Weinstein category for shifted symplectic structures via groupoid & symm TFT

[Cueca-Dorsch-Sjamaar-Z, H. Xu]   June 2026, IHP, Paris, Chenchang Zhu, Göttingen

1. Alan Weinstein (75, 81, 82) proposed a "cat" for symp manifolds (motivated by classical mechanics / Fourier Integral Operators)

Obj: symp manifolds (M_1, ω_1), (M_2, ω_2), etc.

Morp: Lagrangian correspondence

$$\begin{tikzcd} & L_{12} \arrow[dl] \arrow[dr] & \\ (M_1, \omega_1) \arrow[rr, dashed] & & (M_2, \omega_2) \end{tikzcd}$$

$$L_{12} \xrightarrow{\text{Lagrangian submfd}} (\bar{M}_1 \times M_2, (-\omega_1, \omega_2))$$

E.g. $(M_1, \omega_1) \xrightarrow[\text{isom}]{f \cong} (M_2, \omega_2)$

$gr(f) \hookrightarrow (\bar{M}_1 \times M_2, (-\omega_1, \omega_2))$ as Lagrangian submfd.

Composition of Lag. Corr is a problem because of singularities.

Shifted symp $\rightsquigarrow$ correct symp structure for higher derived stacks

[Pantev – Toën – Vaquié – Vezzosi '13] (PTVV)

- Singularity —

1. Quotient (CoLimit) : $M/G \rightsquigarrow$ Lie Gpd $\begin{matrix} M \times G \\ \downarrow\downarrow \\ M \end{matrix} \rightsquigarrow$ Lie ∞ Gpd $X_0 \Leftarrow X_1 \Lleftarrow X_2 \Lleftarrow \cdots$

Lie ∞ Gpd

$\boxed{\text{Lie } \infty \text{ Gpd} = \text{simp mfd} + \text{Kan}}$

$\boxed{\text{Kan}(m,j): \quad X_m \xrightarrow{s.s.} \Lambda^m_j(X)}$

A simplicial manifold =(locally)= Lie ∞ Gpd (Dorsch 23).

(special for continuous symm)

$\downarrow$ differentiate

tang cx : $TX_0 \leftarrow A_{-1} \leftarrow A_{-2} \leftarrow \cdots$ N.Q.mfd

$0 \qquad \downarrow X_0$

2. Singular value (Limit) : $\mu^{-1}(0) \rightsquigarrow$ quasi-smooth mfd $\begin{matrix} M \times \mathfrak{g}^* \\ \downarrow \uparrow \mu \\ M \end{matrix} \rightsquigarrow$ derived mfd $\begin{matrix} L_1 \oplus L_2 \oplus \cdots \\ \downarrow \\ M \end{matrix}$

derived mfd $\qquad \mu : M \to \mathfrak{g}^*$

tang. cx : $TM \to L_1 \to L_2 \to \cdots$

0

Lie Gpd : Lie ∞ Gpd (Getzler, Henriques, Z.)

quasi-smooth mfd : derived mfd (Spivak '08, Joyce/Borisov, ~~Roytenberg~~/Carchedi-Steffens, BLX)
quasi C$^\infty$-rings

derived higher algebraic geom: Lurie, Toën/Vezzosi, Pridham, ...

1 + 2 $\Rightarrow$ $\underbrace{\text{derived Lie } \infty \text{ Gpd}}$ $\rightsquigarrow$ handle all singularities

[CDSZ] form an incomplete cat. of fibrant objects (iCFO) $\rightsquigarrow$ $(\infty, 1)$-cat

- m-shifted symp structures on these groupoids:

$\#\omega$: tang cx $\xrightarrow{\simeq}$ cotang cx $[m]$ is a quasi isom.

Ep 0 $(*, 0)$ is m-shifted symp for all m.

$$
\begin{tikzcd}
\text{tang cx} & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[d, "0"'] \\
\text{cotang cx} & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

--> still non-degenerate.

Ep 0' (M, ω) symp mfd is 0-shifted symp.

Ep 1 $\quad [G/G] \leftrightsquigarrow$

$$\begin{tikzcd} G \rtimes G \arrow[loop above, "\text{conj.}"] \arrow[d, shift left] \arrow[d, shift right] \\ G \end{tikzcd}$$

$\omega = -\frac{1}{2}\left[\langle Ad_x\, g^*\theta^l, g^*\theta^l\rangle + \langle g^*\theta^l, x^*(\theta^l + \theta^r)\rangle\right]$

G lie gp, $\langle , \rangle$ on $\mathfrak{g}$.

$\frac{1}{2}\Theta$: Cartan 3-form

1-shifted symp: integration of Dirac str. on G Xu'04] [Burstyne-Crainic-Weinstein-Z'04]

$$\begin{tikzcd} T(G\times G) \arrow[d, shift left] \arrow[d, shift right] \arrow[r, "\text{Dold-Kan}"] & -1 & G \rtimes \mathfrak{g} \arrow[d] \arrow[r, "\langle , \rangle"] & T^*G \\ TG & 0 & TG \arrow[r, "\langle , \rangle"] & G \times \mathfrak{g}^* \arrow[u] \end{tikzcd}$$

$0 - 1 = -1$

$1 - 1 = 0$

tang cx $\qquad$ cotang cx $[-1]$

$$\begin{tikzcd} \Omega = \frac{1}{2}\langle\theta^l, \theta^r\rangle & G \times G \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \Omega G \arrow[l, "\sim"'] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & : \omega \text{ segal} \\ \textbf{Ep 2} \quad BG \leftrightsquigarrow \Theta & G \arrow[d, shift left] \arrow[d, shift right] & P_eG \arrow[l, "\sim"'] \arrow[d, shift left] \arrow[d, shift right] \\ & * & * \arrow[l, "\sim"'] \end{tikzcd}$$

$$\begin{tikzcd} -1 & \mathfrak{g} \arrow[d] \arrow[dr, "\langle , \rangle"] & 0 \arrow[d] \\ 0 & 0 & \mathfrak{g}^* \end{tikzcd}$$

tang cx $\qquad$ cotang cx $[2]$

$0 - 2 = -2$

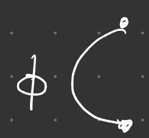
$1 - 2 = -1$

classical TFT (Calaque–Haugseng–Scheimbauer '21, Safronov, AKSZ)

$\mathrm{Cob}_2 \xrightarrow{Z^{BG}}$ Weinstein "Cat"

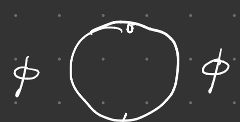
continuous symm TFT

$\bullet \longmapsto BG$ 2-shifted symp



$\longmapsto * \overset{BG}{\dashrightarrow} \begin{matrix}\overline{BG} \\ \times \\ BG\end{matrix} \rightsquigarrow BG \xrightarrow{\Delta} \overline{BG} \times BG$

2-shifted lag



$\longmapsto [G/G] = BG \underset{\overline{BG}\times BG}{\overset{h}{\times}} \overline{BG}$ 1-shifted symp

$\longmapsto * \overset{BG}{\dashrightarrow} [G/G]$

$\longmapsto * \overset{[G/G]}{\dashrightarrow} \begin{matrix}\overline{[G/G]} \\ \times \\ [G/G]\end{matrix}$

$\longmapsto [G/G] \overset{[G\times G/G]}{\dashrightarrow} [G/G] \times [G/G]$

1-shifted Lag

quasi-Hamiltonian $G \curvearrowright M$.

$[M/G]$

$[M/G]$	Bulk $[G/G]$	BG

$[M/G]$ $\|$ phy. boundary

BG $\|$ symm. boundary

$[M/G] \underset{[G/G]}{\overset{h}{\times}} BG = [M/G] \underset{[G/G]}{\times} [\mathfrak{g}/G\times\mathfrak{g}] = [M /\!/ G]$

[CDSZ]

quasi-Hamiltonian reduction

$\xrightarrow{\text{Atiyah-Bott}}$ flat conn / gauge 0-shifted symp

symp mfd.

(higher) Hamiltonian reduction as condensation in continuous symm TFT. [H. Xu]
[Calaque]

E.g.

$M \curvearrowleft G$ Hamiltonian action

phys. bd — Bulk — symm. boundary

$L^{phy} = [M/G]$

$T^*[1]BG$

$L_1^{symm} = \mathfrak{g}^*$ ⇝ Dirichlet

domain wall $\uparrow$

$D = 0$

$\downarrow$

$L_2^{symm} = BG$ ⇝ Von Neumann

can already observe here

```latex
\begin{tikzcd}
M\times G \arrow[r, "\mu\times id"] \arrow[d, shift left] \arrow[d, shift right] & T^*G \arrow[d, shift left] \arrow[d, shift right] & \mathfrak{g}^* \arrow[l] \arrow[d, shift left] \arrow[d, shift right] & & M \\
M \arrow[r, "\mu"] & \mathfrak{g}^* & \mathfrak{g}^* \arrow[l] & \rightsquigarrow & \\
\cup \arrow[r] & \cup & 0 \arrow[l] \arrow[u] \arrow[d] & & \mu^{-1}(0) \arrow[uu] \arrow[dd] \\
 & & G \arrow[d, shift left] \arrow[d, shift right] & & \\
\cup \arrow[r] & \cup & \bullet \arrow[l] & & {[M /\!/ G]}
\end{tikzcd}
```

Lagrangian corr.

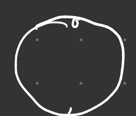
$\mathrm{Cob}_2 \xrightarrow{Z^{BG}}$ Weinstein "Cat" $\xrightarrow{\text{pre quan}}$ Contact "Cat" $\xrightarrow{\text{quan}}$?

$\bullet \longmapsto BG$ 2-shifted symp $\longrightarrow$ $\mathrm{String}(G)$ $\Uparrow$ $H^4(BG, \mathbb{Z})$ p_1 ?? TMF

ϕ  $\longmapsto$ $* \overset{BG}{\dashrightarrow} \overline{BG} \times BG$ $\rightsquigarrow$ $BG \xrightarrow{\Delta} \overline{BG} \times BG$

2-shifted Lag

transgression

ϕ  ϕ $\longmapsto$ $[G/G] = BG \times^h_{\overline{BG} \times BG} \overline{BG}$ 1-shifted symp $\longrightarrow$ S^1-gerbe $\Uparrow$ $H^3_G(G, \mathbb{Z})$

Meinrenken '10 twisted K-homology

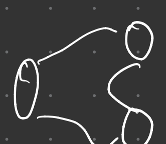
$\longrightarrow$

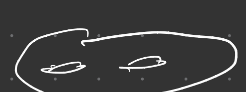
Freed-Hopkins-Teleman '05 twisted K-theory

ϕ  $\longmapsto$ $* \overset{[G/G]}{\dashrightarrow} \overline{[G/G]} \times [G/G]$

1-shifted Lag

$M \longmapsto Q(M) \in R_k(G)$

 $\longmapsto [G/G] \overset{[G\times G/G]}{\dashrightarrow} [G/G] \times [G/G]$

 $\xrightarrow{\text{Atiyah-Bott}}$ flat conn / gauge 0-shifted symp $\longrightarrow$ S^1 bd $L \downarrow M$ $\xrightarrow{\text{geom quan}}$ $\mathcal{H}$

symp mfd

def. quan $\mathcal{A}$

conformal blocks for affine VOA of $\mathbb{Z}_{\geq 0}$