Non-invertible Symmetries of Weyl Fermions, and Fermion-Boundary Scattering

Yunqin Zheng
School of Quantum & Kavli Institute for Theoretical Sciences, UCAS

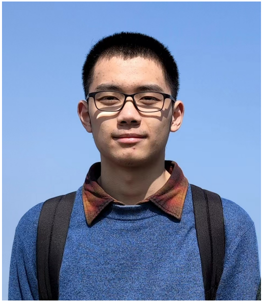
Symmetries 2026, Paris

[Pengcheng Wei, YZ, 2605.19363]

June 25th, 2026

Acknowledgements

Pengcheng Wei



- Our results are heavily inspired by [Ohmori,Watanabe,22'], [Beest,Boyle-Smith,Delmastro,Komargodski,Tong,23'].
- Recent overlapping works: [Arias-Tamargo,Boyle-Smith,Mouland,Hutt,26'],[Antinucci,Copetti,Galati,Rizi,26'], [Talks by A.Antinucci and L.Fidkowski]

Weyl Fermions in 1+1d

- n left moving complex Weyl fermions in 1+1d

$$\int d^2x \sum_{i=1}^{n} i\psi_i^\dagger(\partial_t - \partial_x)\psi_i$$

Weyl Fermions in 1+1d

- *n* left moving complex Weyl fermions in 1+1d

$$\int d^2x \sum_{i=1}^{n} i\psi_i^\dagger(\partial_t - \partial_x)\psi_i$$

- Simplest quantum field theories in 1+1d
    - Free field theory.
    - Chiral
    - Built-up from smallest central charge $c = \frac{1}{2}$.
- Ubitqutious in condensed matter physics, e.g. edge of integer quantum hall states.

Symmetries in Weyl Fermions

$$\int d^2x \sum_{i=1}^{n} i\psi_i^\dagger(\partial_t - \partial_x)\psi_i$$

- **Main question**: What are the global symmetries?

Symmetries in Weyl Fermions

$$\int d^2x \sum_{i=1}^{n} i\psi_i^\dagger(\partial_t - \partial_x)\psi_i$$

- **Main question**: What are the global symmetries?
- Obvious symmetry: $U(n) \rtimes \mathbb{Z}_2^C$

$$U(n): \quad \psi_i \to U_{ij}\psi_j; \qquad \qquad \mathbb{Z}_2^C: \quad \psi_i \to \psi_i^\dagger$$

Symmetries in Weyl Fermions

$$\int d^2x \sum_{i=1}^{n} i\psi_i^\dagger(\partial_t - \partial_x)\psi_i$$

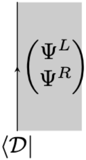
- **Main question**: What are the global symmetries?

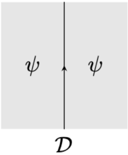
- Obvious symmetry: $U(n) \rtimes \mathbb{Z}_2^C$

$$U(n): \quad \psi_i \to U_{ij}\psi_j; \qquad\qquad \mathbb{Z}_2^C: \quad \psi_i \to \psi_i^\dagger$$

- **Main claim**: The theory enjoys a lot more non-invertible symmetries.

Unfolding Trick





- *n* Dirac fermions, with a conformal boundary condition $|\mathcal{D}\rangle$.
  $(L_n - \tilde{L}_{-n})\,|\mathcal{D}\rangle = 0$.

Unfolding Trick

- n Dirac fermions, with a conformal boundary condition $|\mathcal{D}\rangle$. $(L_n - \tilde{L}_{-n})\,|\mathcal{D}\rangle = 0$.

- After unfolding, we get n Weyl fermions on both sides.

Unfolding Trick

- n Dirac fermions, with a conformal boundary condition $|\mathcal{D}\rangle$.
  $(L_n - \tilde{L}_{-n})\,|\mathcal{D}\rangle = 0$.

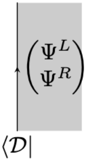
- After unfolding, we get n Weyl fermions on both sides.
  Conformal boundary $\Rightarrow$ conformal defect.

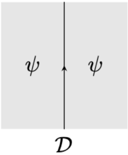
Unfolding Trick

- n Dirac fermions, with a conformal boundary condition $|\mathcal{D}\rangle$. $(L_n - \tilde{L}_{-n})\,|\mathcal{D}\rangle = 0$.

- After unfolding, we get n Weyl fermions on both sides. Conformal boundary $\Rightarrow$ conformal defect. For chiral theory, conformal defect $=$ topological defect.

Unfolding Trick





- n Dirac fermions, with a conformal boundary condition $|\mathcal{D}\rangle$.
  $(L_n - \tilde{L}_{-n})\,|\mathcal{D}\rangle = 0$.

- After unfolding, we get n Weyl fermions on both sides.
  Conformal boundary $\Rightarrow$ conformal defect. For chiral theory, conformal defect $=$ topological defect. $\Rightarrow \mathcal{D}$ is topological.

Unfolding Trick

- n Dirac fermions, with a conformal boundary condition $|\mathcal{D}\rangle$. $(L_n - \tilde{L}_{-n})\,|\mathcal{D}\rangle = 0$.
- After unfolding, we get n Weyl fermions on both sides. Conformal boundary $\Rightarrow$ conformal defect. For chiral theory, conformal defect $=$ topological defect. $\Rightarrow \mathcal{D}$ is topological. $\Rightarrow$ $\mathcal{D}$ is a symmetry of n Weyl fermions.

Unfolding Trick

- n Dirac fermions, with a conformal boundary condition $|\mathcal{D}\rangle$. $(L_n - \tilde{L}_{-n})\,|\mathcal{D}\rangle = 0$.

- After unfolding, we get n Weyl fermions on both sides. Conformal boundary $\Rightarrow$ conformal defect. For chiral theory, conformal defect $=$ topological defect. $\Rightarrow \mathcal{D}$ is topological. $\Rightarrow$ $\mathcal{D}$ is a symmetry of n Weyl fermions.

- Conformal boundary conditions is a “source” for global symmetries.

Conformal Boundary Conditions

- What are the conformal boundary conditions of n Dirac fermions?

Conformal Boundary Conditions

- What are the conformal boundary conditions of n Dirac fermions?

- Obvious boundary conditions (linear in ψ)

$$(\psi_i^L - U_{ij}\psi_j^R)|_{\mathrm{bdy}} = 0, \qquad\qquad (\psi_i^L - U_{ij}\psi_j^{R\dagger})|_{\mathrm{bdy}} = 0$$

  Unfolding such boundary conditions gives $U(N_f) \rtimes \mathbb{Z}_2^C$.

Conformal Boundary Conditions

- What are the conformal boundary conditions of n Dirac fermions?

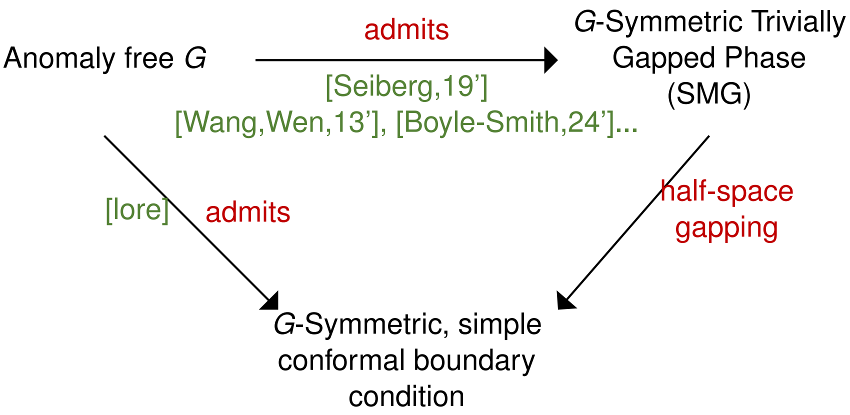
- Obvious boundary conditions (linear in ψ)

$$(\psi_i^L - U_{ij}\psi_j^R)|_{\text{bdy}} = 0, \qquad (\psi_i^L - U_{ij}\psi_j^{R\dagger})|_{\text{bdy}} = 0$$

  Unfolding such boundary conditions gives $U(N_f) \rtimes \mathbb{Z}_2^C$.

- To find more exotic boundary conditions, symmetry helps!

Lore on Anomaly, Gappability, Boundaribility



G Symmetric Conformal Boundary Conditions

- Despite anomaly free, G can act chirally.

$$\psi^L \to U_g \psi^L U_g^{-1}, \qquad \psi^R \to V_g \psi^R V_g^{-1}$$

G symmetric boundary state $\mathcal{D}$ satisfies

$$U_g V_g \left|\mathcal{D}\right\rangle = \left|\mathcal{D}\right\rangle$$

G Symmetric Conformal Boundary Conditions

- Despite anomaly free, G can act chirally.

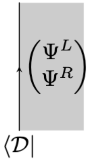
$$\psi^L \to U_g \psi^L U_g^{-1}, \qquad \psi^R \to V_g \psi^R V_g^{-1}$$

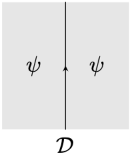
G symmetric boundary state $\mathcal{D}$ satisfies

$$U_g V_g \left|\mathcal{D}\right\rangle = \left|\mathcal{D}\right\rangle$$

- $G = U(1)^n$: the boundary states are constructed explicitly by [Boyle-Smith, Tong, 19']
- $G =$ non-abelian Lie group: hard to construct, but believed to exist due to the Lore.

G-Intertwining Topological Defect





G-symmetric boundary

G-intertwining defect

$$U_g V_g |\mathcal{D}\rangle = |\mathcal{D}\rangle$$

$$V_g \mathcal{D} = \mathcal{D} U_g$$

G-Intertwining Topological Defect

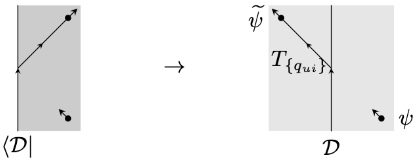
Upshot:

- Consider an anomaly free subgroup of $G \subset (U(n) \rtimes \mathbb{Z}_2^C)^2$ of n Dirac fermions. Each of them should give rise to G-symmetric boundary condition $|\mathcal{D}\rangle$.
- Unfolding along $|\mathcal{D}\rangle$ gives n Weyl fermions with a G-intertwining topological defect $\mathcal{D}$.

$$G \quad \Rightarrow \quad \mathcal{D}$$

- We will construct $\mathcal{D}$ without using explicit form of $|\mathcal{D}\rangle$.

Fermion-Boundary Scattering

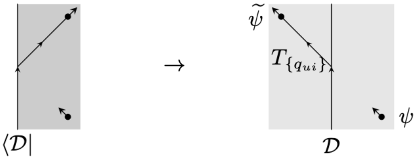


- A boundary condition preserving an non-anomalous chiral symmetry often gives rise to categorical scattering—a fermion is scattered to excitations in the twisted sector. [Beest,Boyle-Smith,Delmastro,Komargodski,Tong,23'], [Beest,Boyle-Smith,Delmastro,Mouland,Tong,23'], [Antinucci talk]...

- After unfolding, it means $\mathcal{D}$ maps a fermion to a twisted sector operator—$\mathcal{D}$ is a non-invertible symmetry. [1]

[1] In [Ohmori,Watanabe,22'], they first realized $G = U(1) \times SU(n)$, and $\mathcal{D}$ is $TY(\mathbb{Z}_n)$ (odd n) or $TY(\mathbb{Z}_{n/2})$ (even n).

Fermion-Boundary Scattering



- Conversely, determining non-invertible symmetry $\mathcal{D}$ helps to construct $|\mathcal{D}\rangle$. The former task is intuitively simpler, since it is purely kinematic.

- Finding $\mathcal{D}$ opens up the possibility of solving fermion-boundary scattering problems, some of which are still open.

Example 1: $G = U(1)^n$

	ψ_i^L	ψ_i^R
$U(1)_a$	Q_{ai}	$\tilde{Q}_{ai}$

$$U_{g_a} \psi_i^L U_{g_a}^{-1} = e^{-2\pi i Q_{ai} \lambda_a} \psi_i^L, \qquad V_{g_a} \psi_i^R V_{g_a}^{-1} = e^{-2\pi i \tilde{Q}_{ai} \lambda_a} \psi_i^R$$

Example 1: $G = U(1)^n$

	ψ_i^L	ψ_i^R
$U(1)_a$	Q_{ai}	$\tilde{Q}_{ai}$

$$U_{g_a}\psi_i^L U_{g_a}^{-1} = e^{-2\pi i Q_{ai}\lambda_a}\psi_i^L, \qquad V_{g_a}\psi_i^R V_{g_a}^{-1} = e^{-2\pi i \tilde{Q}_{ai}\lambda_a}\psi_i^R$$

- $U(1)^n$ Anomaly free condition: $\sum_{i=1}^n Q_{ai}Q_{bi} = \sum_{i=1}^n \tilde{Q}_{ai}\tilde{Q}_{bi}$

Example 1: $G = U(1)^n$

	ψ_i^L	ψ_i^R
$U(1)_a$	Q_{ai}	$\tilde{Q}_{ai}$

$$U_{g_a}\psi_i^L U_{g_a}^{-1} = e^{-2\pi i Q_{ai}\lambda_a}\psi_i^L, \qquad V_{g_a}\psi_i^R V_{g_a}^{-1} = e^{-2\pi i \tilde{Q}_{ai}\lambda_a}\psi_i^R$$

- $U(1)^n$ Anomaly free condition: $\sum_{i=1}^n Q_{ai}Q_{bi} = \sum_{i=1}^n \tilde{Q}_{ai}\tilde{Q}_{bi}$
- Define R matrix: $QR^T = \tilde{Q}$. Anomaly-free $\Rightarrow R \in O(n, \mathbb{Q})$.

Example 1: $G = U(1)^n$

	ψ_i^L	ψ_i^R
$U(1)_a$	Q_{ai}	$\tilde{Q}_{ai}$

$$U_{g_a}\psi_i^L U_{g_a}^{-1} = e^{-2\pi i Q_{ai}\lambda_a}\psi_i^L, \qquad V_{g_a}\psi_i^R V_{g_a}^{-1} = e^{-2\pi i \tilde{Q}_{ai}\lambda_a}\psi_i^R$$

- $U(1)^n$ Anomaly free condition: $\sum_{i=1}^n Q_{ai}Q_{bi} = \sum_{i=1}^n \tilde{Q}_{ai}\tilde{Q}_{bi}$
- Define R matrix: $QR^T = \tilde{Q}$. Anomaly-free $\Rightarrow R \in O(n, \mathbb{Q})$.
- $U(1)^n$ intertwining defect $\mathcal{D}$

$$V_{g_a}\mathcal{D} = \mathcal{D}U_{g_a}$$

$\mathcal{D}$ is Non-invertible

$$\begin{array}{c|cc} & \psi_i^L & \psi_i^R \\ \hline U(1)_a & Q_{ai} & \tilde{Q}_{ai} \end{array}, \qquad V_{g_a}\mathcal{D} = \mathcal{D}U_{g_a}$$

$$U_{g_a}\psi_i^L U_{g_a}^{-1} = e^{-2\pi i Q_{ai}\lambda_a}\psi_i^L, \qquad V_{g_a}\psi_i^R V_{g_a}^{-1} = e^{-2\pi i \tilde{Q}_{ai}\lambda_a}\psi_i^R$$

- If $\mathcal{D}$ is invertible, $U_{g_a} = \mathcal{D}^\dagger V_{g_a}\mathcal{D}$. $V_{g_a} = 1$ implies $U_{g_a} = 1$ acting on fermions.

$\mathcal{D}$ is Non-invertible

$$\begin{array}{c|cc} & \psi_i^L & \psi_i^R \\ \hline U(1)_a & Q_{ai} & \tilde{Q}_{ai} \end{array}, \qquad V_{g_a}\mathcal{D} = \mathcal{D}U_{g_a}$$

$$U_{g_a}\psi_i^L U_{g_a}^{-1} = e^{-2\pi i Q_{ai}\lambda_a}\psi_i^L, \qquad V_{g_a}\psi_i^R V_{g_a}^{-1} = e^{-2\pi i \tilde{Q}_{ai}\lambda_a}\psi_i^R$$

- If $\mathcal{D}$ is invertible, $U_{g_a} = \mathcal{D}^\dagger V_{g_a}\mathcal{D}$. $V_{g_a} = 1$ implies $U_{g_a} = 1$ acting on fermions.

- $V_{g_a} = 1 \Rightarrow \tilde{Q}^T\lambda \in \mathbb{Z}^n$.
  $U_{g_a} = 1 \Rightarrow Q^T\lambda \in \mathbb{Z}^n$.
  Combining the above, denote $y \equiv Q^T\lambda \in \mathbb{Z}^n$, $Ry \in \mathbb{Z}^n$, i.e. R maps an arbitrary integer vector to an integer vector, it is possible only when $R \in O(n,\mathbb{Z}) \subset O(n,\mathbb{Q})$.

$\mathcal{D}$ is Non-invertible

$$\begin{array}{c|cc} & \psi_i^L & \psi_i^R \\ \hline U(1)_a & Q_{ai} & \tilde{Q}_{ai} \end{array}, \qquad V_{g_a}\mathcal{D} = \mathcal{D}U_{g_a}$$

$$U_{g_a}\psi_i^L U_{g_a}^{-1} = e^{-2\pi i Q_{ai}\lambda_a}\psi_i^L, \qquad V_{g_a}\psi_i^R V_{g_a}^{-1} = e^{-2\pi i \tilde{Q}_{ai}\lambda_a}\psi_i^R$$

- If $\mathcal{D}$ is invertible, $U_{g_a} = \mathcal{D}^\dagger V_{g_a}\mathcal{D}$. $V_{g_a} = 1$ implies $U_{g_a} = 1$ acting on fermions.
- $V_{g_a} = 1 \Rightarrow \tilde{Q}^T\lambda \in \mathbb{Z}^n$.
  $U_{g_a} = 1 \Rightarrow Q^T\lambda \in \mathbb{Z}^n$.
  Combining the above, denote $y \equiv Q^T\lambda \in \mathbb{Z}^n$, $Ry \in \mathbb{Z}^n$, i.e. R maps an arbitrary integer vector to an integer vector, it is possible only when $R \in O(n,\mathbb{Z}) \subset O(n,\mathbb{Q})$.
- R contains fractional matrix elements $\Leftrightarrow$ $\mathcal{D}$ is non-invertible.

$\mathcal{D}$ is a Duality Defect

$$\begin{array}{c|cc} & \psi_i^L & \psi_i^R \\ \hline U(1)_a & Q_{ai} & \tilde{Q}_{ai} \end{array}, \qquad V_{g_a}\mathcal{D} = \mathcal{D}U_{g_a}$$

- Simplest non-invertible defect is duality defect, associated with gauging a finite abelian symmetry.

[2] See [Wei,YZ,26'] for a more convoluted derivation. We thank A.Antinucci for telling us this simpler derivation, which is used in [Antinucci,Copetti,Galati,Rizi, 26'].

$\mathcal{D}$ is a Duality Defect

$$\begin{array}{c|cc} & \psi_i^L & \psi_i^R \\ \hline U(1)_a & Q_{ai} & \tilde{Q}_{ai} \end{array}, \qquad V_{g_a}\mathcal{D} = \mathcal{D}U_{g_a}$$

- Simplest non-invertible defect is duality defect, associated with gauging a finite abelian symmetry.
- Set $V_{g_a} = 1$, ask which group Γ does U_{g_a} form.

$$\Gamma = \{\lambda | \tilde{Q}^T\lambda \in \mathbb{Z}^n\}/\sim$$

$\sim$ is an equivalence relation: $\lambda \sim \lambda'$ if $\lambda - \lambda' \in \mathbb{Z}^n$ or $Q^T(\lambda - \lambda') \in \mathbb{Z}^n$.[2]

[2] See [Wei,YZ,26'] for a more convoluted derivation. We thank A.Antinucci for telling us this simpler derivation, which is used in [Antinucci,Copetti,Galati,Rizi, 26'].

$\mathcal{D}$ is a Duality Defect

$$\begin{array}{c|cc} & \psi_i^L & \psi_i^R \\ \hline U(1)_a & Q_{ai} & \tilde{Q}_{ai} \end{array}, \qquad V_{g_a}\mathcal{D} = \mathcal{D}U_{g_a}$$

- Simplest non-invertible defect is duality defect, associated with gauging a finite abelian symmetry.
- Set $V_{g_a} = 1$, ask which group Γ does U_{g_a} form.

$$\Gamma = \{\lambda | \tilde{Q}^T\lambda \in \mathbb{Z}^n\} / \sim$$

$\sim$ is an equivalence relation: $\lambda \sim \lambda'$ if $\lambda - \lambda' \in \mathbb{Z}^n$ or $Q^T(\lambda - \lambda') \in \mathbb{Z}^n$.[2] After simplification,

$$\Gamma = \frac{R^T\mathbb{Z}^n}{R^T\mathbb{Z}^n \cap \mathbb{Z}^n}$$

It is a finite abelian group.

[2] See [Wei,YZ,26'] for a more convoluted derivation. We thank A.Antinucci for telling us this simpler derivation, which is used in [Antinucci,Copetti,Galati,Rizi, 26'].

$\mathcal{D}$ is a Duality Defect

$$\Gamma = \frac{R^T\mathbb{Z}^n}{R^T\mathbb{Z}^n \cap \mathbb{Z}^n}$$

- The explicit form of Γ can be extracted from R.

$$\Gamma = \prod_{u=1}^{n} \mathbb{Z}_{N_u}$$

where

$$N_u = \frac{d}{\gcd(d, s_u)}, \qquad q_{ui} = \frac{s_u}{\gcd(s_u, d)} U_{iu}$$

Here d is the minimal positive integer such that dR^T is an integer matrix. s_u and U_{iu} come from the Smith normal form $dR^T = USV$, where $S = \mathrm{diag}(s_1, s_2, ..., s_n)$.

$\mathcal{D}$ is a Duality Defect

$$\Gamma = \frac{R^T\mathbb{Z}^n}{R^T\mathbb{Z}^n \cap \mathbb{Z}^n} = \prod_{u=1}^{n} \mathbb{Z}_{N_u}$$

- Self-duality under gauging Γ can be explicitly checked at the level of torus partition function.

$$\frac{1}{|\Gamma|}\sum_{\{a_u,b_u\}}\prod_{i=1}^{n}\frac{1}{\eta(\tau)}\theta\begin{bmatrix}\sum_{u=1}^{n}\frac{q_{ui}}{N_u}a_u \\ -\sum_{v=1}^{n}\frac{q_{vi}}{N_v}b_v - Q_{\alpha i}\lambda_\alpha\end{bmatrix}(0,\tau)$$
$$= \prod_{i=1}^{n}\frac{1}{\eta(\tau)}\theta\begin{bmatrix}0 \\ -\widetilde{Q}_{\alpha i}\lambda_\alpha\end{bmatrix}(0,\tau).$$

Example 2: $G = SU(2)$ in 15789 Model

	ψ^L	ψ^R
$SU(2)$	$\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{9}$	$\mathbf{7} \oplus \mathbf{8}$

Example 2: $G = SU(2)$ in 15789 Model

	ψ^L	ψ^R
$SU(2)$	$\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{9}$	$\mathbf{7} \oplus \mathbf{8}$

- $SU(2)$ chiral anomaly is classified by an integer $a \in \mathbb{Z}$.

Example 2: $G = SU(2)$ in 15789 Model

	ψ^L	ψ^R
$SU(2)$	$\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{9}$	$\mathbf{7} \oplus \mathbf{8}$

- $SU(2)$ chiral anomaly is classified by an integer $a \in \mathbb{Z}$.
- A multiplet of Weyl fermions in n dimensional representation of $SU(2)$ contributes to anomaly $a_n = n(n^2 - 1)/6 \in \mathbb{Z}$.

Example 2: $G = SU(2)$ in 15789 Model

	ψ^L	ψ^R
$SU(2)$	$\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{9}$	$\mathbf{7} \oplus \mathbf{8}$

- $SU(2)$ chiral anomaly is classified by an integer $a \in \mathbb{Z}$.
- A multiplet of Weyl fermions in n dimensional representation of $SU(2)$ contributes to anomaly $a_n = n(n^2 - 1)/6 \in \mathbb{Z}$.
- Anomaly of ψ^L cancels the anomaly of ψ^R.

$$(a_1 + a_5 + a_9) - (a_7 + a_8) = 0$$

Example 2: $G = SU(2)$ in 15789 Model

	ψ^L	ψ^R
$SU(2)$	$\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{9}$	$\mathbf{7} \oplus \mathbf{8}$

- $SU(2)$ chiral anomaly is classified by an integer $a \in \mathbb{Z}$.
- A multiplet of Weyl fermions in n dimensional representation of $SU(2)$ contributes to anomaly $a_n = n(n^2 - 1)/6 \in \mathbb{Z}$.
- Anomaly of ψ^L cancels the anomaly of ψ^R.

$$(a_1 + a_5 + a_9) - (a_7 + a_8) = 0$$

- The Lore predicts the existence of a $SU(2)$ preserving conformal boundary condition $|\mathcal{D}\rangle$. Hasn't been constructed yet.

$\mathcal{D}$ is Non-invertible

- Focus on the $U(1)$ cartan subgroup of $SU(2)$.

$$Q_i = (0, 2, 1, 0, -1, -2, 4, 3, 2, 1, 0, -1, -2, -3, -4)$$

$$\tilde{Q}_i = (3, 2, 1, 0, -1, -2, -3, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2})$$

$\mathcal{D}$ is Non-invertible

- Focus on the $U(1)$ cartan subgroup of $SU(2)$.

$$Q_i = (0, 2, 1, 0, -1, -2, 4, 3, 2, 1, 0, -1, -2, -3, -4)$$

$$\tilde{Q}_i = (3, 2, 1, 0, -1, -2, -3, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2})$$

- Define R matrix $\sum_{j=1}^{15} R_{ij} Q_j = \tilde{Q}_i$. Since $\tilde{Q}_i$ is fractional, R_{ij} can not be all integers. $\Rightarrow \mathcal{D}$ is non-invertible.

$\mathcal{D}$ is Duality Defect?

$$\sum_{j=1}^{15} R_{ij} Q_j = \tilde{Q}_i$$

- this relation does not uniquely determine R, since we only make use of one $U(1)$.

$\mathcal{D}$ is Duality Defect?

$$\sum_{j=1}^{15} R_{ij} Q_j = \tilde{Q}_i$$

- this relation does not uniquely determine R, since we only make use of one $U(1)$.

- But one can pick an orthogonal matrix R, and the theory is invariant under gauging Γ,

$$\Gamma = \frac{R^T \mathbb{Z}^n}{R^T \mathbb{Z}^n \cap \mathbb{Z}^n} = \prod_{u=1}^{n} \mathbb{Z}_{N_u}$$

$\mathcal{D}$ is Duality Defect?

$$\sum_{j=1}^{15} R_{ij} Q_j = \tilde{Q}_i$$

- this relation does not uniquely determine R, since we only make use of one $U(1)$.

- But one can pick an orthogonal matrix R, and the theory is invariant under gauging Γ,

$$\Gamma = \frac{R^T \mathbb{Z}^n}{R^T \mathbb{Z}^n \cap \mathbb{Z}^n} = \prod_{u=1}^{n} \mathbb{Z}_{N_u}$$

- However, such Γ-duality defect does not intertwin $SU(2)$!

$\mathcal{D}$ can not be $TY(\Gamma)$ for finite abelian Γ

- Intertwining relation implies $[U_g, A] = 0$, $A = \oplus_{h\in\Gamma} h$ is algebra.

Since $g \in SU(2)$ is simply connected, while Γ is discrete, $[U_g, h] = 0$ for any $h \in \Gamma$.

$\mathcal{D}$ can not be $TY(\Gamma)$ for finite abelian Γ

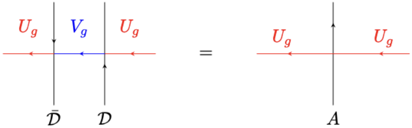
- Intertwining relation implies $[U_g, A] = 0$, $A = \oplus_{h \in \Gamma} h$ is algebra.

  Since $g \in SU(2)$ is simply connected, while Γ is discrete, $[U_g, h] = 0$ for any $h \in \Gamma$.

- It can be shown that $[U_g, h] = 0$ is not compatible with any choice of Γ.

$\mathcal{D}$ can not be $TY(\Gamma)$ for finite abelian Γ

- Intertwining relation implies $[U_g, A] = 0$, $A = \oplus_{h \in \Gamma} h$ is algebra.



  Since $g \in SU(2)$ is simply connected, while Γ is discrete, $[U_g, h] = 0$ for any $h \in \Gamma$.

- It can be shown that $[U_g, h] = 0$ is not compatible with any choice of Γ.

- As a consequence, in the fermion-boundary scattering, the out-going state is in the twist sector of certain non-invertible line.

Details of Proof 1

- The action of Γ should be proportional to identity in each multiplet of $SU(2)$.

$$q_{u2} = ... = q_{u6} \text{ mod } N_u$$

$$q_{u7} = ... = q_{u,15} \text{ mod } N_u$$

Details of Proof 1

- The action of Γ should be proportional to identity in each multiplet of $SU(2)$.

$$q_{u2} = ... = q_{u6} \text{ mod } N_u$$

$$q_{u7} = ... = q_{u,15} \text{ mod } N_u$$

- q_{ui} and N_u depends on the R matrix from Smith normal form decomposition.

$$R_{ij} = \sum_{u=1}^{n} V_{ui} \frac{q_{uj}}{N_u}$$

Fractional part of R_{ij}, denoted by C_{ij}, determined by q_{uj}/N_u.

Details of Proof 1

- The action of Γ should be proportional to identity in each multiplet of $SU(2)$.

$$q_{u2} = ... = q_{u6} \text{ mod } N_u$$

$$q_{u7} = ... = q_{u,15} \text{ mod } N_u$$

- q_{ui} and N_u depends on the R matrix from Smith normal form decomposition.

$$R_{ij} = \sum_{u=1}^{n} V_{ui} \frac{q_{uj}}{N_u}$$

  Fractional part of R_{ij}, denoted by C_{ij}, determined by q_{uj}/N_u.
- Split R into integer and fractional parts, $R_{ij} = Z_{ij} + C_{ij}$.

$$C_{i2} = ... = C_{i6}, C_{i7} = ... = C_{i,15}$$

Details of Proof 2

$$R_{ij} = Z_{ij} + C_{ij}$$

$$C_{i2} = ... = C_{i6}, \quad C_{i7} = ... = C_{i,15}$$

Details of Proof 2

$$R_{ij} = Z_{ij} + C_{ij}$$

$$C_{i2} = ... = C_{i6}, \quad C_{i7} = ... = C_{i,15}$$

- However,

$$\begin{aligned}
\tilde{Q}_i &= \sum_{j=1}^{15} R_{ij} Q_j \\
&\overset{mod 1}{=} \sum_{j=1}^{15} C_{ij} Q_j \\
&= C_{i1} \underbrace{Q_1}_{=0} + C_{i2}(\underbrace{Q_2 + ... + Q_6}_{=0}) + C_{i7}(\underbrace{Q_7 + ... + Q_{15}}_{=0}) \\
&= 0 \bmod 1
\end{aligned}$$

Contradicts with $\tilde{Q}_i$ contains half integer!

Summary

- Symmetries in n Weyl fermions is still far from understood, despite being a free field theory.
- Symmetric conformal boundary condition of Dirac fermions is a "source" for symmetries of Weyl fermions.
- G symmetric boundary condition $\overset{\textit{unfolding}}{\Rightarrow}$ G-intertwining defect.

  1. $G = U(1)^n$, $\mathcal{D} = TY(\Gamma)$, $\Gamma =$ finite abelian group.
  2. G general nonabelian group, the G intertwining defect can not be TY type, and is yet to be determined.

Thank you!