What are the possible anomalies of 4d TFTs?

Arun Debray — June 23, 2026

joint work with Weicheng Ye and Matthew Yu:
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Outline

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- Background and the main theorem

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- Where to next?

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  - (This isn't a comprehensive definition of everything called an anomaly!)

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- **Anomalies are computable:** Freed–Hopkins '21 and Grady '23 prove a theorem classifying reflection-positive IFTs using homotopy theory
  - Computations range from trivial to very hard, but **lots of tractable, applicable computations** are possible

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- So: **given a 5d IFT α, can we find a 4d TFT Z whose anomaly is α?**
    - We will assume $\dim Z(S^3) = 1$ (mild assumption, for symmetry preservation reasons)

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- Wan–Wang '19: higher symmetries
- ... and more

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- There is a nice class of IFTs whose actions are built from something called **twisted (extended) supercohomology** similarly to Freed–Quinn '93
- We (D.–Ye–Yu, '26) showed that a 5d IFT α is the anomaly of a 4d unitary TFT Z with $\dim Z(S^3) = 1$ iff it is in the supercohomology class of examples

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- Example: an **Enriques surface** is a quotient of a **K3 surface** by a free involution, but $\mathbb{Z}/2$ acting on K3 extends to $\mathbb{Z}/4$ acting on the spin frames, and the Enriques surface isn't spin, but is $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$

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- The double cover $\widetilde{G} \to G$ is classified by some $b \in H^2(BG; \{\pm 1\})$, so we call a $\mathrm{Spin} \times_{\{\pm 1\}} G$-structure a **$(BG, b)$-twisted spin structure**

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- We expect the same symmetry on the anomaly theory, leading to (BG, b)-twisted spin IFTs

Twisted spin IFTs

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- We will call this group $\mho^n_{\mathrm{Spin}}(BG; b)$

Twisted supercohomology

- Replacing spin bordism with a Postnikov truncation, keeping only the homotopy groups in degrees 2 and below, produces **twisted (extended) supercohomology** $SH^*(BG, b)$ (Wang–Gu '18, '20, Kapustin–Thorngren '20, Décoppet '24)

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- The Anderson dual of the truncation map $\mathrm{MTSpin} \to \tau_{\le 2}\mathrm{MTSpin}$ gives a map from supercohomology to IFTs; the image of this map is the IFTs with supercohomology actions mentioned a while back

Main theorem

Theorem (D.–Ye–Yu, '26)

Let G be a finite group, $b \in H^2(BG; \{\pm 1\})$, and α be a reflection-positive 5d IFT of (BG, b)-twisted spin manifolds. The following are equivalent:

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4. The partition function of α is the integral of a twisted supercohomology class.

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- Upshot: there are exactly two options:
  1. Twisted supercohomology IFTs admit **algorithmically constructed boundary TFTs**
  2. Everything else admits **no boundary TFT**
- The AHSS obstruction is tractable (and fun!) to compute
- Examples: see Córdova–Ohmori '20, D.–Ye–Yu '26, Wan–Wang '25, Wan–Wang–Yau '26

Gauge theory examples

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- This $\mathbb{Z}/4N_f\delta(V)$-symmetry can still carry an anomaly

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- The anomaly of this gauge theory is $(22, 0)$
- Conclusion: this anomaly *cannot* be realized by a TFT

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- The anomaly of this gauge theory is $2(2n+1) \in \mathbb{Z}/16$
- Conclusion: this anomaly *can* be realized by a TFT (in fact a $\mathbb{Z}/4$ gauge theory)

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- When I say "describing a 4d TFT algorithmically," I mean giving an algorithm for constructing a **fusion 2-category** (Douglas–Reutter '18)

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- The details involve some cocycle shenanigans, making this hard to generalize to other IFTs

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- At a high level, they avoid cocycles and instead attack the problem using classification results for fusion 2-categories and their symmetries
- So given $\alpha \in SH^5(BG, b)$, we want a surjective map $p\colon H \to G$, H finite, such that $p^*\alpha = 0$

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- We also need to upgrade "$p^*\alpha = 0$" to an algorithmic construction of a trivialization — but since these are finite groups one can just **try every possible trivialization**

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- For $n < 7$, the only nonzero summands have $q = -1$, 1, 2, and 3 (Milnor '65)

The Atiyah–Hirzebruch spectral sequence

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- A homomorphism $(-)_{p+ip} \colon \mho^5_{\mathrm{Spin}}(BG) \to H^2(BG;\mathbb{Z})$
- (Note: if $b \neq w_2(\rho)$ for a representation ρ, one must give a more complicated definition)

The Atiyah–Hirzebruch spectral sequence

3			$H^2(BG;\mathbb{Z})$ $\leftarrow p + ip$ layer					
2				$H^3(BG;\mathbb{Z}/2)$				
1					$H^4(BG;\mathbb{Z}/2)$			
0								
-1							$H^6(BG;\mathbb{Z})$	
	0	1	2	3	4	5	6	7

A proof that builds character

- Recall: we aim to show that for $\alpha \in \mho^5_{\mathrm{Spin}}(BG, b)$, G finite, if $\alpha_{p+ip} \neq 0$, there is no 4d TFT Z with $\dim Z(S^3) = 1$ and anomaly α

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- For now assume $b = 0$
- Use α to define a character $\chi_\alpha \colon G \to \mathbb{C}^\times$: choose an isomorphism $\alpha(\mathrm{K3}) \cong \mathbb{C}$, and let $\chi_\alpha(g)$ be the value of α on the bordism which is $\mathrm{K3} \times [0, 1]$ with trivial G-bundle, attached by e at 0 and g at 1

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  - Choose the spin structure on the boundary such that $[0,1]/\partial \cong S^1_b$
  - When $b \neq 0$, the construction is the same but more annoying

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- Córdova–Ohmori proved $\chi_\alpha \neq 0$ implies there can't be a TFT Z with anomaly α (and $\dim Z(S^3) = 1$)
- Together, these imply the nonexistence part of our theorem

Part 1: $\chi_\alpha \neq 0$ iff $\alpha_{p+ip} \neq 0$

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- There are four cases ($p = 2$, twisted; $p = 2$, untwisted; $p = 3$, untwisted, $p = 5$, untwisted)
- In all four cases, you can just compute the AHSS, the $p + ip$ layer, and χ_α directly

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- So if $\chi_\alpha \neq 0$, then $Z(\mathrm{K3}) = 0$

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- Wall '64 showed that for any simply-connected closed 4-manifold X, there is an $\ell \geq 0$ such that

$$X \# (-X) \# (S^2 \times S^2)^{\#\ell} \cong (S^2 \times S^2)^{\#\ell+\chi(X)-2}$$

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- $\dim Z(S^3) = 1$ implies $Z(X \# Y) = Z(X)Z(Y)Z(S^4)^{-1}$ (write the connect sum as a bordism)
- $Z(S^2 \times S^2) \neq 0$ (unitarity + explicit bordism decomposition of $S^2 \times S^2$)

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- Plug all these into Wall's formula:

$$|Z(X)|^2 = Z(S^2 \times S^2)^{\chi(X)-2} Z(S^4)^{4-\chi(X)},$$

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  which in particular is nonzero when $X = \mathrm{K3}$ ($\chi = 24$)
- But two slides ago, we proved $Z(\mathrm{K3}) = 0$

Future directions: Lie groups?

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- E.g. Witten's old SU_2 anomaly, and Wang–Wen–Witten's new SU_2 anomaly
- Progress would require generalizing Décoppet–Yu's work, and indeed quite a bit of the theory of fusion 2-categories, to incorporate compact Lie group symmetries

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- Progress would require developing the theory of **unitary fusion 2-categories**
- We conjecture that the $p + ip$ layer will still be a complete obstruction

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  - Stay tuned for our upcoming work using the **hastened Adams spectral sequence** of Behrens–Hill–Hopkins–Mahowald to solve this problem!