Self-dual Higgs transitions: Toric Code and Beyond

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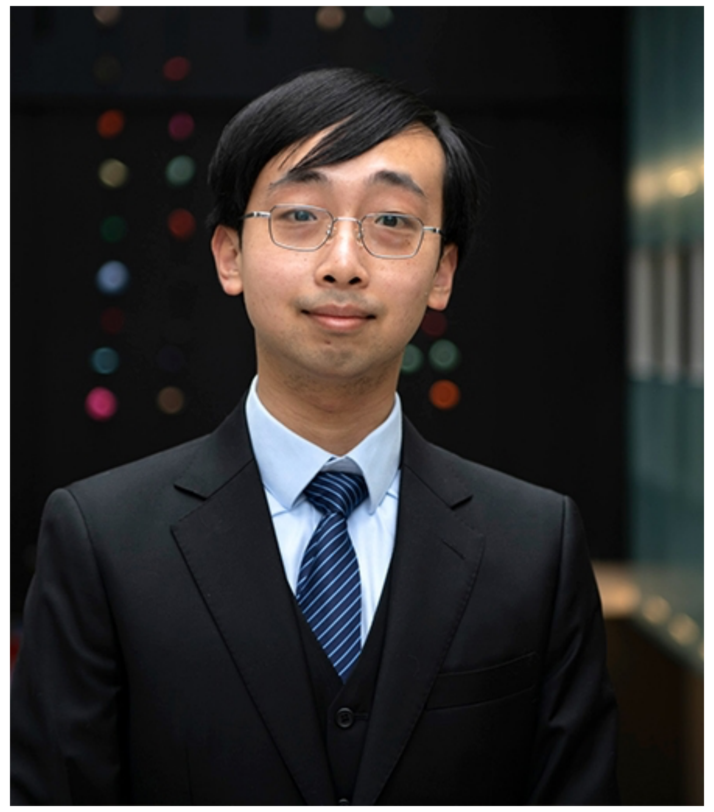

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Chong Wang, PI

arXiv: 1601.20945

Institut Henri Poincaré June 2026

Phase Transitions out of Topological Order

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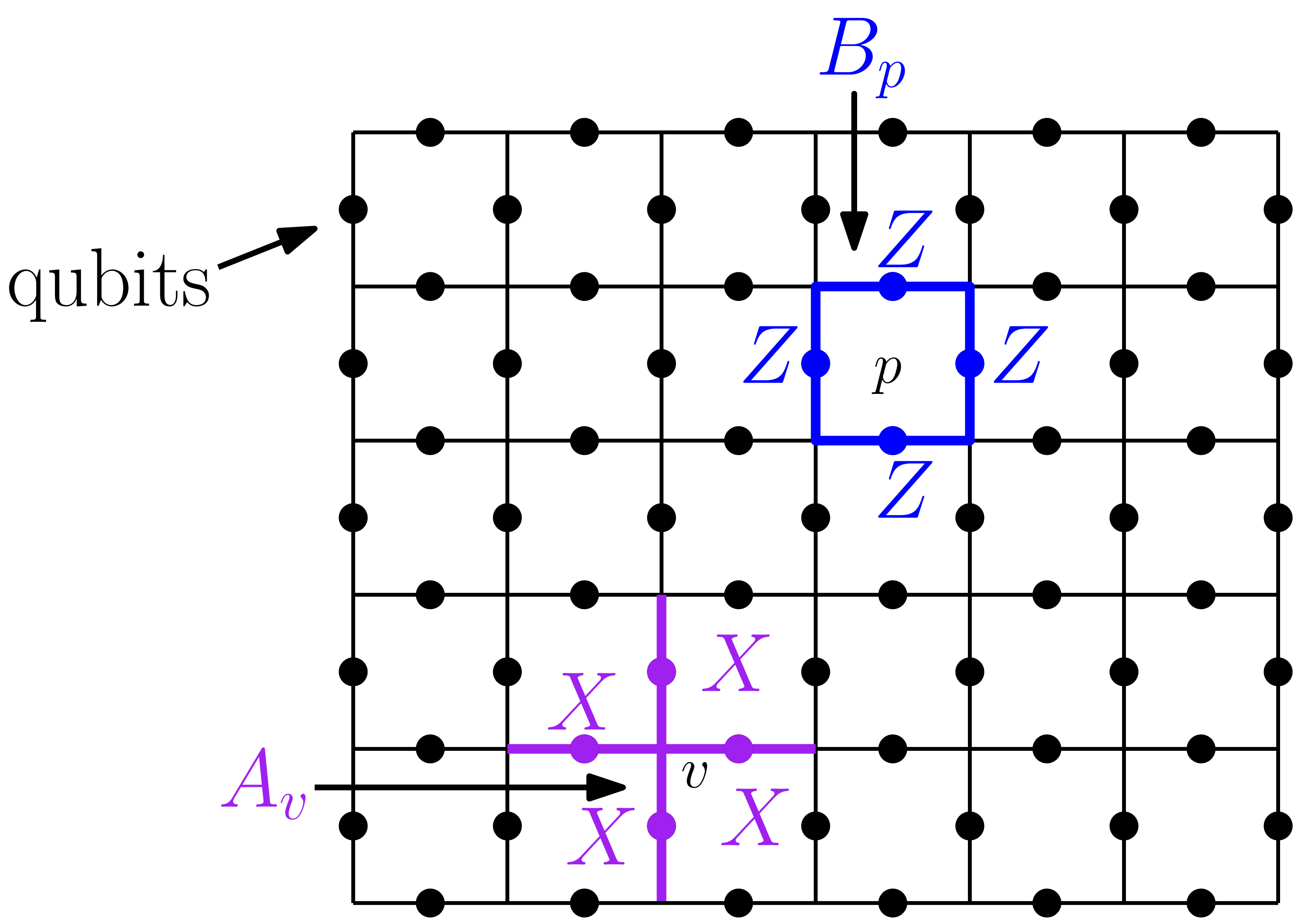


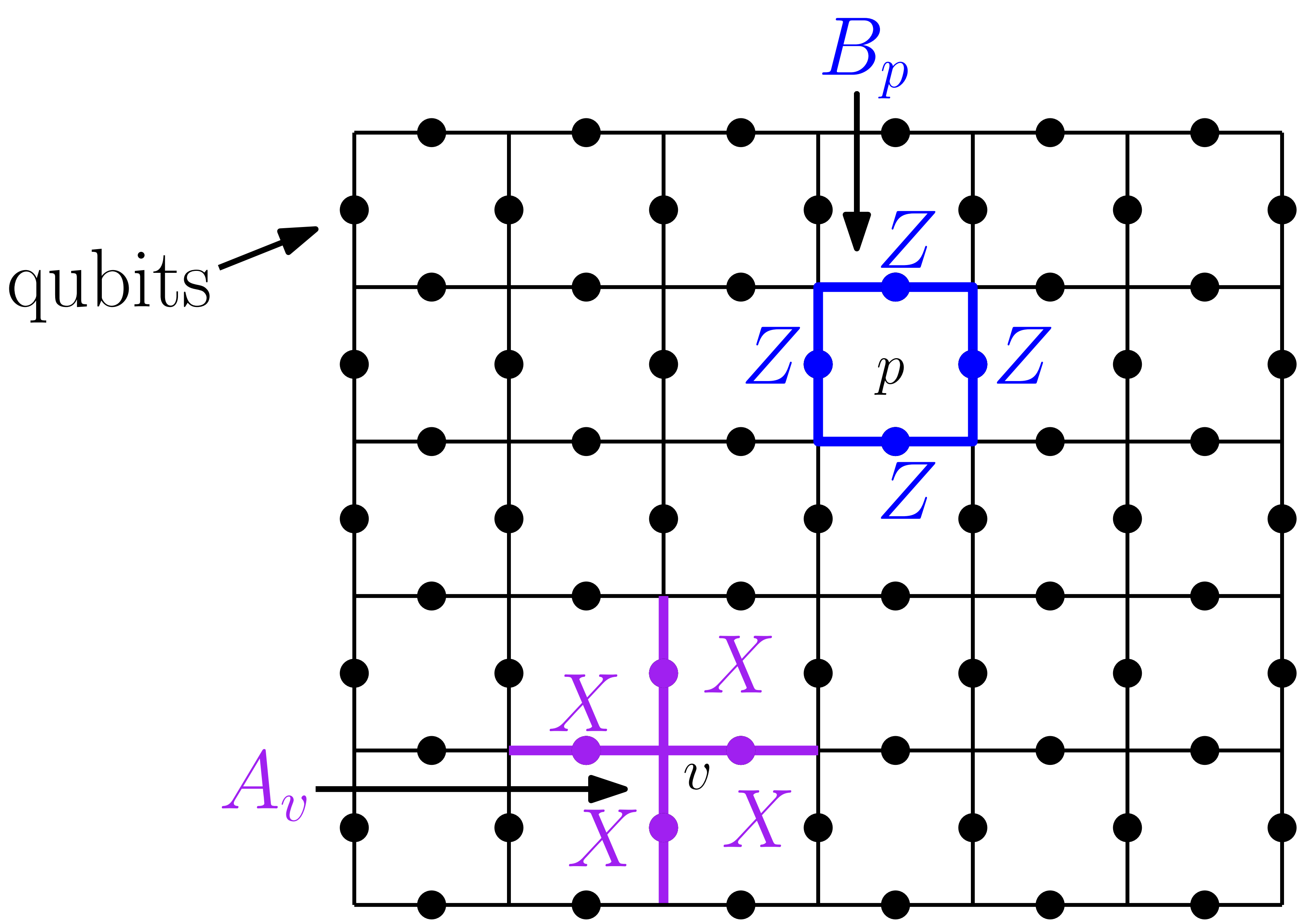
Toric code phase

$$H = -\sum_v A_v - \sum_p B_p$$

- Deconfined phase of the $\mathbb{Z}_2$ gauge theory

Wagner; Kogut; Fradkin, Shenker; Read, Sachdev; Wen; Kitaev...

Phase Transitions out of Topological Order



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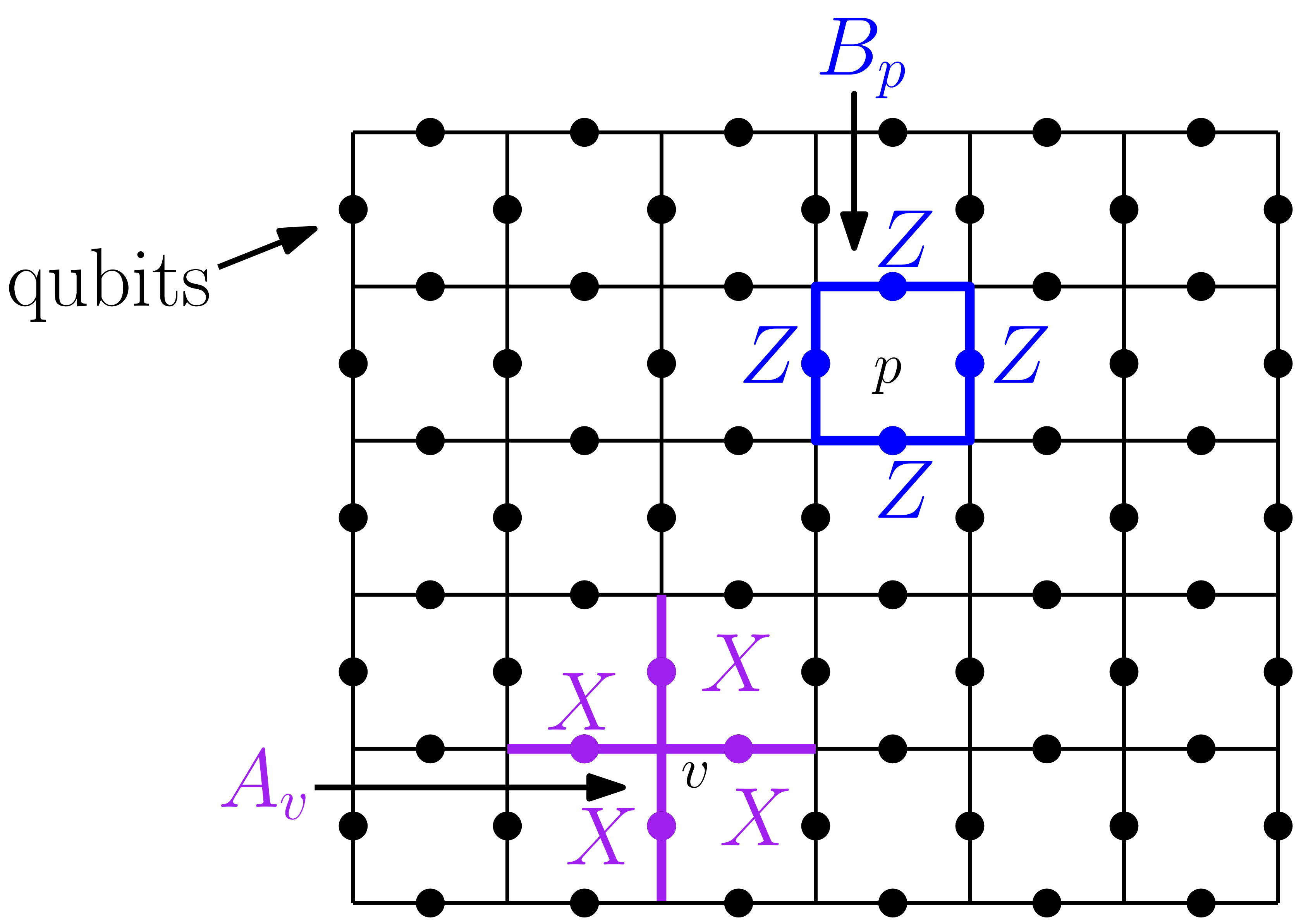
- Gapped topological excitations $\{e, m, \epsilon = e \times m\}$

$$m \; e \qquad \theta_{em} = -1$$

- ϵ : self-fermion

Phase Transitions out of Topological Order



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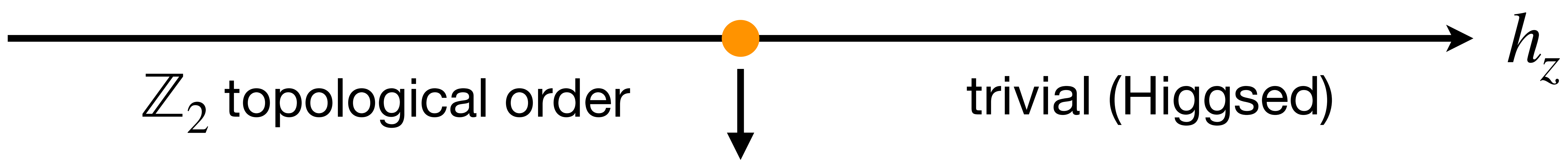
- ϵ : self-fermion
- $\mathbb{Z}_2^{\mathrm{SD}}$ **self-duality symmetry:** $e \leftrightarrow m$

Phase Transitions out of Topological Order

Transitions

$$H = H_{T.C.} - h_z \sum_e Z_e$$



Ising* transition

- 2+1D Ising CFT coupled to $\mathbb{Z}_2$ gauge field / 2+1D $\mathbb{Z}_2$ gauged Ising CFT
- Physically: gap of e excitation ($\mathbb{Z}_2$ gauge charge) closes $\rightarrow$ $\mathbb{Z}_2$ Higgs transition

  "e anyon condenses"

Wagner; Kogut; Fradkin, Shenker...

Phase Transitions out of Topological Order

Transitions

Apply $\mathbb{Z}_2^{SD} : e \leftrightarrow m$

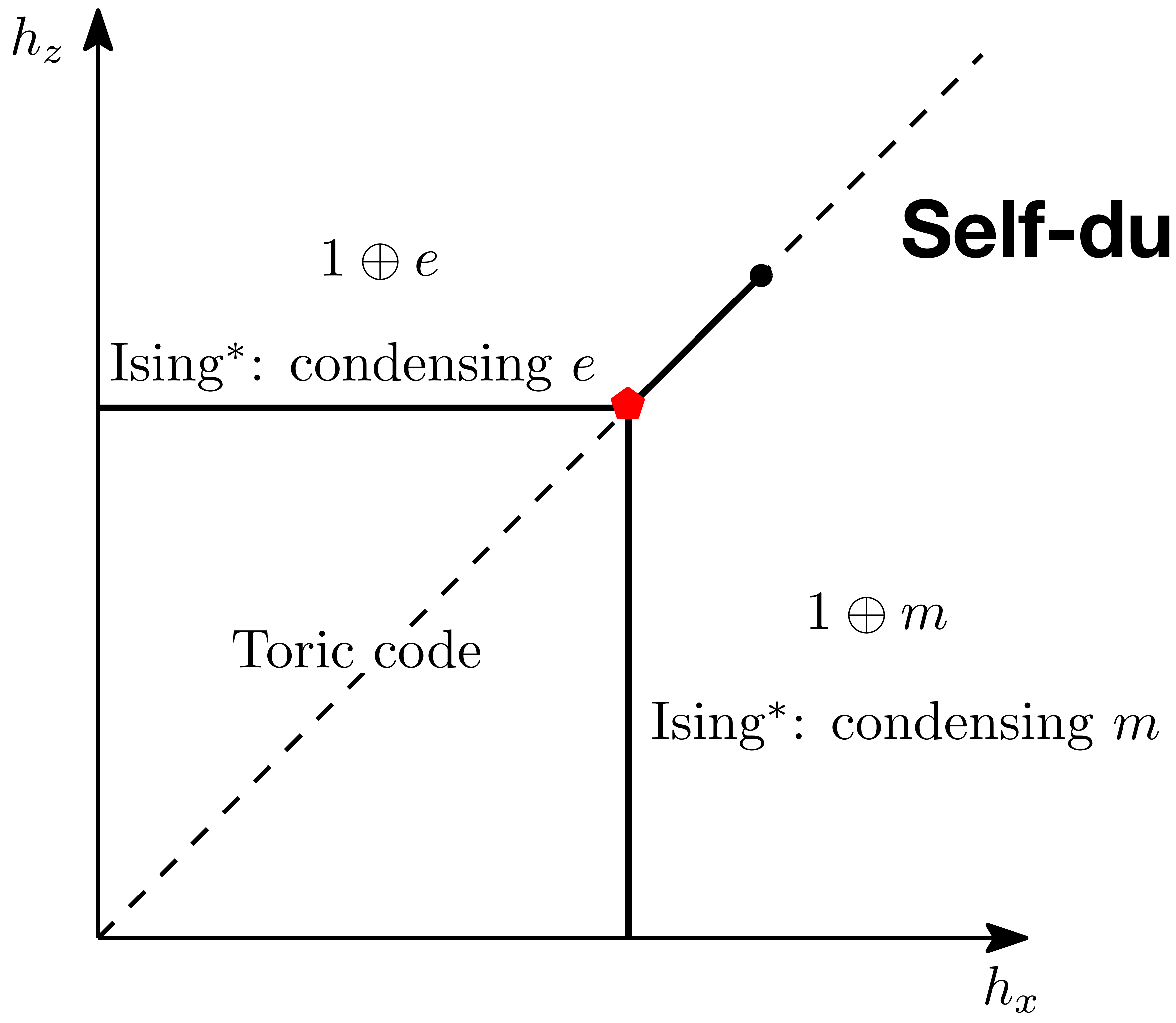
$$H = H_{T.C.} - h_x \sum_e X_e$$

- Still Ising* transition ("m anyon condenses")

Phase Transitions out of Topological Order

Transitions

Generally, $$H = H_{T.C.} - h_z \sum_e Z_e - h_x \sum_e X_e$$



Self-dual line $h_z = h_x$

Phase transition with self-duality symmetry

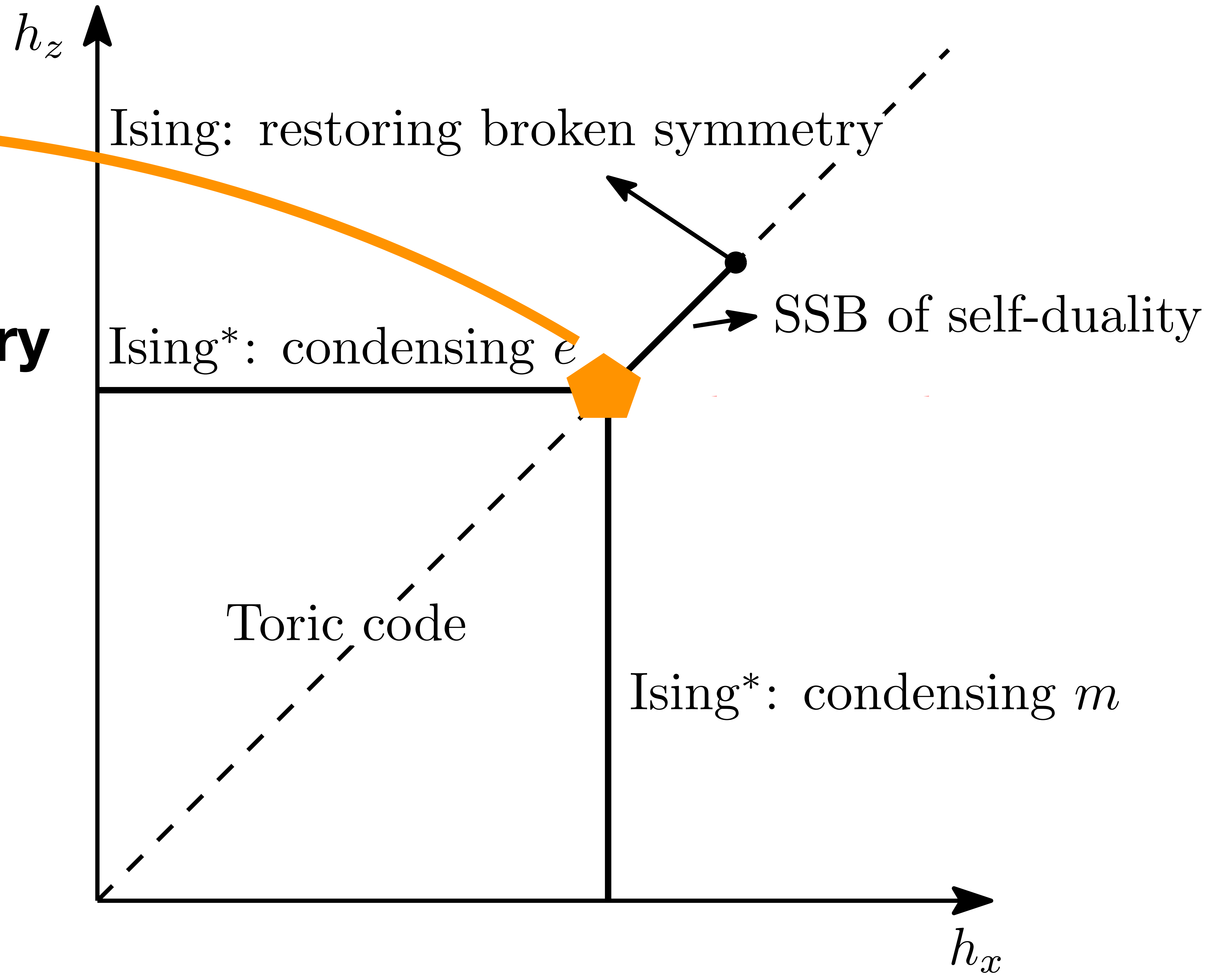
$$H = -\sum_v A_v - \sum_p B_p - \sum_e (h_z Z_e + h_x X_e)$$

Self-dual line $h_z = h_x$

A direct, continuous transition from $\mathbb{Z}_2$ topological order to a trivial state with spontaneously broken self-duality symmetry

Numerical evidence from

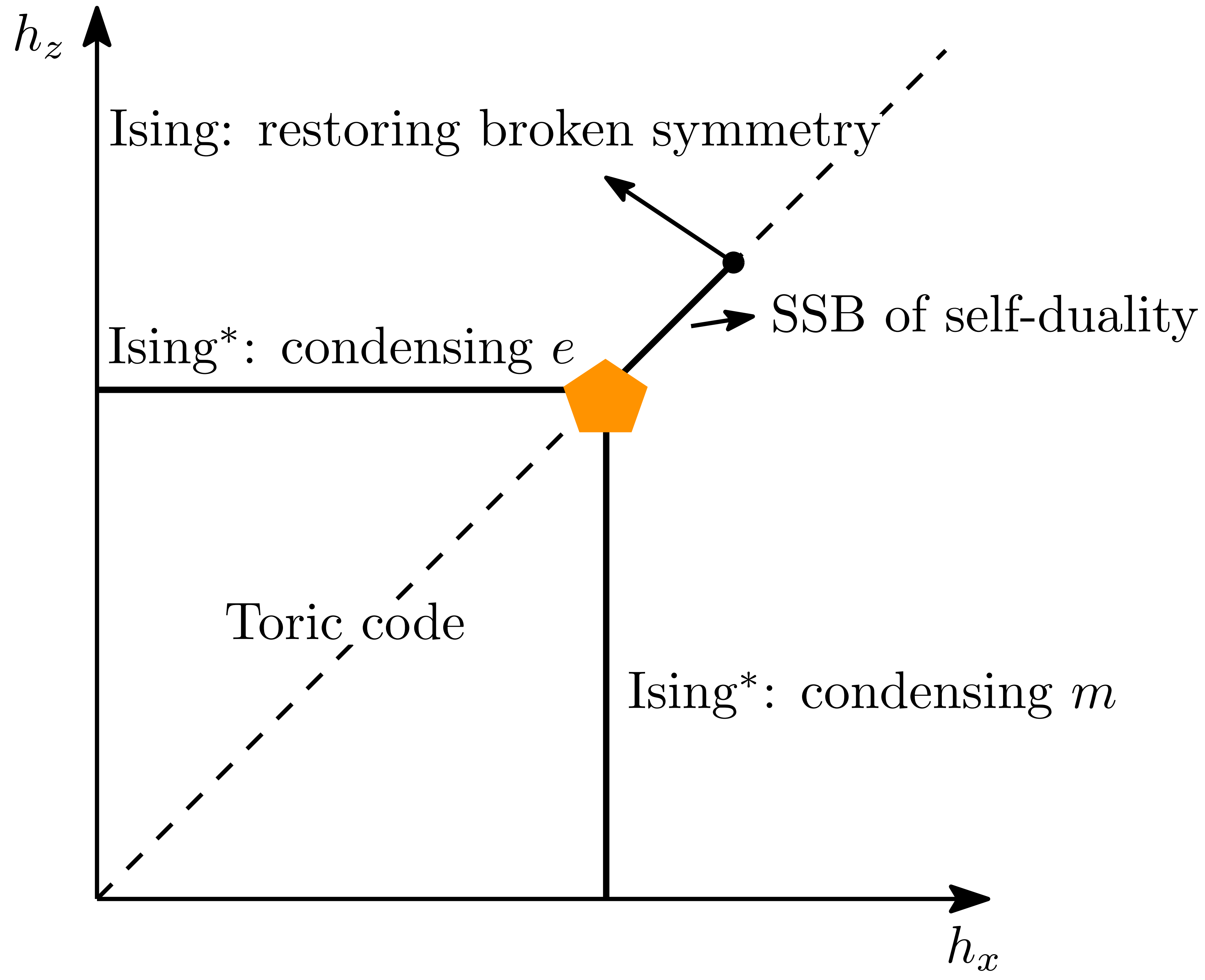
Tupitsyn-Kitaev-Prokof'ev-Stamp, '08
Vidal-Dusuel-Schmidt, '08
Wu-Deng-Prokof'ev, '12
Somoza-Serna-Nahum, '20
Oppenheim-Koch-Janusz-Gazit-Ringel, '23 ...



Phase transition with self-duality symmetry

Challenge: Nature of the transition?

- not conventional anyon condensation
  e and m have nontrivial mutual statistics
  cannot condense simultaneously



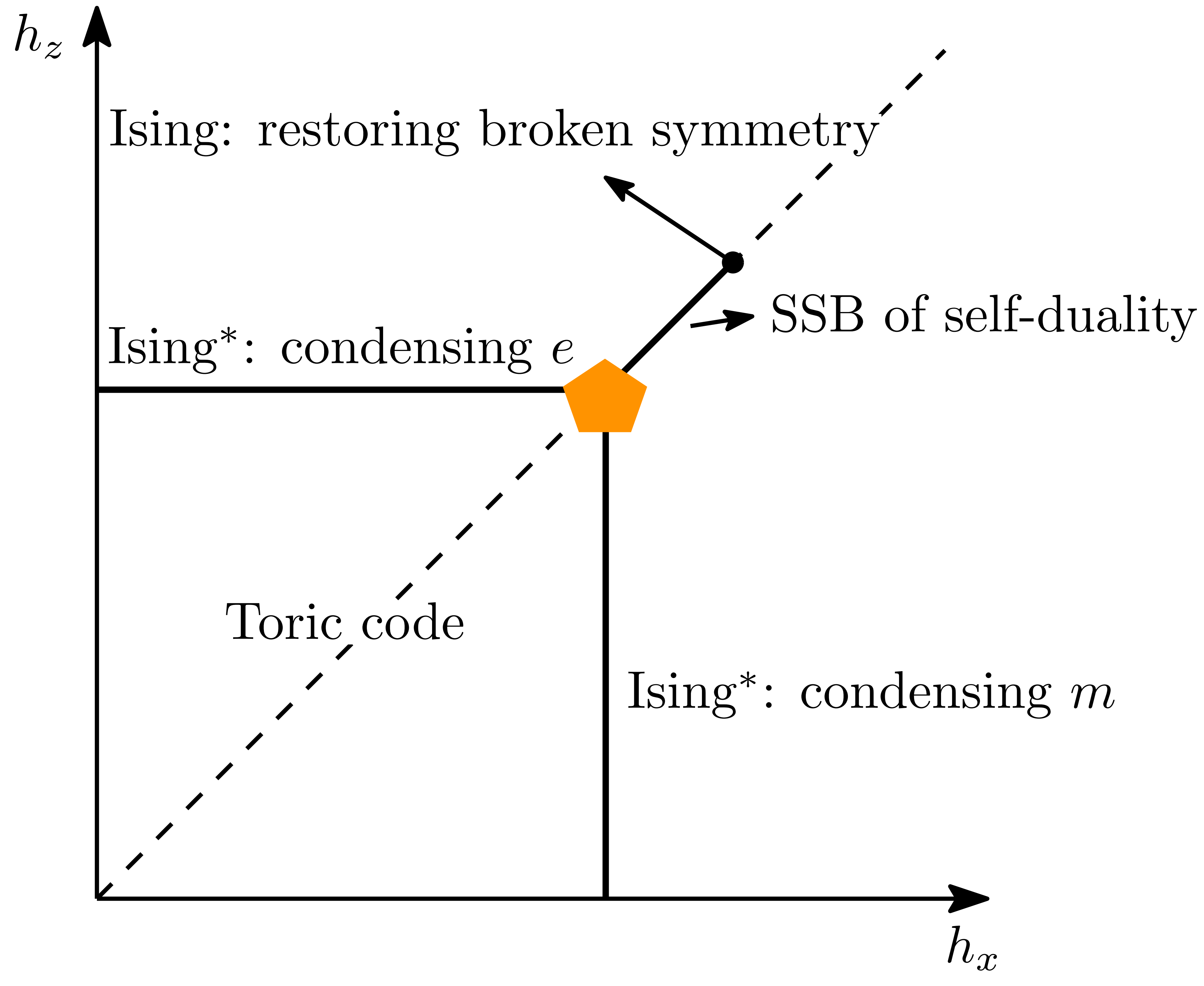
Phase transition with self-duality symmetry

Question: A field theory for the self-dual transition

(1) d.o.f. with mutual statistics

(2) naturally gives

topological order $\leftrightarrow$ $\mathbb{Z}_2^{SD}$ **SSB**



Intuition/Hint: Gauging $\mathbb{Z}_2^{SD}$

$\mathbb{Z}_2$ topological order w/ $\mathbb{Z}_2^{SD}$	$\xrightarrow{}$ e, m become gapless	trivial $\mathbb{Z}_2^{SD}$ SSB

Intuition/Hint: Gauging $\mathbb{Z}_2^{SD}$

Ising x $\overline{\text{Ising}}$

$\{1, \sigma, \psi\}$ x $\{1, \bar{\sigma}, \bar{\psi}\}$

gauge $\mathbb{Z}_2^{SD}$

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trivial
$\mathbb{Z}_2^{SD}$ SSB

Teo-Hughes-Fradkin, '15

Intuition/Hint: Gauging $\mathbb{Z}_2^{SD}$

$\{1, \sigma, \psi\}$ x $\{1, \bar{\sigma}, \bar{\psi}\}$

$$
\begin{tikzcd}
\text{Ising x } \overline{\text{Ising}} \arrow[r, "\sigma\bar{\sigma} \sim e \oplus m"] & \text{trivial: confined, } \mathbb{Z}_2^{(1)} \text{ symmetric} \\
\begin{array}{c} \mathbb{Z}_2 \text{ topological order} \\ \text{w/ } \mathbb{Z}_2^{SD} \end{array} \arrow[u, "\text{gauge } \mathbb{Z}_2^{SD}"'] \arrow[r, "e, m \text{ become gapless}"'] & \begin{array}{c} \text{trivial} \\ \mathbb{Z}_2^{SD} \text{ SSB} \end{array} \arrow[u, "\text{gauge } \mathbb{Z}_2^{SD}"']
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$$

Teo-Hughes-Fradkin, '15

Intuition/Hint: Gauging $\mathbb{Z}_2^{SD}$

Non-Abelian boson can condense!

$\{1,\sigma,\psi\}$ x $\{1,\bar{\sigma},\bar{\psi}\}$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
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\end{tikzcd}$$

Claim: this theory works

$$(D_{\mathscr{A}}\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}^2 + i\mathrm{CS}[\mathscr{A}]_{2,-2}$$

$\mathscr{A}$: dynamical $SO(4)$ gauge field

Φ : real scalar in $SO(4)$ vector rep (4 component)

Recall $SO(4) = \dfrac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}$

$\mathscr{A} \sim (\mathscr{A}_L, \mathscr{A}_R)$ $\Phi \sim$ bi-fundamental rep

$\mathrm{CS}[\mathscr{A}]_{2,-2}$: Chern-Simons term, level 2 for $\mathscr{A}_L$; level -2 for $\mathscr{A}_R$

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- a Higgs transition of $SO(4)$ gauge theory with standard Φ^4 and Yang-Mills term except with a specific CS term

"$SO(4)_{2,-2}$ Chern-Simons Higgs (CSH) theory"

$SO(4)_{2,-2}$ CSH theory

$$(D_{\mathscr{A}}\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}^2 + i\mathsf{CS}[\mathscr{A}]_{2,-2}$$

Global Symmetries

- Time-reversal $\mathbb{Z}_2^T : SU(2)_L \leftrightarrow SU(2)_R$, improper $\mathbb{Z}_2$ on Φ

$SO(4)_{2,-2}$ CSH theory

$$(D_{\mathscr{A}}\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}^2 + i\mathsf{CS}[\mathscr{A}]_{2,-2}$$

Global Symmetries

- Unitary $\mathbb{Z}_2$: flux conservation from $\pi_1(SO(4)) = \mathbb{Z}_2$

  $\mathbb{Z}_2$ charged operator: monopole of $SO(4)$ gauge field

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  Conserved $\mathbb{Z}_2$ flux through a closed surface Σ: $\int_\Sigma \omega_2^{\mathscr{A}} \in \mathbb{Z}_2$

  Couple to background $\mathbb{Z}_2$ gauge field $A^{\mathbb{Z}_2}$:

$$i\pi A^{\mathbb{Z}_2} \cup \omega_2^{\mathscr{A}}$$

Phase diagram

$$(D_{\mathscr{A}}\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}^2 + i\mathsf{CS}[\mathscr{A}]_{2,-2}$$

$r > 0$

$\mathbb{Z}_2$ T.O.

$\mathbb{Z}_2$ flux conservation symmetry acts as $\mathbb{Z}_2^{SD}$ exchanging e and m

$r < 0$

No T.O.

$\mathbb{Z}_2$ flux conservation symmetry SSB

Topological order

$$i\text{CS}[\mathscr{A}^{SO(4)}]_{2,-2}$$

$r > 0$: Φ gapped, only CS term left $\rightarrow$ pure $SO(4)_{2,-2}$ CS theory

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Suppose $\mathscr{A} \in SU(2)_L \times SU(2)_R \rightarrow SU(2)_2 \times SU(2)_{-2}$ CS theory

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<u>$\mathbb{Z}_2$ **Topologically ordered**</u>

$$\left\{0,\frac{1}{2},1\right\}_L \times \left\{0,\frac{1}{2},1\right\}_R \sim \{1,\sigma,\psi\} \text{ x } \{1,\bar{\sigma},\bar{\psi}\}$$

$$\theta_\sigma = e^{i\frac{3}{8}\pi}$$

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then $\mathscr{A} \in SO(4) = \dfrac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}$ $\rightarrow$ gauge $\mathbb{Z}_2^{(1)}$ generated by $\psi\bar{\psi}$ line (or condense $\psi\bar{\psi}$)

$$\{1, e, m, \epsilon\}$$

Topological order

$$i\text{CS}[\mathscr{A}^{SO(4)}]_{2,-2} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$$

$r > 0$: pure $SO(4)_{2,-2}$ CS theory, $\mathbb{Z}_2$ T.O.

$\mathbb{Z}_2$ flux conservation symmetry

Topological order

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<u>$\mathbb{Z}_2$ flux conservation symmetry = $\mathbb{Z}_2^{SD}$</u>

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Suppose we gauge the symmetry

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$A^{\mathbb{Z}_2}$ becomes dynamical $a^{\mathbb{Z}_2} \to w_2^{SO(4)} = 0 \to SU(2)_2 \times SU(2)_{-2}$ CS theory

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So it is $\mathbb{Z}_2^{SD}$ in toric code that we have gauged.

Higgs phase

$r < 0$: $\langle \Phi \rangle \neq 0$, $SO(4)$ Higgsed down to $SO(3) \sim SU(2)_{diag}/\mathbb{Z}_2$

$\mathscr{A}_L = \mathscr{A}_R$, CS terms cancel $\rightarrow$ Pure $SO(3)$ Yang-Mills

$$\frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}^2 + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(3)}$$

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<u>$\mathbb{Z}_2^{SD}$ **SSB**</u>

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<u>$\mathbb{Z}_2^{SD}$ **SSB**</u>

Suppose we gauge the symmetry

$A^{\mathbb{Z}_2}$ becomes dynamical $a^{\mathbb{Z}_2} \rightarrow w_2^{SO(3)} = 0 \rightarrow$ Pure $SU(2)$ Yang-Mills

should confine to a unique vacuum

Higgs phase

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$\mathbb{Z}_2^{SD}$ SSB

Suppose we gauge the symmetry

$A^{\mathbb{Z}_2}$ becomes dynamical $a^{\mathbb{Z}_2} \rightarrow w_2^{SO(3)} = 0 \rightarrow$ Pure $SU(2)$ Yang-Mills

should confine to a unique vacuum

So before gauging, $\mathbb{Z}_2^{SD}$ was spontaneous broken.

Phase diagram matches

$$(D_{\mathscr{A}}\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}^2 + i\mathrm{CS}[\mathscr{A}]_{2,-2} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$$

r

Trivial
$\mathbb{Z}_2^{SD}$ SSB

$\mathbb{Z}_2$ Topological order
with self-duality

Phase diagram matches

$$\begin{tikzcd}[column sep=huge, row sep=huge]
SU(2)_2 \text{ x } SU(2)_{-2}\ \text{CS} \arrow[r, "\sigma\bar{\sigma}\ \text{condense}"] \arrow[d, shift left, "\text{gauge } \mathbb{Z}_2^{(1)}"] & \begin{array}{c} SU(2)\ \text{YM} \\ \text{trivially confined} \end{array} \arrow[d, shift left, "\text{gauge } \mathbb{Z}_2^{(1)}"] \\
\begin{array}{c} SO(4)_{2,-2}\ \text{CS} \\ \mathbb{Z}_2\ \text{T.O.} \end{array} \arrow[u, shift left, "\text{gauge } \mathbb{Z}_2^{SD}"] \arrow[r, "\begin{array}{c} \langle\Phi\rangle \neq 0\ \text{Higgs transition} \\ \left(\frac{1}{2},\frac{1}{2}\right) \end{array}"'] & \begin{array}{c} SO(3)\ \text{YM} \\ \text{no T.O.}\quad \mathbb{Z}_2^{SD}\ \text{SSB} \end{array} \arrow[u, shift left, "\text{gauge } \mathbb{Z}_2^{SD}"]
\end{tikzcd}$$

- This Higgs transition aligns with the $\sigma\bar{\sigma}$ anyon condensation theory.

Generalization: $SO(4)_{k,-k}$ CSH

$$(D_{\mathscr{A}}\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}^2 + i\mathrm{CS}[\mathscr{A}]_{k,-k} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$$

even k

r

Trivial
$\mathbb{Z}_2^{SD}$ SSB

some topological order
with self-duality

Generalization: $SO(4)_{k,-k}$ CSH

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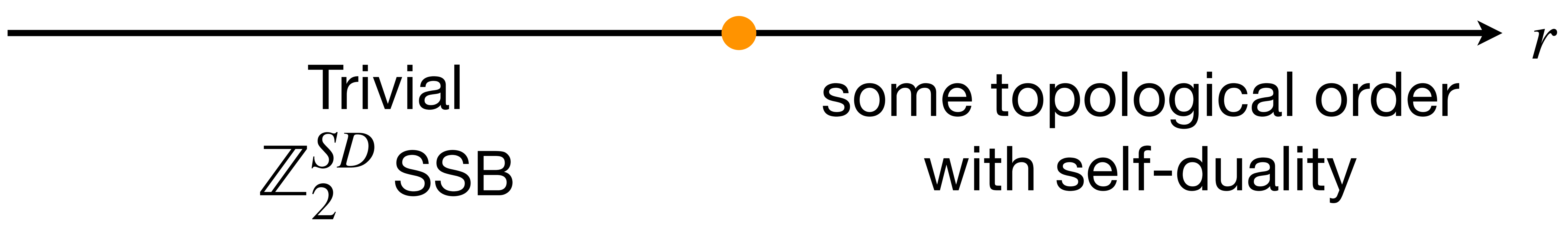


At critical r_c, for large enough k, gauge fluctuations suppressed

some critical exponent approaches that of $O(4)$ WF

$$\Delta_{\Phi^2} \to \Delta_\epsilon^{O(4)\,WF} \sim 1.66,$$

$$\Delta_{\Phi^2,k=2}\ \text{(numerics)}\ \in [1.45, 1.55]$$

Generalization: $SO(4)_{k,-k}$ CSH

k=4 r

Trivial, $\mathbb{Z}_2^{SD}$ SSB

S_3 T.O. w/ $\mathbb{Z}_2^{SD}$

exchanging non-Abelian chargeon and fluxeon

Generalization: $SO(4)_{k,-k}$ CSH

k=4 $\longrightarrow r$

Trivial, $\mathbb{Z}_2^{SD}$ SSB

S_3 T.O. w/ $\mathbb{Z}_2^{SD}$
exchanging non-Abelian chargeon and fluxeon

$$\begin{tikzcd}
SU(2)_4 \times SU(2)_{-4}\ \text{CS} \arrow[rr] \arrow[d, shift left, "\text{condense (2,2)}", "(1,1) \to C+F"'] & & \text{trivial} \\ \text{single vacuum} \arrow[d, shift left] \\
SO(3)_{4,-4}\ \text{CS} \arrow[u, shift left, "\text{gauge } \mathbb{Z}_2^{SD}"] \arrow[rr, "\langle \Phi \rangle \neq 0"'] & & \text{no T.O.} \arrow[u, shift left]
\end{tikzcd}$$

S_3 T.O. $\quad \mathbb{Z}_2^{SD} : C \leftrightarrow F$

$\langle \Phi \rangle \neq 0$

$\left(\frac{1}{2},\frac{1}{2}\right) \sim D$ **condense**

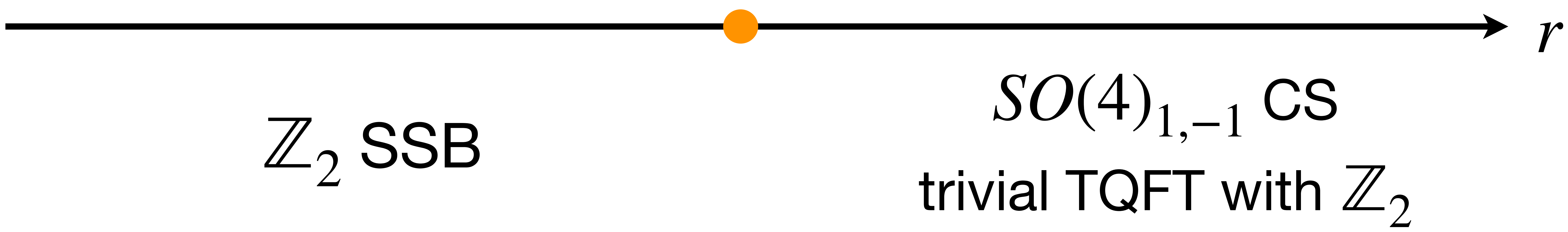
no T.O.
$\mathbb{Z}_2^{SD}$ SSB

$A + C + D$ or $A + F + D$

Chatterjee-Wen, '23

Generalization: $SO(4)_{k,-k}$ CSH

k=1



conjecture: **IR dual to 3d Ising**

Ising spin $\leftrightarrow$ $SO(4)$ monopole

analogous to particle-vortex duality for complex scalar

Generalization: $SO(4)_{k,-k}$ CSH

k=3 r

Trivial, $\mathbb{Z}_2$ SSB

$SO(4)_{3,-3}$ CS

Double Fibonacci w/ $\mathbb{Z}_2$

$SU(2)_3$ x $SU(2)_{-3}$ CS $\longrightarrow$ trivial

single vacuum

gauge $\mathbb{Z}_2$ $\Big\uparrow\Big\downarrow$ condense $\left(\frac{3}{2},\frac{3}{2}\right)$

$SO(3)_{3,-3}$ CS $\xrightarrow[\langle\Phi\rangle \neq 0]{}$ no T.O.

$\mathbb{Z}_2$ SSB

Fib x $\overline{\text{Fib}}$ T.O.

w/ $\mathbb{Z}_2$

$\langle\Phi\rangle \neq 0$

$\left(\frac{1}{2},\frac{1}{2}\right) \sim \tau\bar{\tau}$ condense

Summary and Further Questions

$$(D_{\mathscr{A}}\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}^2 + i\mathrm{CS}[\mathscr{A}]_{2,-2} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$$

r

Trivial
$\mathbb{Z}_2^{SD}$ SSB

$\mathbb{Z}_2$ Topological order
with self-duality

- Numerical verification for no further IR symmetries in perturbed toric code at self-dual criticality?
- Large N calculations for monopole?
- Numerics on transitions out of S_3 quantum double?
- Defects?

Thank you!