr}\otimes} \xrightarrow{\sigma^{n+1}_{BF(G)}} \mathrm{Mor}_{n-1}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

is factorization homology.

Theorem (Scheimbauer)

G a compact Lie groupoid (or a proper Lie groupoid). The composite

$$\mathrm{Bord}_{n-1}^{\mathrm{fr}\otimes} \hookrightarrow \mathrm{Bord}_n^{\mathrm{fr}\otimes} \xrightarrow{\sigma^{n+1}_{BF(G)}} \mathrm{Mor}_{n-1}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

is factorization homology.

Can we construct $\sigma^{n+1}_{BF(G)}$ on all of $\mathrm{Bord}_n^{\mathrm{fr}\otimes}$, by quantization of classical BF?

Theorem (Scheimbauer)

G a compact Lie groupoid (or a proper Lie groupoid). The composite

$$\mathrm{Bord}_{n-1}^{\mathrm{fr}\otimes} \hookrightarrow \mathrm{Bord}_{n}^{\mathrm{fr}\otimes} \xrightarrow{\sigma^{n+1}_{BF(G)}} \mathrm{Mor}_{n-1}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

is factorization homology.

Can we construct $\sigma^{n+1}_{BF(G)}$ on all of $\mathrm{Bord}_{n}^{\mathrm{fr}\otimes}$, by quantization of classical BF? Yes, with some more machinery.

Theorem (Scheimbauer)

G a compact Lie groupoid (or a proper Lie groupoid). The composite

$$\mathrm{Bord}_{n-1}^{\mathrm{fr}\otimes} \hookrightarrow \mathrm{Bord}_n^{\mathrm{fr}\otimes} \xrightarrow{\sigma_{BF(G)}^{n+1}} \mathrm{Mor}_{n-1}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

is factorization homology.

Can we construct $\sigma_{BF(G)}^{n+1}$ on all of $\mathrm{Bord}_n^{\mathrm{fr}\otimes}$, by quantization of classical BF? Yes, with some more machinery.

Definition

For $n > 1$, there is a sym. monoidal $(\infty, n+1)$-category $n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$ of *n-presentable* $\mathcal{A}\mathrm{Mod}$*-enriched ∞-categories*.

Theorem (Scheimbauer)

G a compact Lie groupoid (or a proper Lie groupoid). The composite

$$\mathrm{Bord}_{n-1}^{\mathrm{fr}\otimes} \hookrightarrow \mathrm{Bord}_{n}^{\mathrm{fr}\otimes} \xrightarrow{\sigma^{n+1}_{BF(G)}} \mathrm{Mor}_{n-1}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

is factorization homology.

Can we construct $\sigma^{n+1}_{BF(G)}$ on all of $\mathrm{Bord}_{n}^{\mathrm{fr}\otimes}$, by quantization of classical BF? Yes, with some more machinery.

Definition

For $n > 1$, there is a sym. monoidal $(\infty, n+1)$-category $n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$ of *n-presentable $\mathcal{A}$Mod-enriched ∞-categories*.

Morally

$$n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}} = \mathrm{Pr}^{\mathrm{L}}_{(n-1)\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}}$$

up to set-theoretic technicalities [Stefanich, '22]. In particular, $(n-1)\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$ is the monoidal unit of $n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$.

Theorem (Scheimbauer)

G a compact Lie groupoid (or a proper Lie groupoid). The composite

$$\mathrm{Bord}_{n-1}^{\mathrm{fr}\otimes} \hookrightarrow \mathrm{Bord}_{n}^{\mathrm{fr}\otimes} \xrightarrow{\sigma^{n+1}_{BF(G)}} \mathrm{Mor}_{n-1}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}})$$

is factorization homology.

Can we construct $\sigma^{n+1}_{BF(G)}$ on all of $\mathrm{Bord}_{n}^{\mathrm{fr}\otimes}$, by quantization of classical BF? Yes, with some more machinery.

Definition

For $n > 1$, there is a sym. monoidal $(\infty, n+1)$-category $n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$ of *n-presentable $\mathcal{A}$Mod-enriched ∞-categories*.

Morally

$$n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}} = \mathrm{Pr}^{\mathrm{L}}_{(n-1)\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}}$$

up to set-theoretic technicalities [Stefanich, '22]. In particular, $(n-1)\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$ is the monoidal unit of $n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$.

Remark

By taking iterated module categories, there is a fully faithful sym. monoidal embedding

$$\mathrm{Mor}_{n}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}) \subset (n+1)\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}.$$

Sheaf theories

Theorem (S, QCoh is a *six-functor formalism*)

The following hold true.

(1) QCoh *extends to* derived manifolds *(think NQ-supermanifolds in degrees* ≤ 0*) and to* derived differentiable stacks *(think NQ-supermanifolds in unbounded degrees).*

(2) *For any map* $f : X \to Y$ *of such, there is a functor*

$$f_! : \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(Y),$$

the pushforward with proper support*. This turns* QCoh *into a functor*

$$\mathrm{Span}(\mathrm{d}C^\infty\mathrm{St}) \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}.$$

Based on hard work by many people (Gaitsgory-Rozenbyum, Liu-Zheng, Mann,...)

Sheaf theories

Theorem (S, QCoh is a *six-functor formalism*)

The following hold true.

(1) QCoh *extends to* derived manifolds *(think NQ-supermanifolds in degrees* ≤ 0*) and to* derived differentiable stacks *(think NQ-supermanifolds in unbounded degrees).*

(2) *For any map* $f : X \to Y$ *of such, there is a functor*

$$f_! : \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(Y),$$

the pushforward with proper support*. This turns* QCoh *into a functor*

$$\mathrm{Span}(\mathrm{d}C^\infty\mathrm{St}) \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}.$$

Based on hard work by many people (Gaitsgory-Rozenbyum, Liu-Zheng, Mann,...)
E.g. for

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \\ Z \arrow[r, "k"] & W \end{tikzcd}$$

a pullback in $\mathrm{d}C^\infty\mathrm{St}$, there is base change equivalence

$$f_! h^* \simeq g^* k_!.$$

+ an infinite collection of intricate coherence data.

Linearizing spans

Theorem (In progress)

Define $n\mathrm{QCoh} \in n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$ *as*

$$n\mathrm{QCoh}(\_) := \mathrm{Mod}_{\mathrm{Mod}_{\cdots \mathrm{QCoh}(\_)}}$$

on derived manifolds and on derived stacks by right Kan extension. Then $n\mathrm{QCoh}(\_)$ *extends to a functor*

$$n\mathrm{Span}(\mathrm{d}C^\infty\mathrm{St}) \longrightarrow n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}.$$

Symmetric monoidal after applying some adjectives to $\mathrm{d}C^\infty\mathrm{St}$*.*

For G a Lie groupoid (or proper Lie groupoid), the theory $\sigma^{n+1}_{BF(G)}$ is the composite

$$\mathrm{Bord}_n^{\mathrm{fr}\otimes} \xrightarrow{\mathrm{Map}((\_)_{Betti}, BG)} n\mathrm{Span}(\mathrm{d}C^\infty\mathrm{St}) \xrightarrow{n\mathrm{QCoh}} n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}.$$

Linearizing spans

Theorem (In progress)

Define $n\mathrm{QCoh} \in n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}$ *as*

$$n\mathrm{QCoh}(\_) := \mathrm{Mod}_{\mathrm{Mod}_{\cdots \mathrm{QCoh}(\_)}}$$

on derived manifolds and on derived stacks by right Kan extension. Then $n\mathrm{QCoh}(\_)$ *extends to a functor*

$$n\mathrm{Span}(\mathrm{d}C^{\infty}\mathrm{St}) \longrightarrow n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}.$$

Symmetric monoidal after applying some adjectives to $\mathrm{d}C^{\infty}\mathrm{St}$.

For G a Lie groupoid (or proper Lie groupoid), the theory $\sigma^{n+1}_{BF(G)}$ is the composite

$$\mathrm{Bord}^{\mathrm{fr}\otimes}_{n} \xrightarrow{\mathrm{Map}((\_)_{Betti}, BG)} n\mathrm{Span}(\mathrm{d}C^{\infty}\mathrm{St}) \xrightarrow{n\mathrm{QCoh}} n\mathrm{Pr}^{\mathrm{L}}_{\mathcal{A}\mathrm{Mod}}.$$

Here

$$\mathrm{Map}(M_{Betti}, BG) = \{G\text{-local systems on } M\} = \{\text{Flat } G\text{-bundles on } M\}.$$

Hard work by Calaque-Haugseng-Scheimbauer.

$\sigma^{n+1}_{BF(G)}$ as fully extended geometric quantization

Fully extended (once-categorified) classical BF theory is defined by

$$\mathrm{Map}((\_)_{Betti}, T^{\vee}[n]BG) : \mathrm{Bord}_n \longrightarrow \mathrm{Lag}_n.$$

to shifted symplectic stacks and Lagrangian correspondences between them (Weinstein's category).

$\sigma^{n+1}_{BF(G)}$ as fully extended geometric quantization

Fully extended (once-categorified) classical BF theory is defined by

$$\mathrm{Map}((\_)_{Betti}, T^\vee[n]BG) : \mathrm{Bord}_n \longrightarrow \mathrm{Lag}_n.$$

to shifted symplectic stacks and Lagrangian correspondences between them (Weinstein's category). The map

$$T^\vee[n]BG \longrightarrow BG$$

is an *n-shifted Lagrangian fibration*. Interpret this [Safranov, '23] as a polarization for the (trivial) prequantum gerbe whose curvature is the canonical shifted symplectic structure on $T^\vee[n]BG$.

$\sigma^{n+1}_{BF(G)}$ as fully extended geometric quantization

Fully extended (once-categorified) classical BF theory is defined by

$$\mathrm{Map}((\_)_{Betti}, T^\vee[n]BG) : \mathrm{Bord}_n \longrightarrow \mathrm{Lag}_n.$$

to shifted symplectic stacks and Lagrangian correspondences between them (Weinstein's category). The map

$$T^\vee[n]BG \longrightarrow BG$$

is an *n-shifted Lagrangian fibration*. Interpret this [Safranov, '23] as a polarization for the (trivial) prequantum gerbe whose curvature is the canonical shifted symplectic structure on $T^\vee[n]BG$. Then $n\mathrm{QCoh}(BG)$ are the polarized sections of the prequantum gerbe $\Rightarrow \sigma^{n+1}_{BF(G)}$ is the fully extended *geometric quantization* of classical fully extended BF, as expected from the physics literature.