A discrete 1d TFT with a point defect

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Factorization algebras

The notion was introduced by Costello–Gwilliam to model the algebraic structure formed by observables of a quantum field theory.

Definition – A factorization algebra F on a topological space X consists of the following data:

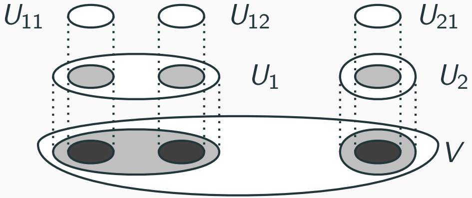
- a cochain complex F_U is assigned to every open subset $U \subset X$;
- a cochain map $\bigotimes_{i \in I} F_{U_i} \longrightarrow F_V$ is assigned to every inclusion $\coprod_{i \in I} U_i \subset V$ of pairwise disjoint opens;

satisfying associativity (this is called a **pre**factorization algebra) as well as factorization and gluing.

Remark – The notion more generally makes sense in a symmetric monoidal (∞-)category.

Factorization algebras

- Here is a picture explaining **associativity**:



$$\begin{tikzcd} F_{U_{11}} \otimes F_{U_{12}} \otimes F_{U_{21}} \arrow[d] \arrow[dd, bend left] \\ F_{U_1} \otimes F_{U_2} \arrow[d] \\ F_V \end{tikzcd}$$

- **Factorization** means that the map $F_U \otimes F_V \to F_{U \coprod V}$ associated with the identity inclusion $U \coprod V \subset U \coprod V$ is a quasi-isomorphism (this allows to restrict to connected open subsets).
- The **gluing** property is a local-to-global property. We will mostly ignore it, as it is automatic in the constructible case (that we are interested in), thanks to results of Karlsson–Scheimbauer–Walde.

Locally constant factorization algebras

Observable of a topological quantum field theory form a locally constant factorization algebra, meaning that if $U \subset V$ is a homotopy equivalence then $F_U \to F_V$ is a quasi-isomorphism.

A natural question is: *How much local information is needed in order to reconstruct a factorization algebra?*

On $\mathbb{R}^n$, an answer is given by a result of Lurie:

Theorem – Locally constant factorization algebras on $\mathbb{R}^n$ are equivalent (as an ∞-category) to $\mathbb{E}_n$-algebras, where $\mathbb{E}_n$-algebras are algebras over the little disks operad.

Very rough idea:

$\boxed{\longrightarrow}$ restrict F to small open disks and rescale.

$\boxed{\longleftarrow}$ extend from disks to general open subsets by gluing and factorization.

$\mathbb{E}_n$-algebras

Examples:

- $\mathbb{E}_1$-algebras in $(Vect, \otimes)$ are associative algebras.
- For $n \geqslant 2$, $\mathbb{E}_n$-algebras in $(Vect, \otimes)$ are associative algebras.
- $\mathbb{E}_1$-algebras in $(Cpx, \otimes)$ "are" A_∞-algebras.
- $\mathbb{E}_2$-algebras in $(Cat, \times)$ are braided monoidal categories.
- $\mathbb{E}_3$-algebras in $(2\text{-}Cat, \times)$ are sylleptic monoidal 2-categories.

Weird examples:

- In $(Cpx, \oplus)$, every object is a canonically an $\mathbb{E}_n$-algebra for every $n \geqslant 1$.
- Not true any more in the category of complexes equipped with a pairing (still with monoidal product being $\oplus$).

Finite-scale version: loc. const. prefact. alg. on “not too little disks”

Fix $R > 0$ (the “scale”).

Main intuition: thanks to the locally constant property, the whole structure (i.e. the $\mathbb{E}_n$-algebra structure) shall be recovered from what happens on disks of radius $> R$. This seems to be an incarnation that for topological field theories, renormalization is quite “easy”.

Theorem [C–Carmona] – There is an equivalence between $\mathbb{E}_n$-algebras and locally constant prefactorization algebras defined on open disks of radius $> R$ in $\mathbb{R}^n$.

How to use it?

a) Consider a lattice approximation of your favorite topological field theory on flat space.

b) Compute the locally constant prefactorization algebra structure of observables at scale $> R$ (here R is some appropriate multiple of the mesh lattice).

c) Get an $\mathbb{E}_n$-algebra.

In dimension 1

Let F be a locally constant factorization on not too little intervals (say of radius $> R$). Assume $b - a > R$.

Consequence (of the theorem) – $F_{(a,b)}$ carries an A_∞-algebra structure, that doesn't depend on the choice of (a, b).

The induced algebra structure on the cohomology $H^\bullet(F_{(a,b)})$ is computed as follows: assume $b - c > R$ and $c - a > R$, then

$$H^\bullet(F_{(a,b)}) \otimes H^\bullet(F_{(a,b)}) \underset{\text{loc. const.}}{\overset{\simeq}{\longleftarrow}} H^\bullet(F_{(a,c)}) \otimes H^\bullet(F_{(c,b)}) \underset{\text{fact. prod.}}{\longrightarrow} H^\bullet(F_{(a,b)})$$

given by

$$\begin{array}{ccccc} (a,b) & \hookleftarrow & (a,c) & & \\ \coprod & & \coprod & \hookrightarrow & (a,b) \\ (a,b) & \hookleftarrow & (c,b) & & \end{array}$$

A $1d$ example

Let V be a vector space with a skew-symmetric pairing $\langle\,,\rangle : \wedge^2 V \to \mathbb{R}$ (a constant Poisson structure on V^*).

Linear observables: For every open interval $(a,b) \subset \mathbb{R}$, $\mathcal{O}^{lin}_{(a,b)}$ is the 2-term complex (in degrees -1 and 0)

$$\{\bar{f} \in V^{\mathbb{Z}} \,|\, supp(f) \in (a, b-1)\} \xrightarrow{Q} \{g \in V^{\mathbb{Z}} \,|\, supp(f) \in (a,b)\}$$

where Q is the discrete de Rham differential: $Q(f)(n) = f(n-1) - f(n)$.

Odd (degree $+1$) symmetric pairing on linear observables:

$$\langle\!\langle \bar{f}, g \rangle\!\rangle \stackrel{\text{def}}{=} \frac{1}{2} \sum_{n\in\mathbb{Z}} \langle f(n-1) + f(n), g(n) \rangle$$

Claim – the pairing is compatible with Q, the assignment $(a,b) \mapsto \mathcal{O}^{lin}_{(a,b)}$ is a prefactorization algebra for $\oplus$, and it is locally constant at scale > 1.

BV/BD quantization of free theories

Let $W = \mathcal{O}^{lin}$ a complex, together with Q and $\langle\langle\, , \rangle\rangle$ as above.

Classical observables: Extend Q and $\langle\langle\, , \rangle\rangle$ to $\mathcal{O}^{cl} \stackrel{\text{def}}{=} \mathrm{Sym}(W)$ by the Leibniz rule; it makes $\mathcal{O}^{cl}$ a dg odd Poisson algebra (a-k-a $\mathbb{P}_0$-algebra).

Quantum observables: Notice that $\mathfrak{g}_W \stackrel{\text{def}}{=} W[-1] \oplus \mathbb{R}[-1]\hbar$ is a Lie algebra, with bracket

$$[u + \lambda\hbar, v + \mu\hbar] \stackrel{\text{def}}{=} \langle\langle u, v \rangle\rangle \hbar$$

Then $\mathcal{O}^q \stackrel{\text{def}}{=} CE(\mathfrak{g}_W) = \mathrm{Sym}(W)[\hbar]$ with differential $d = Q + \hbar\Delta$, where Δ is an order 2 operator (the BV laplacian) determined by $\langle\langle\, , \rangle\rangle$.

The assignment $W \mapsto CE(\mathfrak{g}_W)$ is functorial, exact and symmetric monoidal: it sends locally constant prefactorization algebras (for $\oplus$) to locally constant prefactorization algebras (for $\otimes$).

Back to our $1d$ example

V is a vector space with a skew-symmetric pairing $\langle\, , \rangle : \wedge^2 V \to \mathbb{R}$.

Cohomology of linear observables: Let $n \in (a, b)$ (it exists if $b - a > 1$). The following are quasi-isomorphisms and compose to the identity:

$$\begin{array}{ccccc} V & \to & \mathcal{O}^{lin}_{(a,b)} & \to & V \\ v & \mapsto & (0, \delta_n v) & & \\ & & (\bar{f}, g) & \mapsto & \sum_{n \in \mathbb{Z}} g(n) \end{array}$$

Consequence – The cohomology of $\mathcal{O}^{lin}_{(a,b)}$, and thus of $\mathcal{O}^{q}_{(a,b)}$, is concentrated in degree 0. Hence $\mathcal{O}^{q}_{(a,b)}$ is quasi-isomorphic to a genuine algebra.

Calculation – This algebra is isomorphic to the Weyl algebra

$$W_\hbar(V) \overset{\text{def}}{=} T(V)[\hbar] \big/ (uv - vu = \hbar\langle u, v\rangle)$$

A version of the theorem, with defect

We state the result in dimension 1 with a point defect, for simplicity.

Same game as before, with modified rules: (a) we restric to intervals U satisfying "if $0 \in U$ then 0 is the center of U" ; (b) we replace "locally constant" by "constructible", meaning that inclusions of intervals are sent to quasi-isomorphisms if the intervals are of the same type (i.e. they both contain 0, or they both don't).

Theorem [C-Carmona] – Prefactorization algebras defined on those disks of radius $> R$ and that are constructible (in the above sense), are equivalent to pairs of $\mathbb{E}_1$-algebras together with a bimodule.

Remarks – (1) There is a higher dimensional version, with defects of various codimensions. There is also a version with boundary, as well as with corners.
(2) Constructible factorization algebras on flat space with (iterated) defects have been used by Scheimbauer to construct the Morita category of $\mathbb{E}_n$-algebras.

Example of bimodule: linear coisotropic reduction

Linear algebra – V is a vector space with a skew-symmetric pairing $\langle\, , \rangle : \wedge^2 V \to \mathbb{R}$.
$U \subset V$ is isotropic: $U \subset U^\circ \stackrel{\text{def}}{=} \{v \in V \,|\, \langle v, - \rangle = 0\}$.
$\boxed{\Rightarrow}$ $\langle\, , \rangle$ descends to a pairing on U°/U.

Poisson geometric interpretation – V^* has a (constant) Poisson structure and $U^\perp \in V^*$ is coisotropic, with characteristic foliation given by translation along $(U^\circ)^\perp$.
Hence $(V^*)_{red} = (U^\circ/U)^*$.

Quantization – Consider $W_\hbar(V)$ as above, and J the left ideal generated by U. There is a right action on $M \stackrel{\text{def}}{=} W_\hbar(V)/J$ by $N(J)/J$, that commutes with the left action of $W_\hbar(V)$.

Facts – (1) Universality of $N(J)/J$: any right action that commutes with the left $W_\hbar(V)$-action factors through $N(J)/J$.
(2) $N(J)/J$ is canonically isomorphic to $W_\hbar(U^\circ/U)$.

A $1d$ discrete model with a point defect (joint with Maziar Farahzad)

We now have three possibilities for the linear observables:

- If $a > b > 0$, then $\mathcal{O}^{lin}$ is as before.
- If $0 < a < b$, we simply replace V with U°/U.
- For $(-a, a)$ the complex $\mathcal{O}^{lin}_{(-a,a)}$ is defined in a similar way, with

$$\bar{f}(0) \in U \quad \text{and} \quad \forall n > 0,\ \bar{f}(n), g(n) \in U^\circ/U.$$

Claims – (1) Q and $\langle\langle\, ,\rangle\rangle$ still make sense.
(2) This defines a constructible prefactorization algebra on "not too little intervals" for $\oplus$, and thus applying $CE(-)$ we get a constructible prefactorization algebra on not too little intervals for $\otimes$.
(3) This gives back the bimodule $W_\hbar(V) \circlearrowright W_\hbar(V)/W_\hbar(V)U \circlearrowleft W_\hbar(U^\circ/U)$.

Perspectives

- Incorporate interactions (by tensoring the discrete de Rham complex with an L_∞-algebra). This should allow to recover Fedosov deformation quantization of constant rank Poisson structures.
- Compute Hochschild homology of the quantization (and recover trace formula). Not too little intervals can't work. Need the notion of compatible families w.r.t. refinements (renormalization is back).
- Go to higer dimension (joint work with Giovanni Felder on 2d Yang–Mills: we recover Witten's computation of the partition function using compatible families of discrete factorization algebras – see also work of Benini–Fernández–Schenkel).