Swiss cheese, Poisson groups and topology

Adrien Brochier
Global categorical symmetries 2026

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For any choice of a Drinfeld associator[a] Φ, there exists a functorial quantization $G \mapsto \mathcal{O}_\Phi(G)$ of Poisson groups.

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  - I will review a somewhat folkloric TFT interpretation of their construction.
  - My goal is to emphasize that there is a lot more interesting structures and symmetries than the mere existence of these quantizations, made manifest by this perspective.
  - I will illustrate this in the seemingly trivial case where the Poisson structure is 0 (this is closely related to BF theory).

- Such Poisson structures are in 1 to 1 correspondence with Lie cobrackets

$$\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$$

on the Lie algebra $(\mathfrak{g}, \mu)$ of G, making it into a Lie bialgebra.

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- There exists a unique Lie algebra structure on $\mathfrak{g} := \mathfrak{g} \oplus \mathfrak{g}^\vee$ (the Drinfeld double) such that:

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    - $\mathfrak{g}, \mathfrak{g}^\vee$ are Lie subalgebra
    - the tautological pairing on $\mathfrak{d}$ is ad-invariant.

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- The pairing on $\mathfrak{d}$ induces a canonical Poisson structure on character varieties (really stacks, aka moduli stacks of flat D-bundles)

$$\mathrm{Ch}_D(S) := \{\pi_1(S) \longrightarrow D\}/D$$

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- Compatible with smooth embeddings.

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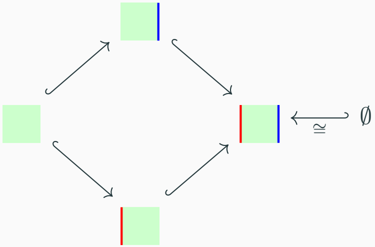
- Those boundary conditions are transverse: the map

 $= G\backslash D/G^\vee \longrightarrow pt$

is an iso.

- Applying this to the diagram



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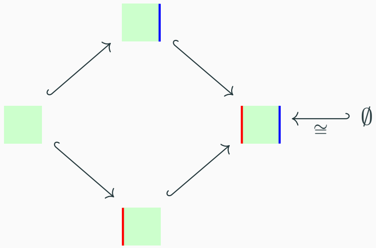

- induces the commutative diagram of monoidal forgetful functors

$$
\begin{tikzcd}
 & \mathrm{Rep}\, G^\vee \arrow[dr] & \\
\mathrm{Rep}\, D \arrow[ur] \arrow[dr] & & \mathrm{Vect} \\
 & \mathrm{Rep}\, G \arrow[ur] &
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we have a monoidal equivalence

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- Taking modules over those, we get (right and left) Swiss–Cheese algebras, i.e. monoidal categories $\mathcal{B}$ and $\mathcal{C}$, together with braided functors

$$\mathcal{A}^{bop} \longrightarrow Z(\mathcal{B}) \qquad\qquad \mathcal{A} \longrightarrow Z(\mathcal{C})$$

where $Z(-)$ is the Drinfeld center and *bop* means braided opposite.

- Via the formalism of factorization homology/skein categories this leads to an oriented 2d TFT with two transverse boundary conditions: the unit

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- The bottom right arrow, in particular, produces an Hopf algebra via Tannakian formalism: this is EK's quantization of G.

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- There is in fact a universal Feynman diagrammatic expression for J_Φ.
- The product on $\mathcal{O}_\Phi(G)$ is then

$$\mathcal{O}(G)^{\otimes 2} \xrightarrow{J_\Phi} \mathcal{O}(G)^{\otimes 2} \xrightarrow{m_0} \mathcal{O}(G)$$

  where m_0 is the original commutative multiplication.

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There exists an infinite order diff. operator u on $S(\mathfrak{g})$ such that

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- Popular interpretation: $S(\mathfrak{g})^G$ is the Poisson center of $S(\mathfrak{g})$, $U(\mathfrak{g})^G$ is the center of $U(\mathfrak{g})$ and "quantization commutes with taking centers".

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- From the general formalism (this is a Lagrangian intersection) we recover that this is 1-shifted Poisson, hence that the underlying variety is Poisson commutative.

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- Then we make crucial use of the $G \leftrightarrow G^\vee$ symmetry.

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  - one by seeing $\mathcal{O}(G)$ as an object in $\operatorname{Rep}_\Phi D$: this one is closely related to the mapping class group action on flat $\mathfrak{g}$-connections on a punctured disc by isomonodromy.

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- That solutions of the KV conjecture were related to these braid groups actions was observed in work of Bar-Natan–Dancso and Alekseev–Enriquez–Torossian: this is my attempt at an explanation.

Thanks for your attention !