Topological order enriched by non-invertible symmetry via anyon condensation

Lea Bottini — IHES

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Based on *2605.28688* with Clement Delcamp, Edmund Heng, Campbell McLauchlan, Dominic Williamson

Introduction

- ☆ Topological order (TO) in (2+1)d: theory of anyons

- ☆ In the presence of global symmetry given by group G (finite): symmetry enriched topological order (SET)

- ☆ For both of these well-established mathematical formulation:

  - TO $\Longleftrightarrow$ Unitary modular tensor category (UMTC) $\mathscr{B}$
  - G-SET $\Longleftrightarrow$ G-crossed braided fusion category $\mathscr{B}_G^{\times}$

- ☆ How do we extend this when the enriching symmetry is *non-invertible*?

[See also Balasubramanian, Buican, Delcamp, Radhakrishnan '25; Eck, Huston, Kawagoe, Penneys '26; Gannon, Rayhaun '26]

  - Notice this is a natural question coming from two perspectives: *condensation defects* and *anyon condensation*

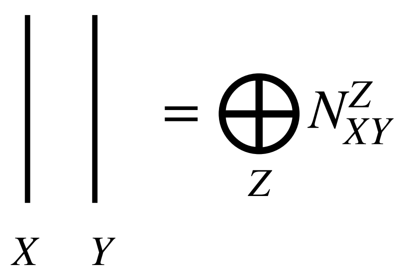
[Gaiotto, Johnson-Freyd '19;
Roumpedakis, Seifnashri, Shao '23]

Plan

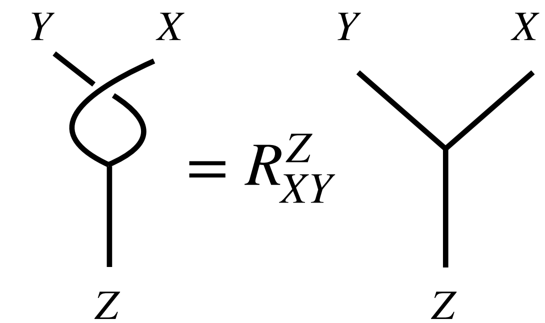
- Review of G symmetry enriched topological order
- Symmetry enriched topological order via anyon condensation
- Hypergroup symmetry enriched topological order
  - Hypergroup grading
  - Hypergroup action
- Conclusions

Symmetry enriched TO

- Given a UMTC $\mathscr{B}$, simple objects $a \in \mathscr{B}$ correspond to anyon species

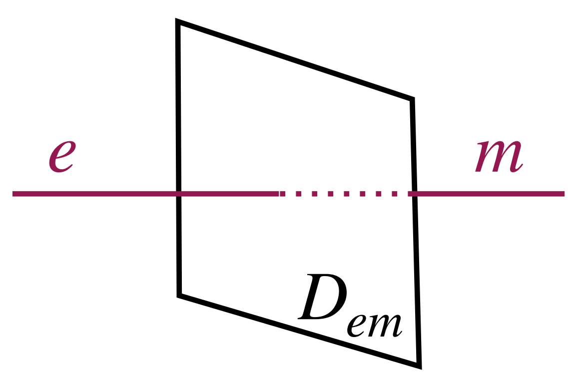


Fusion

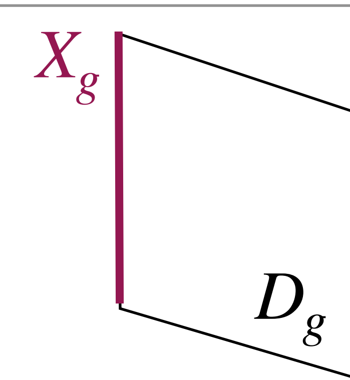


Braiding

- In the presence of global symmetry G
  - Symmetry acts on anyons in a way compatible with fusion structure and braiding (*braided tensor auto-equivalences*)
  - Domain wall D_g for each element $g \in G$



  - Example: *em* duality ($G = \mathbb{Z}_2$) in the toric code with anyons $\{1, e, m, f\}$



  - Moreover, *twisted sector* lines living at the end of a g-domain wall

G-crossed braided fusion categories $\mathscr{B}_G^{\times}$

[Etingof, Nikshych, Ostrik '10;
Barkeshli, Bonderson, Cheng, Wang '14;
Etingof, Gelaki, Nikshych, Ostrik '16]

- G-graded fusion category: $\mathscr{B}_G^{\times} = \bigoplus_{g \in G} \mathscr{B}_g$ with

$$\begin{cases} \mathscr{B}_1 = \mathscr{B} & \text{untwisted anyons (underlying topological order)} \\ \mathscr{B}_g \,, g \neq 1 & g\text{-twisted sector} \end{cases}$$

- G-grading compatible with tensor structure: $\boxed{\mathscr{B}_{g_1} \boxtimes \mathscr{B}_{g_2} \to \mathscr{B}_{g_1 g_2}}$

- G-action $T : G \to \mathrm{Aut}^{\mathrm{br}}_{\otimes}(\mathscr{B})$ such that: $\boxed{T_g(\mathscr{B}_x) \subset \mathscr{B}_{gxg^{-1}}}$

- Braiding twisted by the G-action: $X \in \mathscr{B}_g$ $\boxed{R_{X,Y} : X \otimes Y \cong T_g(Y) \otimes X}$

☆ *G-equivariantisation* (physically: gauging the 0-form symmetry G)

- Once you have $\mathscr{B}_G^{\times}$, can consider $(\mathscr{B}_G^{\times})^G$

- Objects in $(\mathscr{B}_G^{\times})^G$ are labelled by $(X, \{\lambda_g\}_{g\in G})$, with $X \in \mathscr{B}_G^{\times}$ and $\lambda_g : T_g(X) \cong X$

Features of $\mathscr{B}_G^{\times}$

- Symmetry fractionalisation
  - Global symmetry G can act in a fractionalised way on anyons

$$U_{g_1} \circ U_{g_2}(a) = \eta_a(g_1, g_2) U_{g_1 g_2}(a)$$

  - Classified by $\eta \in H_T^2(G, \mathscr{A})$ with $\mathscr{A}$ abelian anyons
  - Decorate junction of 0-fs defects by an abelian anyon $\Longrightarrow$ fractionalised charge η_a due to braiding between a and $\eta(g_1, g_2)$

- Discrete torsion
  - Underlying TO can be stacked with an SPT for G
  - Class in $H^3(G, U(1))$
  - Encoded in associativity of g-defects

G-SET via anyon condensation

[Bais, Slingerland '08; Kong '13]

☆ G-crossed braided $\mathscr{B}_G^\times$ can be obtained via *anyon condensation*

☆ Patterns of anyon condensations classified by *condensable algebras* A in $\mathscr{B}$

- Roughly, set of bosonic anyons that become identified with the identity

☆ Condensing A gives rise to a new topological order $\mathscr{B}_A$ of A-*modules* (pairs (M, ρ) with $M \in \mathscr{B}$ and map $\rho : M \otimes A \to M$)

- *Deconfined* excitations $\mathscr{B}_A^{\mathrm{loc}}$ (local modules of A) braid trivially with A
- The rest, braiding non-trivially with A, are *confined* excitations (non-local modules of A)

☆ If A forms $\mathrm{Rep}(G)$ in $\mathscr{B}$ then $\mathscr{B}_A$ is G-crossed braided

[Drinfeld, Gelaki, Nikshych, Ostrik '10]

$$A \simeq \bigoplus_{\rho \in \mathrm{Irrep}(G)} \dim(\rho)\, \rho$$

☆ Moreover $(\mathscr{B}_A)^G \simeq \mathscr{B}$

Example: S_3 quantum double model

$S_3 = \{r, s \mid r^3 = s^2 = 1,\ sr = r^2 s\}$

- $\mathscr{B} = \mathcal{Z}(\mathrm{Vec}_{S_3})$ (*Drinfeld center*) contains $\mathrm{Rep}(S_3)$ as a subcategory (*pure charges*)
  - $\mathrm{Rep}(S_3) = \{1, e, \pi\}$ with e *sign irrep* and π *2d irrep* of S_3

$$e^2 = 1,\ e \otimes \pi = \pi$$
$$\pi^2 = 1 \oplus e \oplus \pi$$

- Condensable algebra $A(\mathbb{Z}_3) \cong 1 \oplus e$ [generating Rep $(\mathbb{Z}_2)$]

$\mathcal{Z}(\mathrm{Vec}_{S_3})^{\mathrm{loc}}_{A(\mathbb{Z}_3)} \simeq \mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_3})$	Deconfined excitations
$\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_3)} \simeq \mathcal{Z}_{\mathrm{Vec}_{\mathbb{Z}_3}}(\mathrm{Vec}_{S_3})$	*Relative* Drinfel'd center: confined & deconfined

- $\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_3)}$ has a $\mathbb{Z}_2 = S_3/\mathbb{Z}_3$ *charge conjugation* symmetry: $e \leftrightarrow \bar{e}$, $m \leftrightarrow \bar{m}$

- $\mathbb{Z}_2$-equivariantisation recovers $\mathcal{Z}(\mathrm{Vec}_{S_3})$

$$\mathcal{Z}(\mathrm{Vec}_{S_3}) \underset{S_3/\mathbb{Z}_3\text{-equivariantisation}}{\overset{A(\mathbb{Z}_3)\text{ condensation}}{\rightleftarrows}} \mathcal{Z}_{\mathrm{Vec}_{\mathbb{Z}_3}}(\mathrm{Vec}_{S_3})$$

Condensing more general algebras

- $\mathcal{Z}(\mathrm{Vec}_{S_3})$ has another condensable algebra $A(\mathbb{Z}_2) \cong 1 \oplus \pi$

  - $\mathcal{Z}(\mathrm{Vec}_{S_3})^{\mathrm{loc}}_{A(\mathbb{Z}_2)} \simeq \mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_2})$ — Deconfined excitations $\{1, e, m, f\}$
  - $\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)} \simeq \mathcal{Z}_{\mathrm{Vec}_{\mathbb{Z}_2}}(\mathrm{Vec}_{S_3})$ — Confined excitations $\{c_1, c_2\}$

- Question: what is the symmetry enriching $\mathcal{Z}_{\mathrm{Vec}_{\mathbb{Z}_2}}(\mathrm{Vec}_{S_3})$?

  - We propose it is a *hypergroup symmetry* associated with the double coset $\mathbb{Z}_2 \backslash S_3 / \mathbb{Z}_2$

- $\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)}$ naturally decomposes over double cosets $[k] \in \mathbb{Z}_2 \backslash S_3 / \mathbb{Z}_2$

  - $\mathbb{Z}_2 \backslash S_3 / \mathbb{Z}_2$ has 2 components $[1] = \{1, s\}$ and $[r] = \{r, r^2, rs, r^2 s\}$
  - $\{1, e, m, f\} \in (\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})_{[1]}$ $\quad$ $\{c_1, c_2\} \in (\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})_{[r]}$

Hypergroup grading

- Hypergroup: build from a set $\{k_i\}$ with $k_1 \star k_2 = \sum_{k_3} C_{k_1 k_2}^{k_3} k_3 \qquad C_{k_1 k_2}^{k_3} \geq 0, \ \sum_{k_3} C_{k_1 k_2}^{k_3} = 1$

  [Bischoff '16; Bischoff '20; Riesen '25]

- A double coset $H \backslash G / H$ has the structure of a hypergroup

$$[k] \equiv HkH \in H \backslash G / H \qquad [k_1] \star [k_2] := \frac{1}{|H|} \sum_{h \in H} H(k_1 h k_2) H \equiv \sum_{[k_3]} C_{[k_1][k_2]}^{[k_3]} [k_3]$$

- For normal $N < G$, $N \backslash G / N = G/N$ (previous group case)

- For our example

$$\boxed{[r] \star [r] := \frac{1}{2}[1] \oplus \frac{1}{2}[r] \qquad \Longrightarrow \qquad C_{[r][r]}^{[1]} = C_{[r][r]}^{[r]} = \frac{1}{2}}$$

- Compatibility condition $\quad X \in (\mathscr{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})_{[k_1]}, \ Y \in (\mathscr{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})_{[k_2]}$ (simple objects)

$$\boxed{\frac{\mathrm{FPdim}(\Pi_{[k_3]}(X \otimes Y))}{\mathrm{FPdim}(X \otimes Y)} = C_{[k_1][k_2]}^{[k_3]}}$$

[Hempel '23]

- Check with fusion rules

$$c_1 \otimes c_1 = c_2 \otimes c_2 = 1 \oplus e \oplus c_2$$

$$c_1 \otimes c_2 = m \oplus f \oplus c_1$$

Non-invertible symmetry action I

☆ We define the *non-invertible hypergroup symmetry* as a composition of functors

$$T : \mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)} \xrightarrow{F} \mathcal{Z}(\mathrm{Vec}_{S_3}) \xrightarrow{I} \mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)}$$

- Forgetful functor $F: \ (M, \rho) \to M$ (forgets the module structure)
- Induction functor $I: \ M \to (M \otimes A(\mathbb{Z}_2), 1_M \otimes \mu)$ with $\mu : A \otimes A \to A$ (gives canonical module structure)

☆ T decomposes over the hypergroup $T = \bigoplus_{[k] \in \mathbb{Z}_2 \backslash S_3 / \mathbb{Z}_2} T_{[k]}$ so $T = T_{[1]} \oplus T_{[r]}$

☆ Compute $[r]$-action:

- $T_{[r]}(1) = T_{[r]}(e) = 1 \oplus e$
- $T_{[r]}(m) = T_{[r]}(f) = c_1$
- $T_{[r]}(c_1) = m \oplus f \oplus c_1$
- $T_{[r]}(c_2) = c_2 \oplus c_2$

☆ Compatibility condition (R *regular object*)

- $\mathrm{Ad}_{[k_1]}([k_2]) := [k_1] \star [k_2] \star [k_1^{-1}]$
- $\mathrm{Ad}_{[k_1]}([k_2]) \equiv \sum_{[k_3] \in \mathbb{Z}_2 \backslash S_3 / \mathbb{Z}_2} A^{[k_3]}_{[k_1][k_2]} [k_3]$
- $$\frac{\mathrm{FPdim}(\Pi_{[k_3]}(T_{[k_1]}(R_{[k_2]})))}{\mathrm{FPdim}(T_{[k_1]}(R_{[k_2]}))} = A^{[k_3]}_{[k_1][k_2]}$$

Non-invertible symmetry action II

- T can be given the structure of a *monad* [Bruguieres '00; Bruguieres, Lack, Virelizier '11; Cui, Zini, Wang '19]

  - Functors $T : \mathscr{C} \to \mathscr{C}$ with in particular multiplication rule $\mu : T \circ T \to T$ (natural transformation)

  - The multiplication encodes a *generalised symmetry fractionalisation* (for $H = N$ normal reduces to $\mu : T_{[k_1]} \circ T_{[k_2]} \xrightarrow{\sim} T_{[k_1 k_2]}$)

- We can also *gauge* the non-invertible symmetry

  - Requires promoting T to a *Hopf monad*

  - The gauged theory is obtained by taking *T-modules*: $(\mathscr{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})^T \simeq \mathscr{Z}(\mathrm{Vec}_{S_3})$

- Generalised *fixed point theorem*

  - Group case: rank of $\mathscr{B}_g$ = number of simple objects in $\mathscr{B}_1$ fixed by $g \in G$

  - Hypergroup case:

$$\mathrm{rank}(\mathscr{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})_{[k]} = \dim_{\mathbb{C}}\Big(\bigoplus_{X \in \mathrm{Irr}(\mathscr{Z}(\mathrm{Vec}_{\mathbb{Z}_2}))} \mathrm{Hom}_{\mathscr{Z}(\mathrm{Vec}_{\mathbb{Z}_2})}(T_{[k]}(X), X)\Big)$$

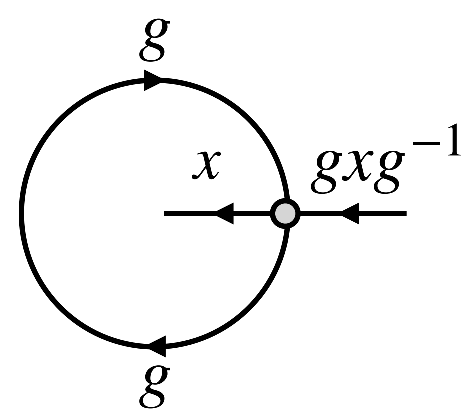
Tube algebra I

[Ocneanu ’94; Koenig, Kuperberg, Reichardt ’12; Kitaev, Kong ’12; Lan, Wen ’14; Bullivant, Delcamp ’19]

- Consider the quantum double model for finite group S_3 [Kitaev ’03]

- Localised excitations captured by *modules* over the tube algebra $\mathrm{Tube}_{S_3}^{S_3}$

$$\mathrm{Tube}_{S_3}^{S_3} = \mathbb{C}\{\mathscr{T}_x^g\}_{g,x\in S_3} = \mathbb{C}\left\{ \quad \right\}$$

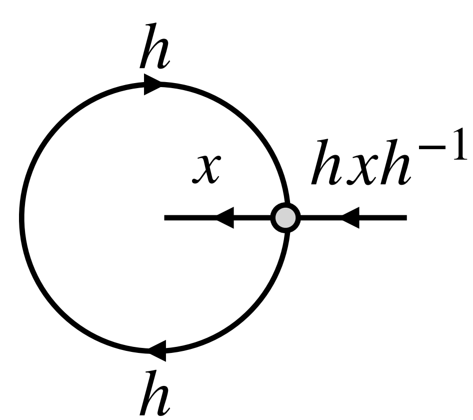


with $\mathscr{T}_{x_1}^{g_1} \cdot \mathscr{T}_{x_2}^{g_2} = \delta_{x_1, g_2 x_2 g_2^{-1}}\, \mathscr{T}_{x_2}^{g_1 g_2}$

  - Simple $\mathrm{Tube}_{S_3}^{S_3}$-modules labeled by $(\mathrm{cl}(f), V)$ with $\mathrm{cl}(f)$ conjugacy class & $V = \mathrm{Irrep}(Z_G(f))$

- Condense algebra $A(\mathbb{Z}_2)$: localised excitations given by modules over $\mathrm{Tube}_{S_3}^{\mathbb{Z}_2}$

$$\mathrm{Tube}_{S_3}^{\mathbb{Z}_2} = \mathbb{C}\{\mathscr{T}_x^h\}_{x\in S_3}^{h\in\mathbb{Z}_2} = \mathbb{C}\left\{ \quad \right\}$$



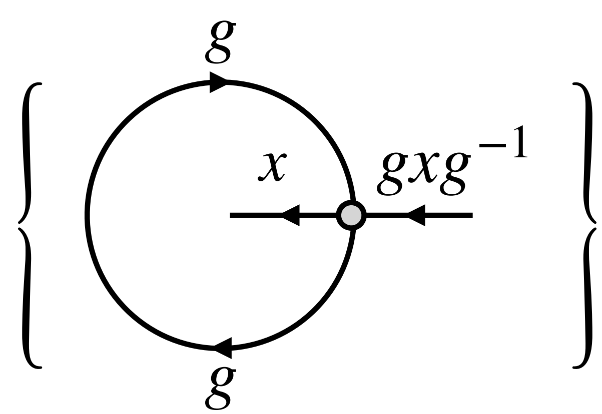
  - $\mathscr{T}_x^h$ w/ $x \in \mathbb{Z}_2$: deconfined
  - $\mathscr{T}_x^h$ w/ $x \in [k]$: $[k]$-twisted

  - Simple $\mathrm{Tube}_{S_3}^{\mathbb{Z}_2}$-modules labeled by $(\mathrm{cl}_H(f), W)$ with $\mathrm{cl}_H(f)$ H-orbit & $W = \mathrm{Irrep}(\mathrm{Stab}_H(f))$

Tube algebra II

☆ Non-invertible symmetry action on localised excitations

- Consider the tubes $M_{[k]} = \mathbb{C}\{\mathcal{T}^{g}_{x}\}^{g\in[k]}_{x\in S_3} = \mathbb{C}$



$g \in [k]$

- Symmetry action by $M_{[k]}$ using multiplication of tubes in $\mathrm{Tube}^{S_3}_{S_3}$

- Given $(\mathrm{cl}_H(f), W) \in \mathrm{Mod}(\mathrm{Tube}^{\mathbb{Z}_2}_{S_3})$ construct *minimal central idempotent* $\mathcal{E}_{(\mathrm{cl}_H(f),W)} \in \mathrm{Tube}^{\mathbb{Z}_2}_{S_3}$

E.g., $\mathcal{E}_1 = \frac{1}{2}(\mathcal{T}^{1}_{1} + \mathcal{T}^{s}_{1})$ $\mathcal{E}_e = \frac{1}{2}(\mathcal{T}^{1}_{1} - \mathcal{T}^{s}_{1})$ $M_{[r]} = \mathbb{C}\{\mathcal{T}^{g}_{x}\}^{g\in[r]}_{x\in S_3}$

Compute $|\mathcal{E}_1 \cdot M_{[r]} \cdot \mathcal{E}_1| = 1$ $|\mathcal{E}_e \cdot M_{[r]} \cdot \mathcal{E}_1| = 1$ $\Longrightarrow$ $T_{[r]}(1) = 1 \oplus e$

Beyond group case

- Our construction for $\mathcal{Z}(\mathrm{Vec}_G) \to \mathcal{Z}_{\mathrm{Vec}_H}(\mathrm{Vec}_G)$ relies on $H \leq G$
  - Hypergroup enrichment of $\mathcal{Z}_{\mathrm{Vec}_H}(\mathrm{Vec}_G)$ via double coset $H\backslash G/H$
- Generalise to any $\mathcal{B} = \mathcal{Z}(\mathcal{C})$ related via anyon condensation to $\mathcal{Z}(\mathcal{D})$ with $\mathcal{D} \subseteq \mathcal{C}$
  - Confined & deconfined excitations captured by the relative center $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$
  - To a fusion ring we can associate a hypergroup and we can take quotients

$$\mathcal{H} = \mathcal{Z}(\mathcal{D})\backslash\mathcal{Z}_{\mathcal{D}}(\mathcal{C})/\mathcal{Z}(\mathcal{D})$$

  - Consider condensable algebra A s.t. $\mathcal{Z}(\mathcal{C})_A \simeq \mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{C})_A^{\mathrm{loc}} \simeq \mathcal{Z}(\mathcal{D})$

Hypergroup action $\quad T : \mathcal{Z}(\mathcal{C})_A \xrightarrow{F} \mathcal{Z}(\mathcal{C}) \xrightarrow{I} \mathcal{Z}(\mathcal{C})_A$ [Riesen '25]

Summarising

	G-crossed braided	Hypergroup enrichment
Compatibility grading-fusion	$\mathcal{B}_{g_1} \boxtimes \mathcal{B}_{g_2} \to \mathcal{B}_{g_1 g_2}$	$\dfrac{\mathrm{FPdim}(\Pi_{[k_3]}(X \otimes Y))}{\mathrm{FPdim}(X \otimes Y)} = C_{[k_1][k_2]}^{[k_3]}$
Compatibility grading-action	$T_g(\mathcal{B}_x) \subset \mathcal{B}_{gxg^{-1}}$	$\dfrac{\mathrm{FPdim}(\Pi_{[k_3]}(T_{[k_1]}(R_{[k_2]})))}{\mathrm{FPdim}(T_{[k_1]}(R_{[k_2]}))} = A_{[k_1][k_2]}^{[k_3]}$
Equivariantisation	$(\mathcal{B}_G^{\times})^G$	T-modules in $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$
Symmetry fractionalisation	$\eta \in H_T^2(G, \mathcal{A})$	Multiplication $\mu : T \circ T \to T$
Discrete torsion	$H^3(G, U(1))$	Associator in $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$
Fixed point thm	$\mathrm{rank}(\mathcal{B}_g) = \lvert \mathrm{Irr}(\mathcal{B}_1)^g \rvert$	$\mathrm{rank}(\mathcal{Z}_{\mathcal{D}}(\mathcal{C}))_{[k]} = \dim_{\mathbb{C}}\Big(\bigoplus_{X \in \mathrm{Irr}(\mathcal{Z}(\mathcal{D}))} \mathrm{Hom}_{\mathcal{Z}(\mathcal{D})}(T_{[k]}(X), X) \Big)$

Conclusions and future directions

- Anyon condensation naturally leads to consider topological order enriched by non-invertible symmetry

- Formalise in terms of a *hypergroup-equivariant hypergroup-graded* fusion category

  - This generalises many features of G-crossed braided extensions

  - We use anyon condensation to construct the hypergroup enriched TO
    This means somehow we start from the final answer to highlight the structure of the non-invertible enrichment

  - These same features should arise if we start more intrinsically with a topological order $\mathscr{B}$ and ask what are the possible hypergroup enrichments

- Classification of such gadgets?

Thank you!