

Invertible Field Theories and the Bott Spiral

Global Categorical Symmetries 2026 | IHP Paris

Cameron Krulewski

joint w/ Arun Debray + Luuk Stehouwer

thanks also to Natalia Pacheco-Tallaj + Ryan Thorngren

Outline

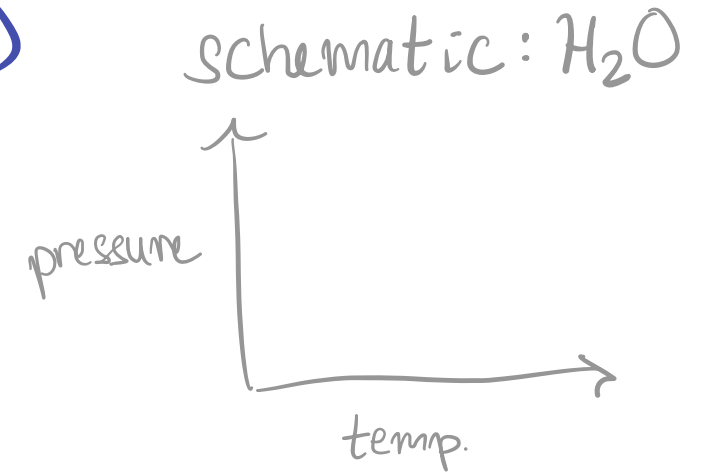
1. The Problem: free versus interacting SPTs
in the 'Bott spiral' (Queiroz-Khalaf-Stern, '16)
2. How well can we model w/ invertible field theories?
3. What new things do we learn?

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Symmetry-protected topological phases (SPTs)

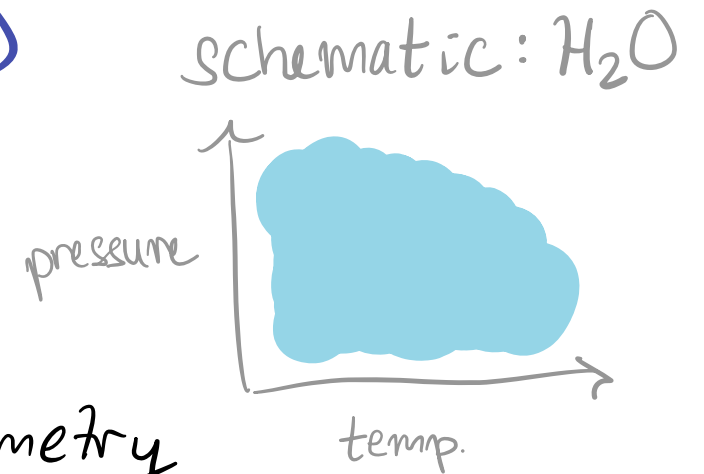
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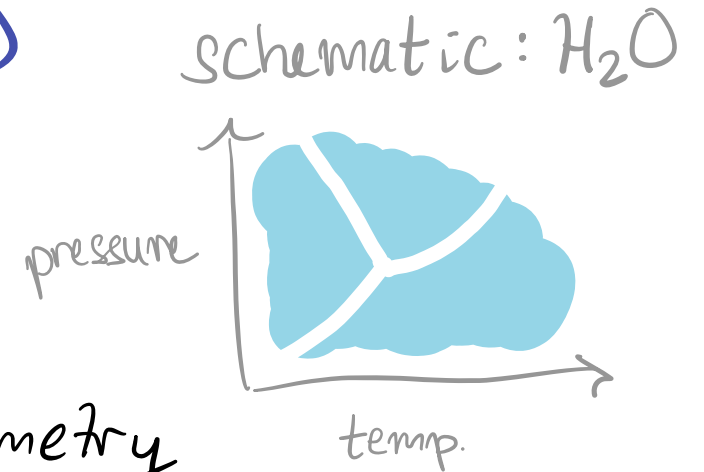


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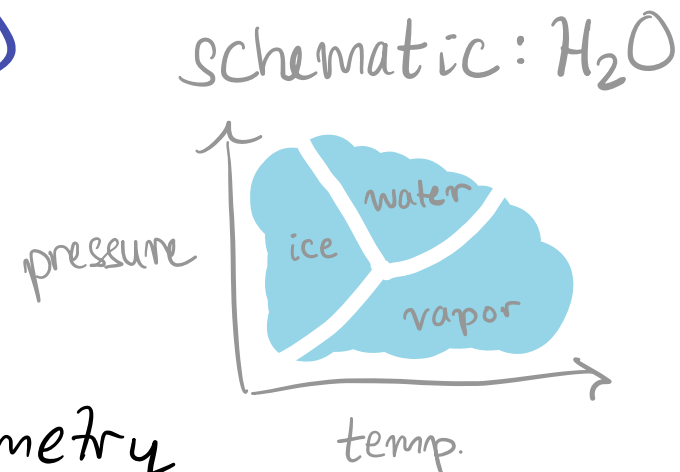
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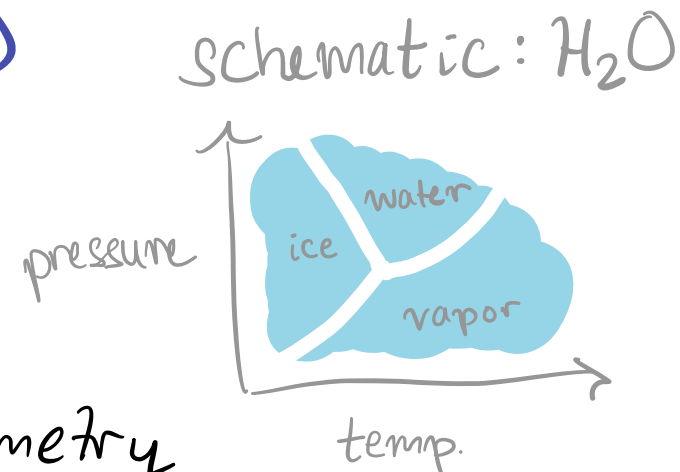
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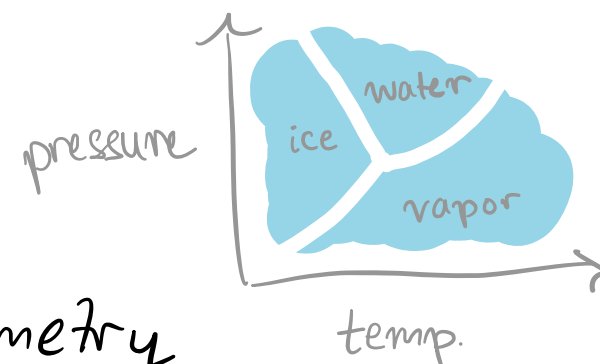
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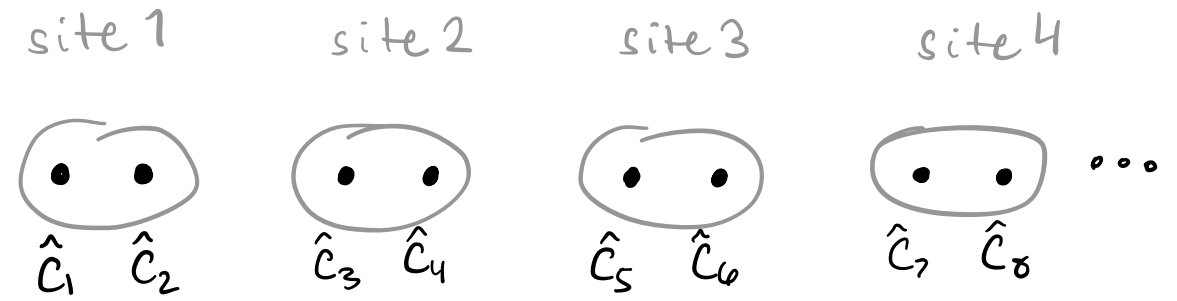
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schematic: H_2O



Question: How does $(\mathcal{M}_{\text{free}} \setminus \Delta) / \sim_{\text{free}}$ compare w/ $(\mathcal{M} \setminus \Delta) / \sim$?

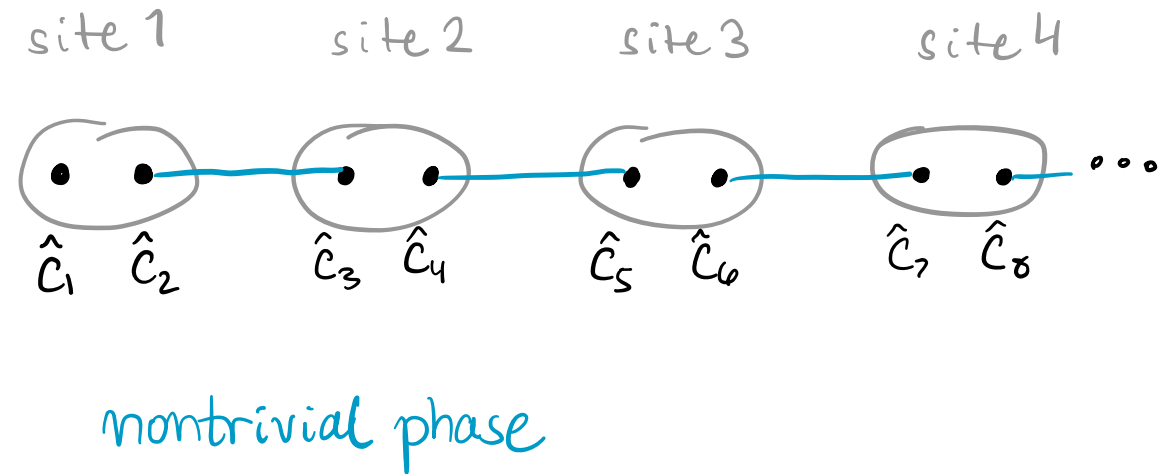
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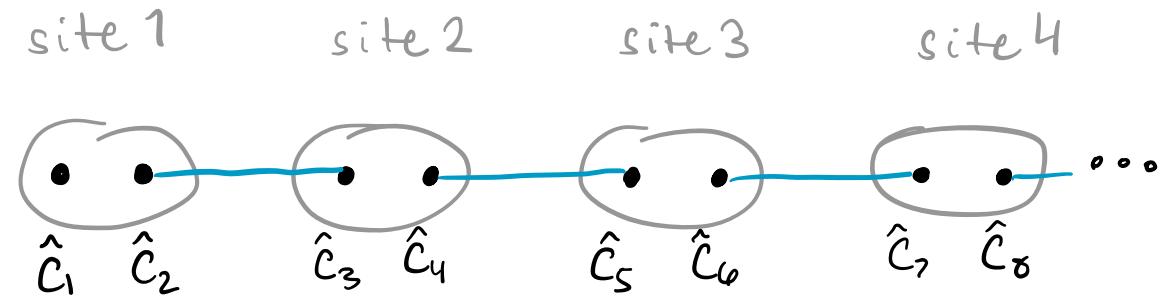
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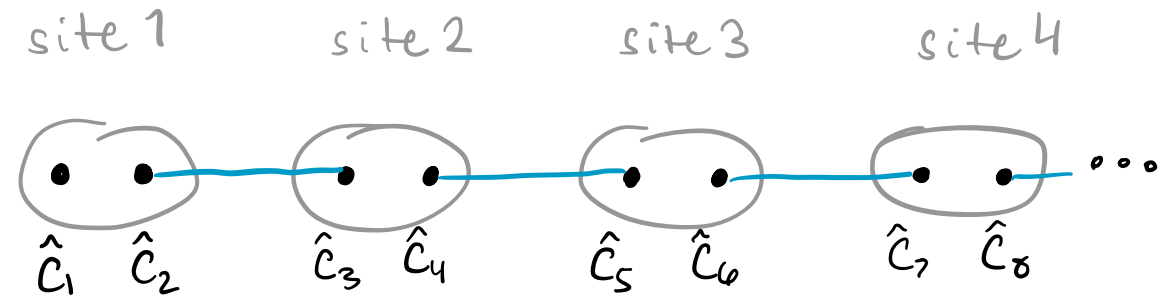
nontrivial phase can be stacked
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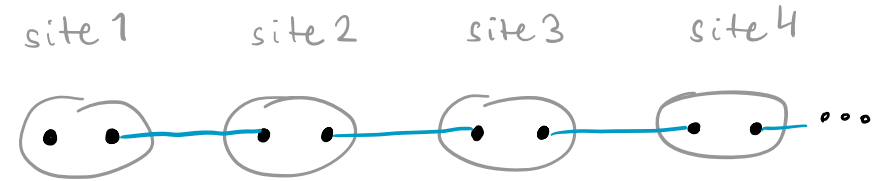
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$\leadsto$ generates $\mathbb{Z} =$ group of free class BDI SPTs

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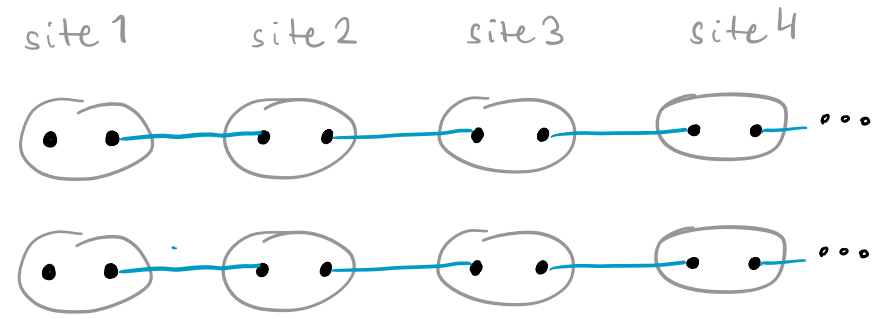


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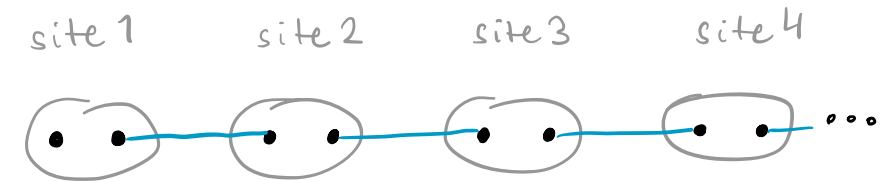
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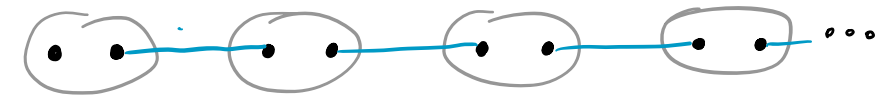
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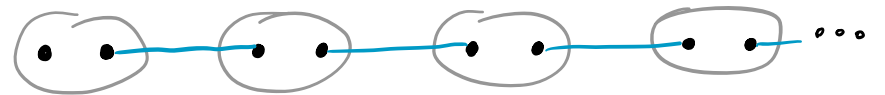
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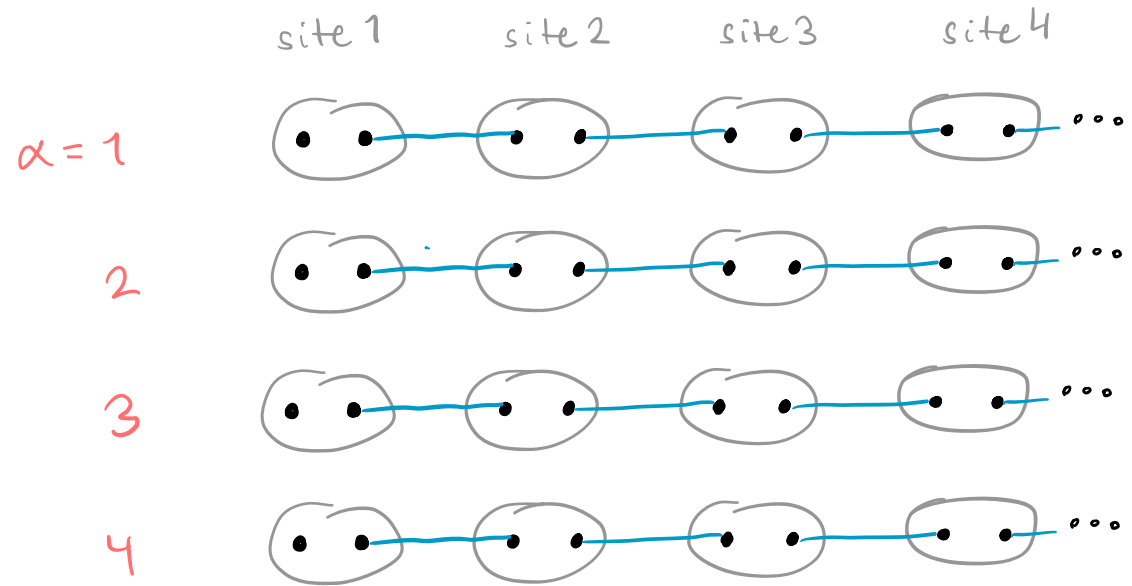


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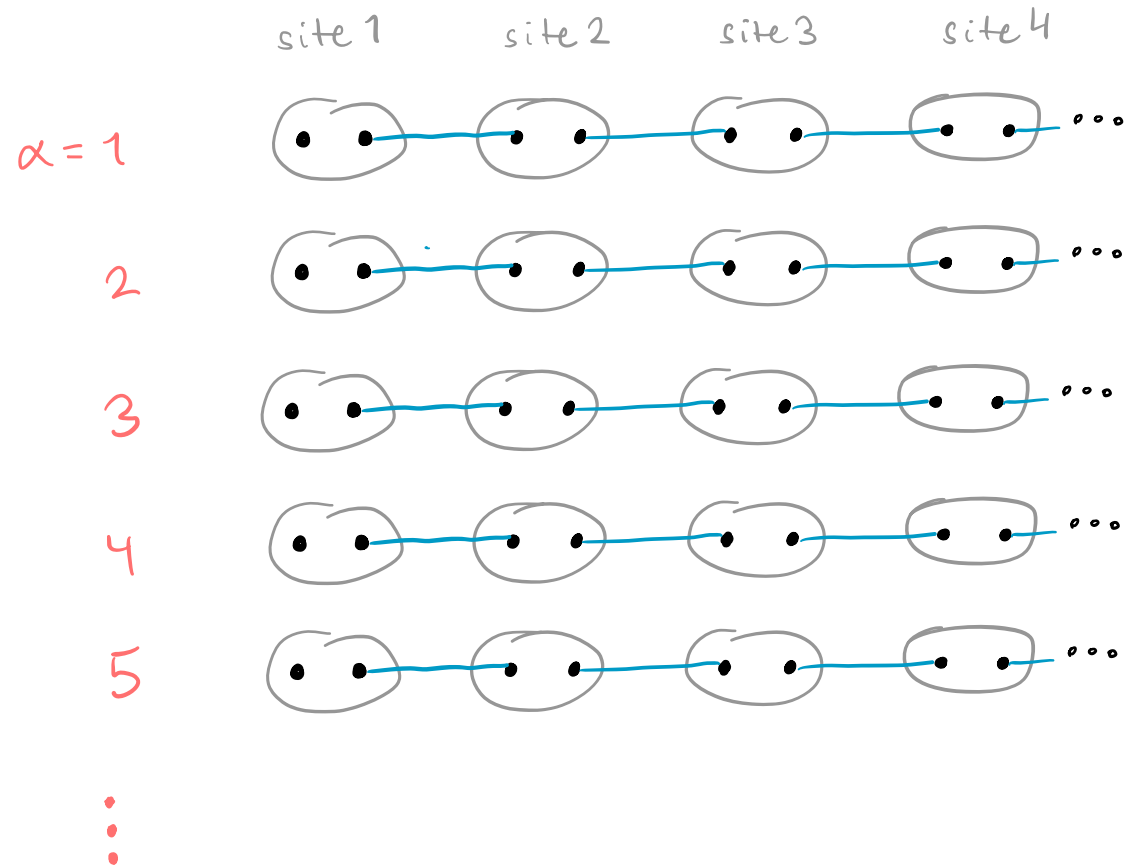
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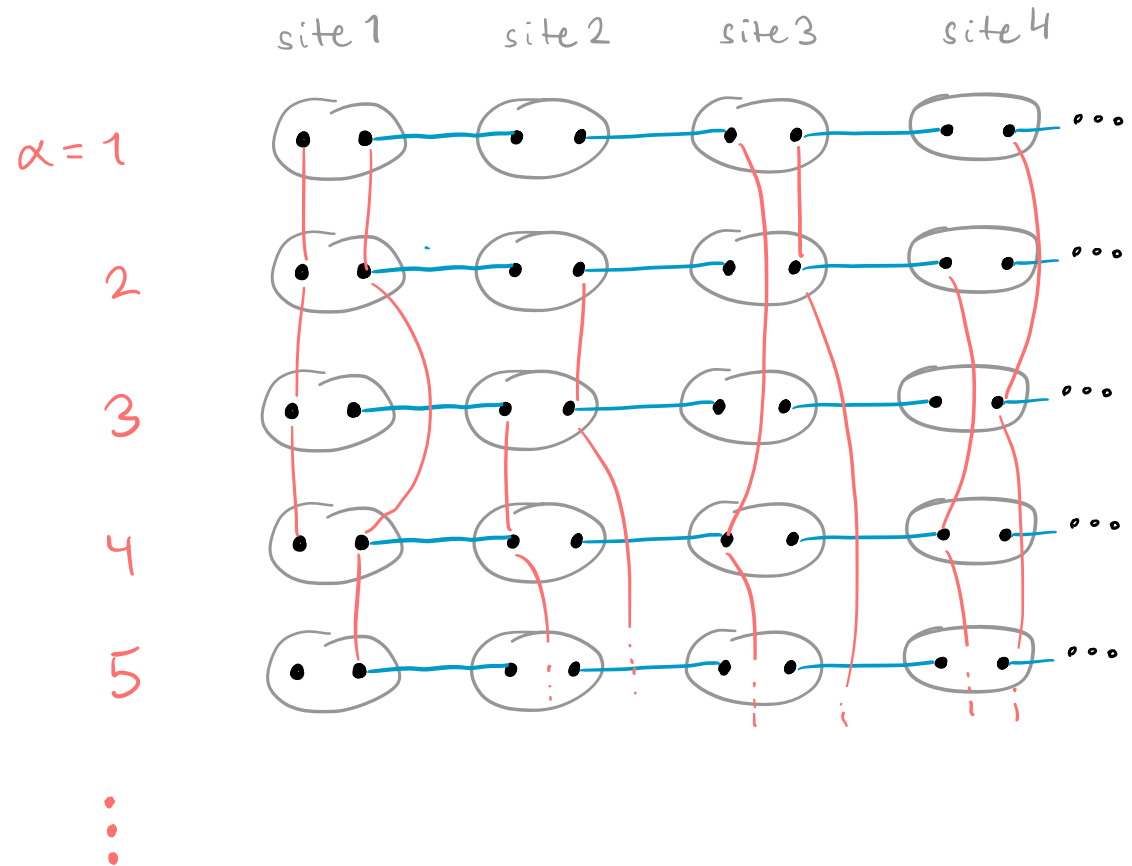
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eight copies can be trivialized via

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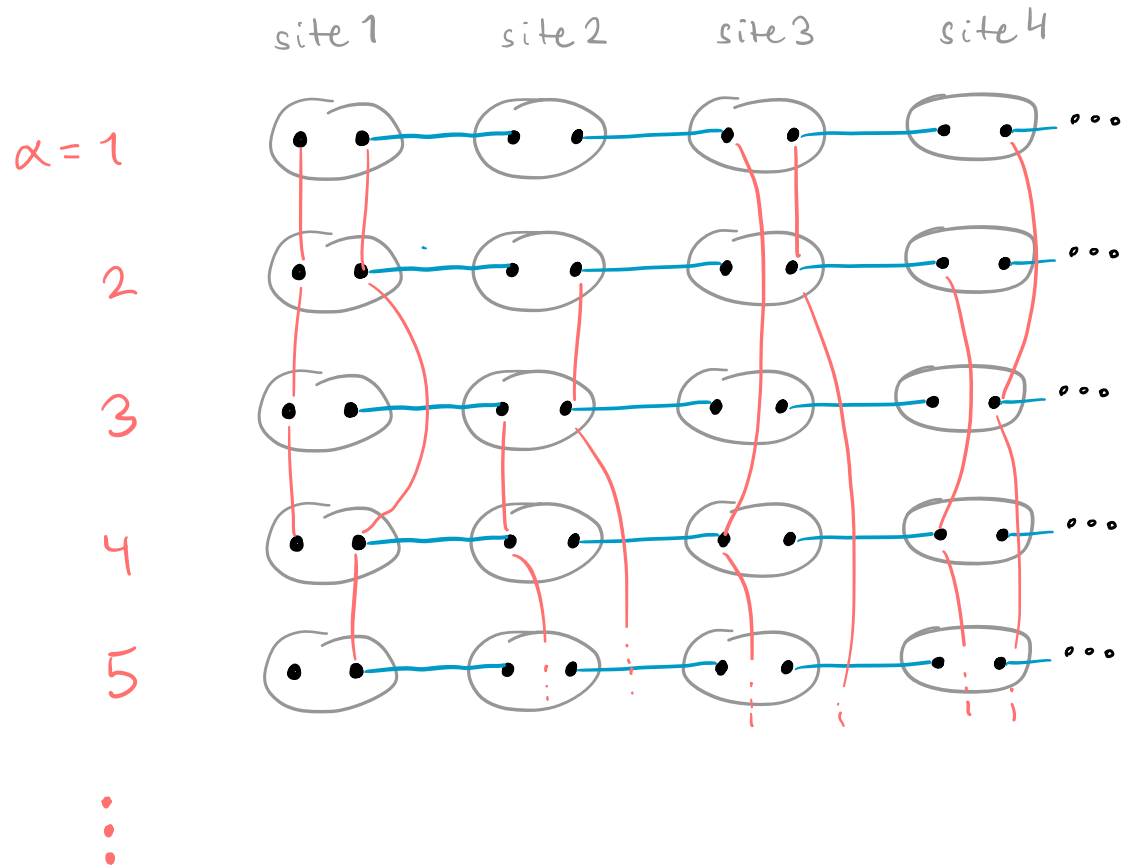
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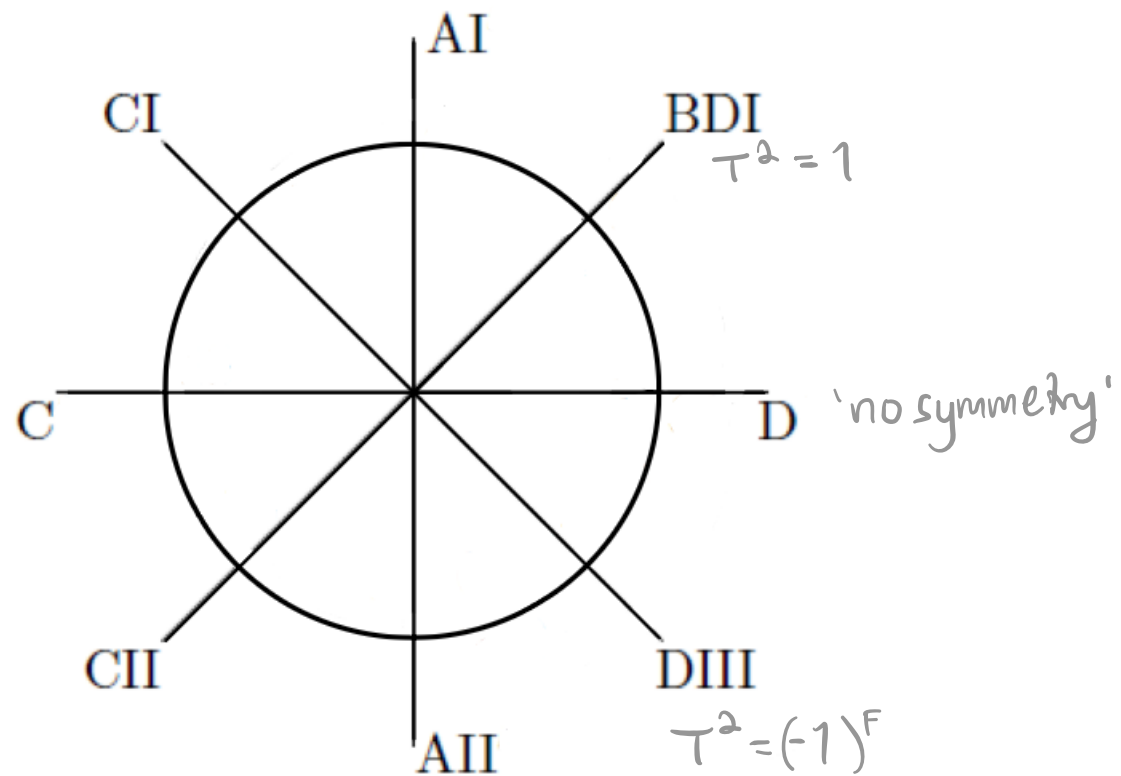
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group of interacting
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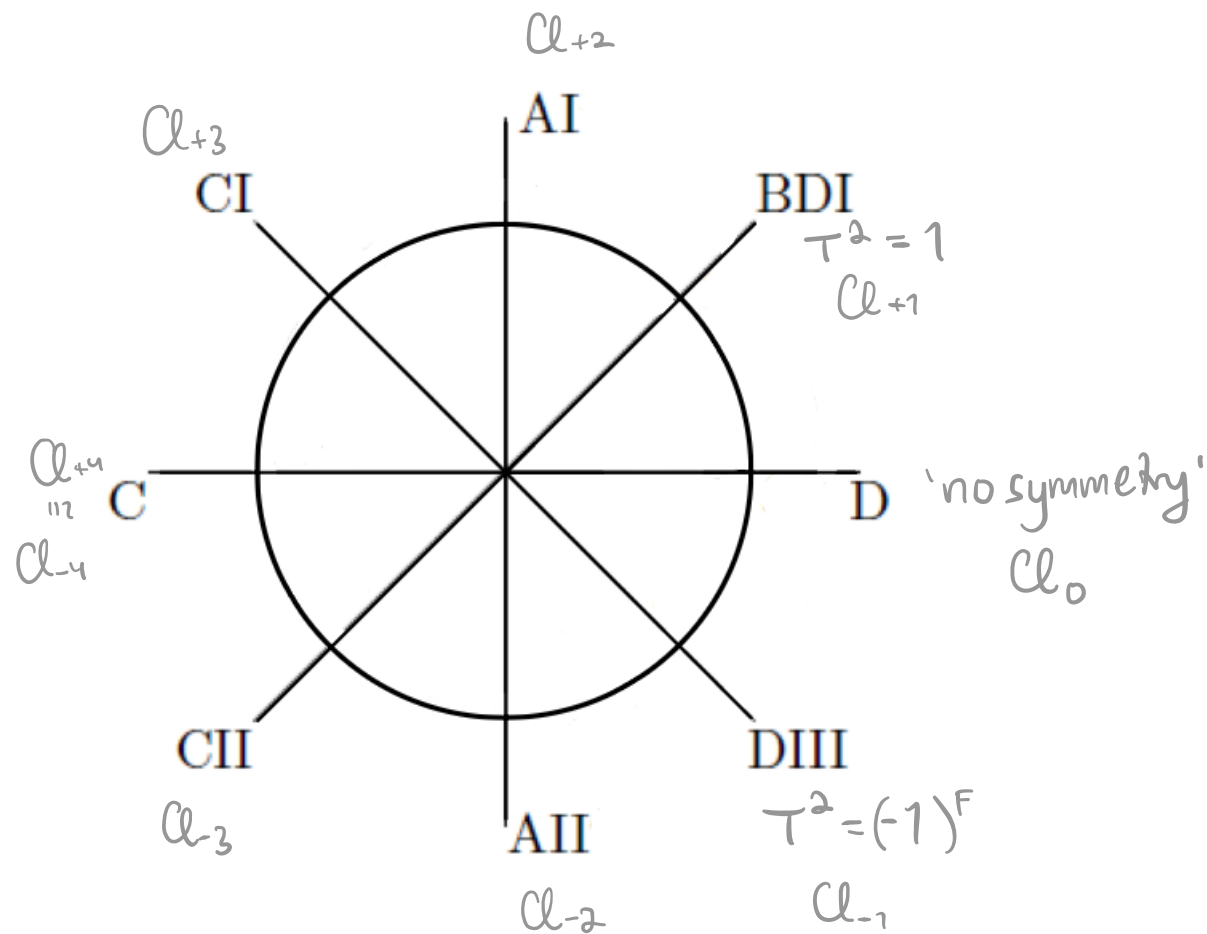
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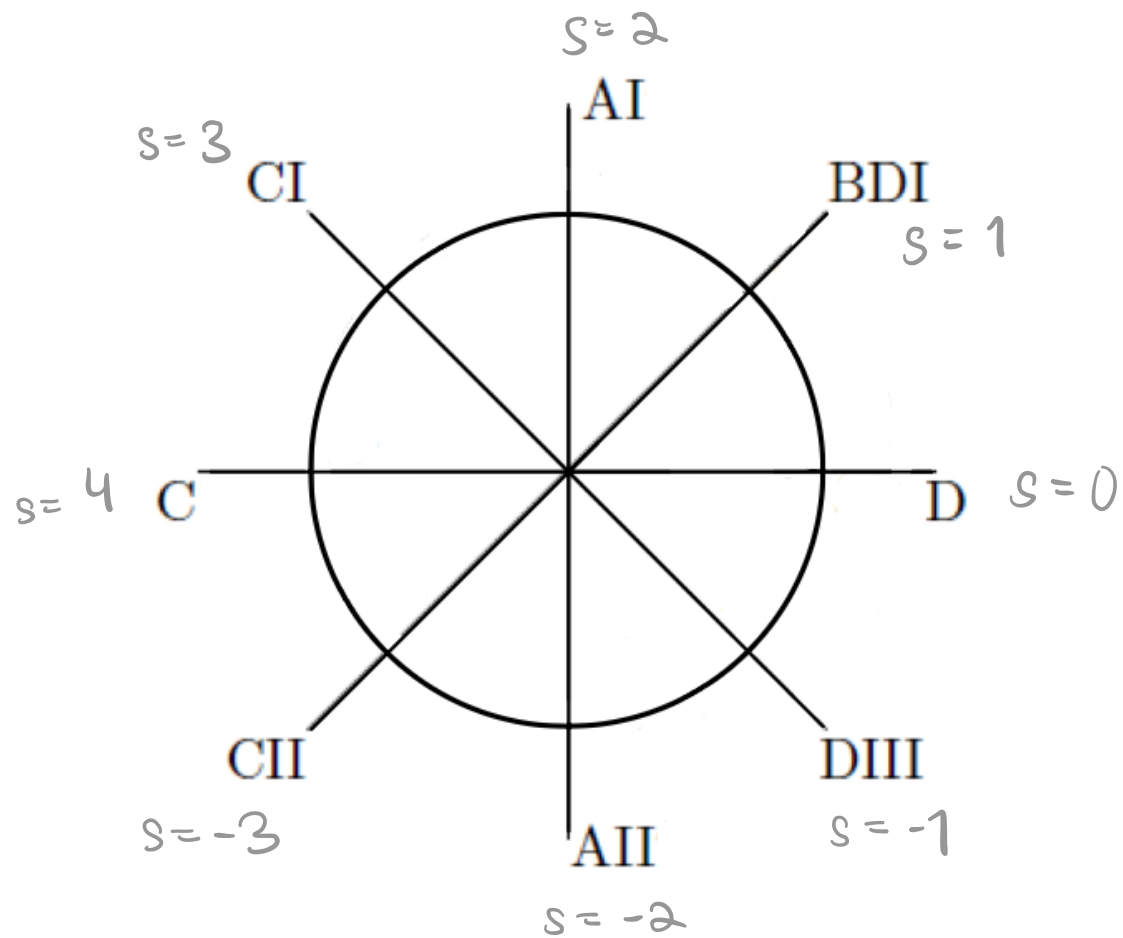
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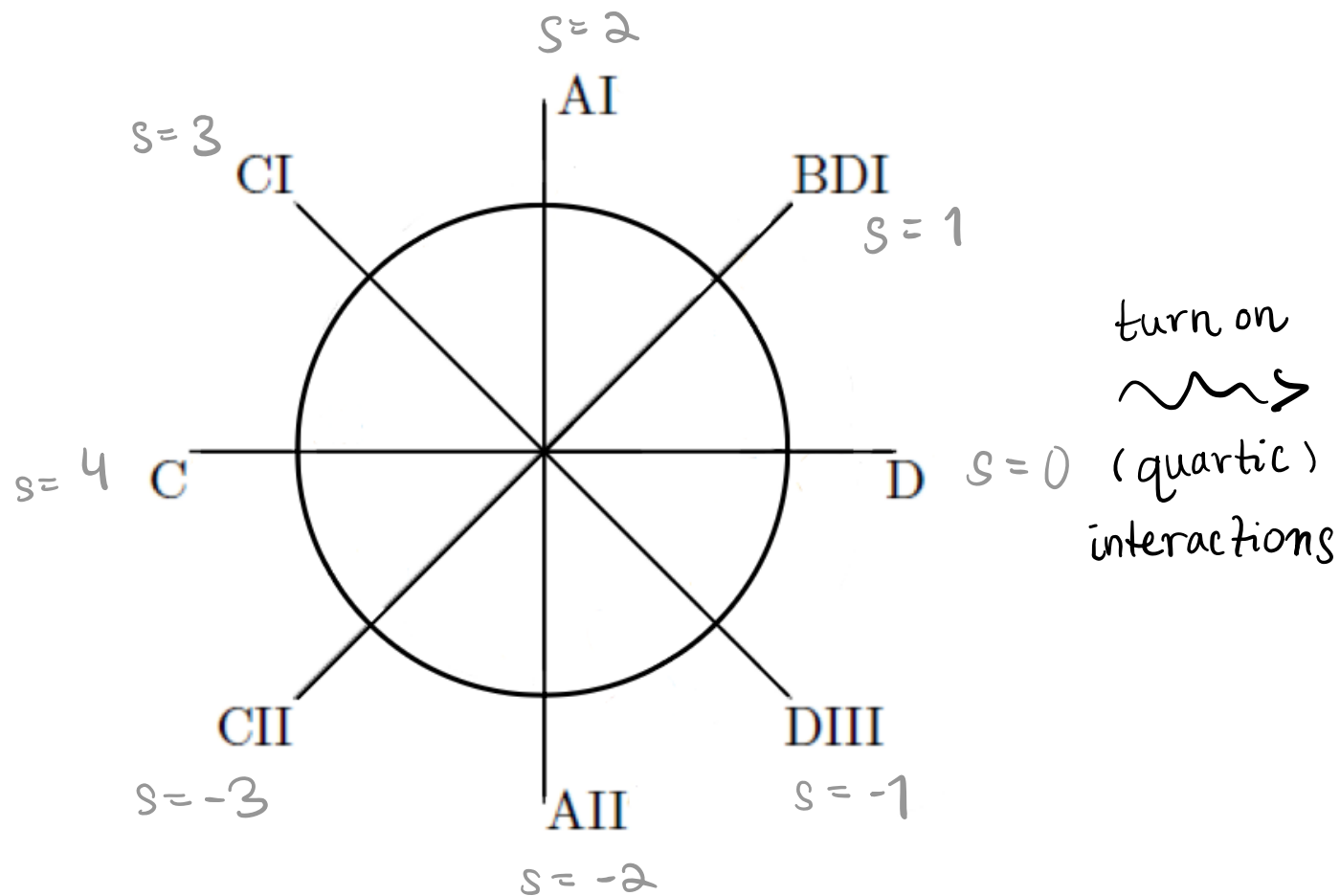
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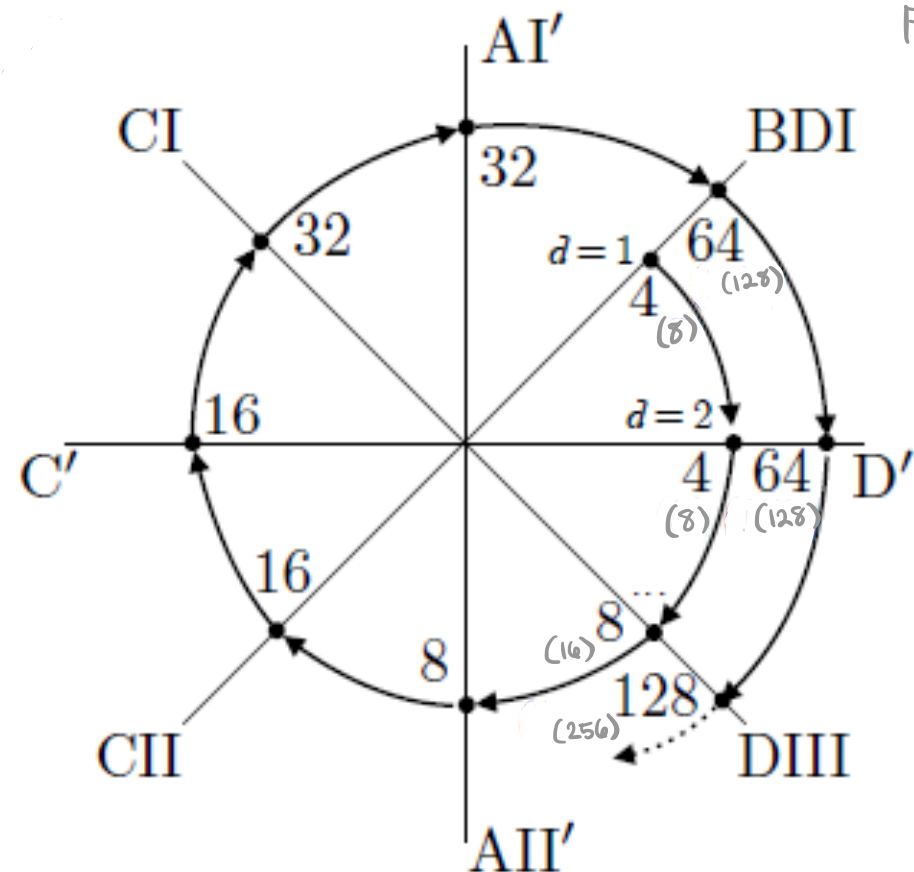
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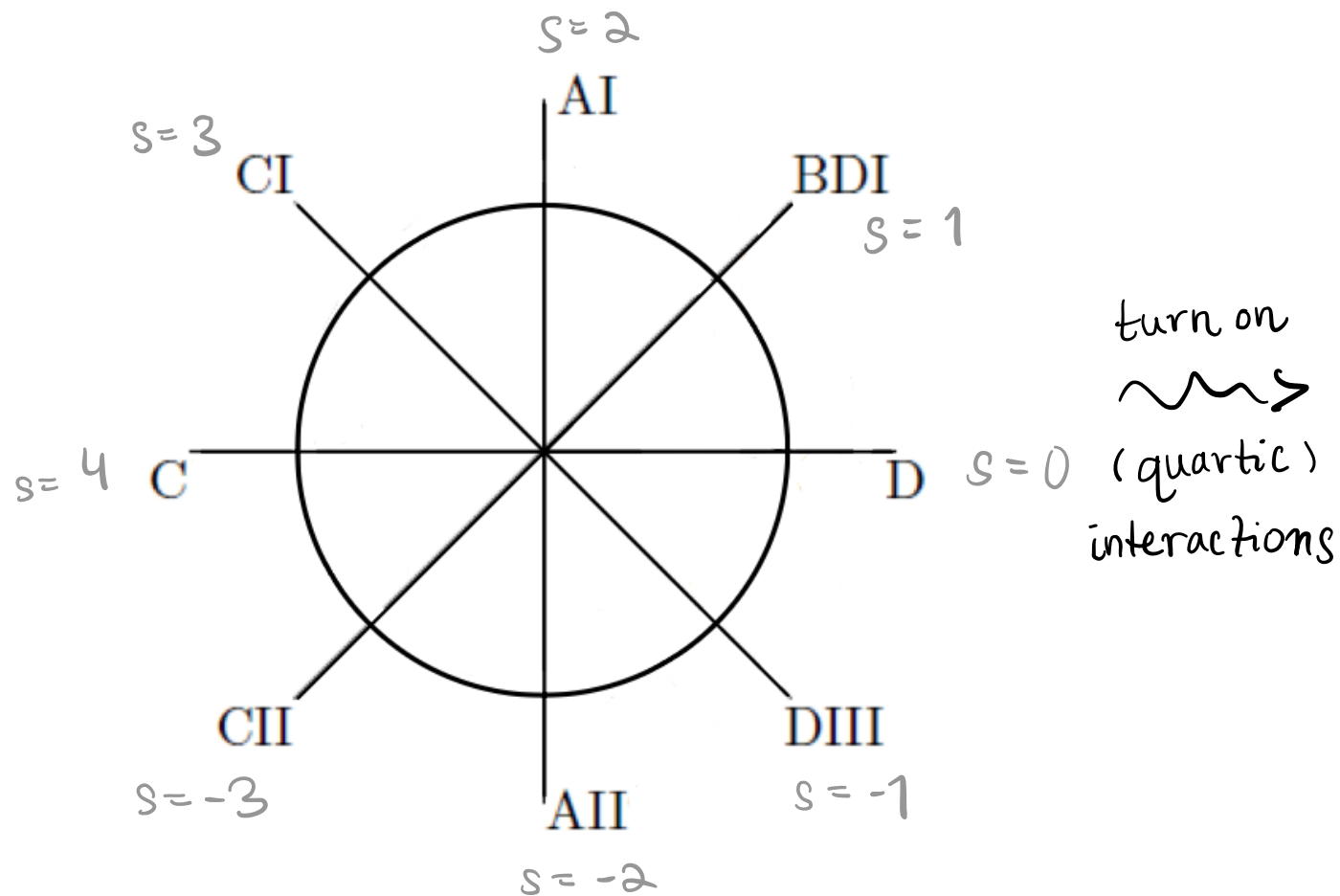
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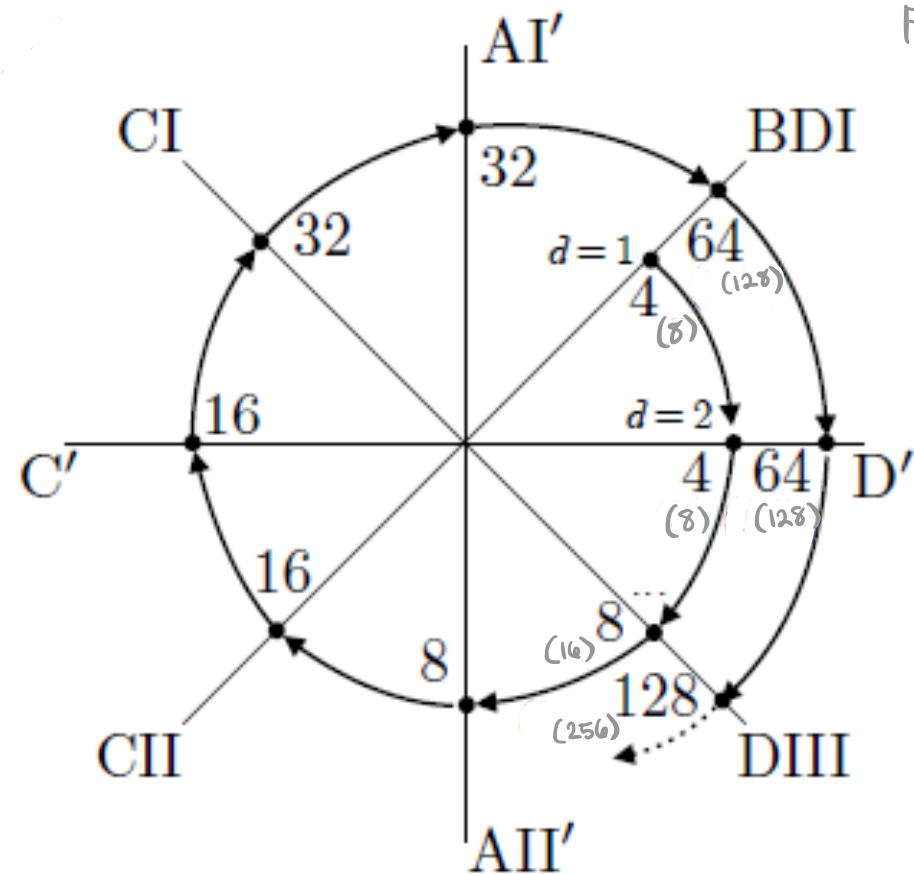
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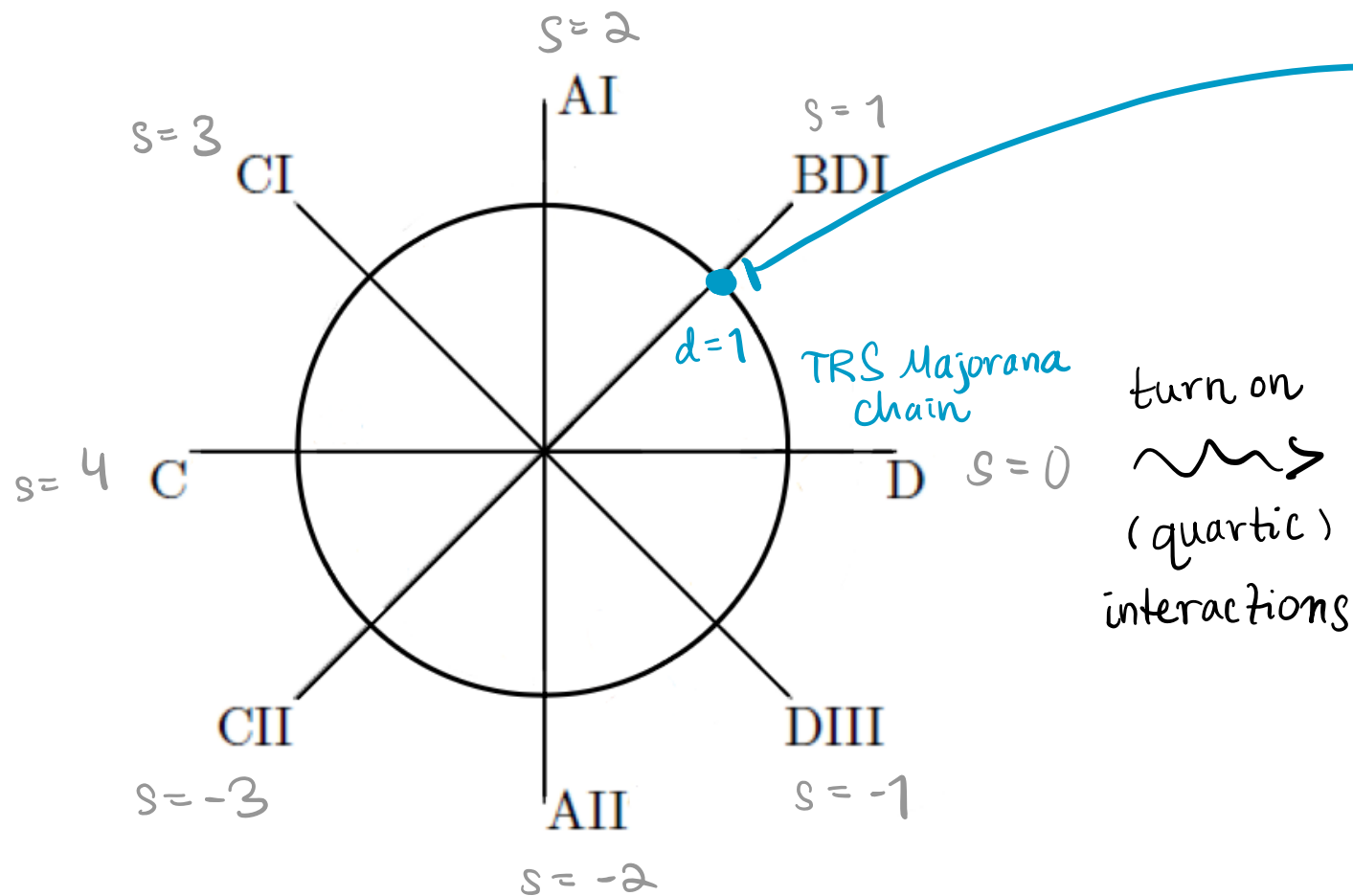
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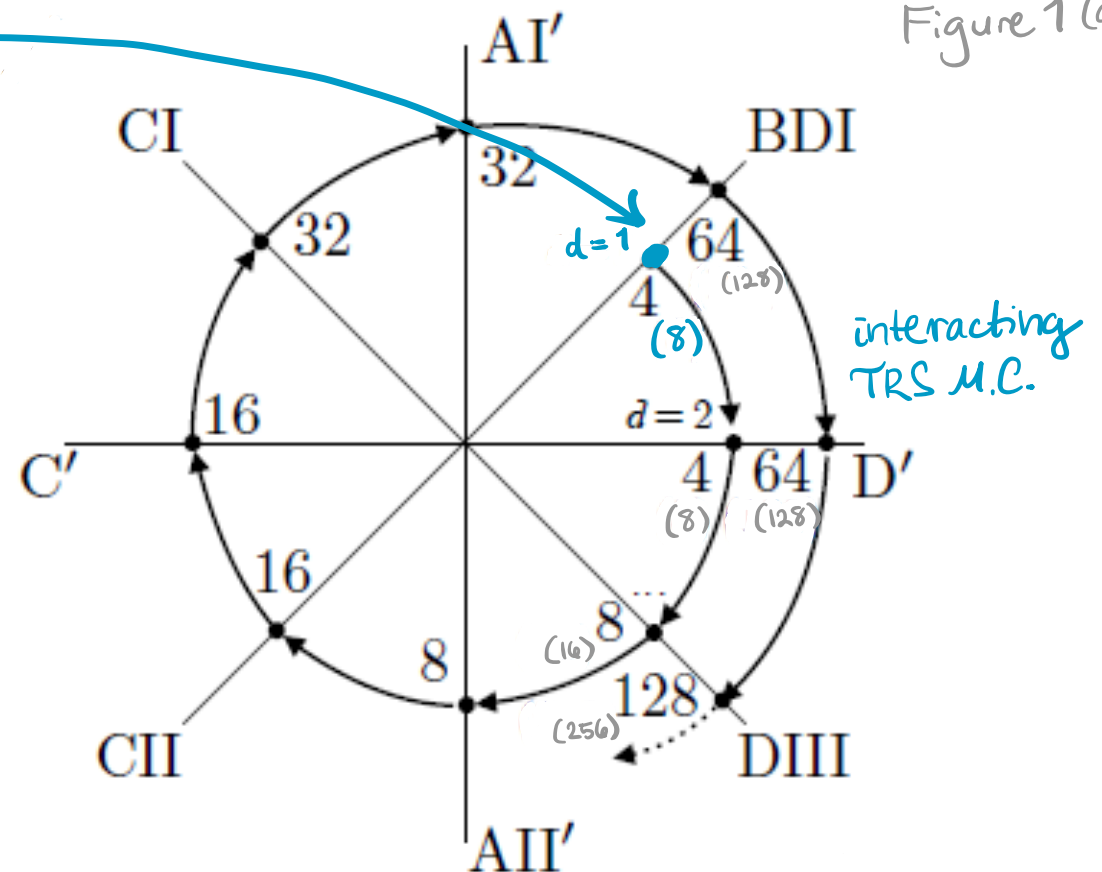
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Our Approach

$$\left\{ \begin{array}{c} \text{free fermion} \\ G\text{-SPTs} \end{array} \right\} \dashrightarrow \left\{ \begin{array}{c} \text{interacting} \\ \text{fermion } G\text{-SPTs} \end{array} \right\}$$

$$(\mathcal{M}_{\text{free}} \setminus \Delta) / \sim \dashrightarrow (\mathcal{M} \setminus \Delta) / \sim$$

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$\begin{matrix} \text{12} \\ \text{KO}_G^{d-2, \tau} \end{matrix}$

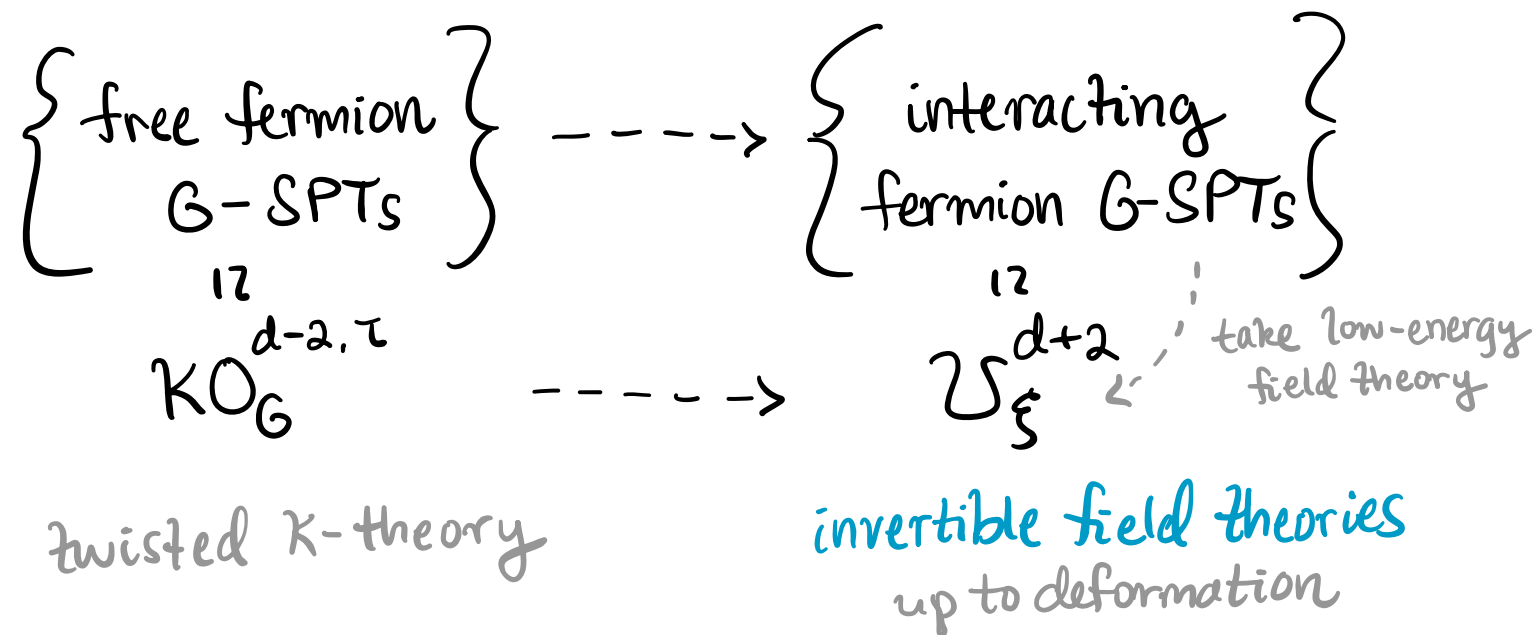
twisted K-theory

see e.g. [Kitaev, '09], [Ryu-Schmyder-Furusaki-Ludwig, '10],
[Freed-Moore, '13], [Thiang, '16], [Gomi, '17], ...

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[Beaudry-Campbell, '18], [Debray, '21], [Grady, '23], ...

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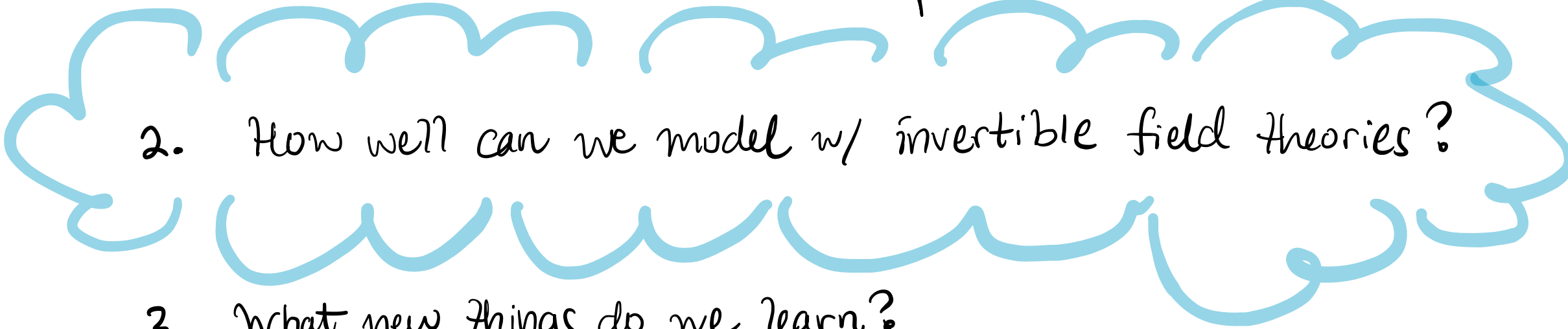
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Today: define F2I for the Bott spiral and study its image

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2. How well can we model w/ invertible field theories?

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Spoiler: our results.

theorem [Debray-K.-Stehouwer]

Let $\tilde{m} := \lfloor (d-l+k+i+1)/8 \rfloor$, where $i \equiv l \pmod{4}$. For d large enough...

$$F2I_{l,k} : KO^{d+l-k-2} \rightarrow \mathcal{U}_{\text{Spin-}(l,k)}^{d+2} \quad \text{has image } \dots$$

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$$\begin{cases} \mathbb{Z}/2^{4+4\tilde{m}-i} \\ \mathbb{Z}/2^{5+4\tilde{m}-i} \\ \mathbb{Z}/2 \end{cases}$$

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* subtleties: making this compatible w/ dimensional reduction

K-theory classification of free fermion SPTs

defn. Let A be a finite-dimensional superalgebra.

Then

$$K_n(A) := \left(\text{Mod}_{A \otimes \text{Cl}_{-n}}^{\text{proj}} \right) / \left(\text{Mod}_{A \otimes \text{Cl}_{-n+1}}^{\text{proj}} \right)$$

action of one more
Clifford generator



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* For class BDI, choose Cl_{+1} to encode $\tau^2 = 1$

The TRS Majorana chain generates $K_{2-1}(\text{Cl}_{+1}) \cong K_0(\mathbb{R}) \cong KO^0 \cong \mathbb{Z}$

Modeling interacting theories I : tangential structures

defn. A **stable tangential structure** $\xi: X \rightarrow BO$ on a

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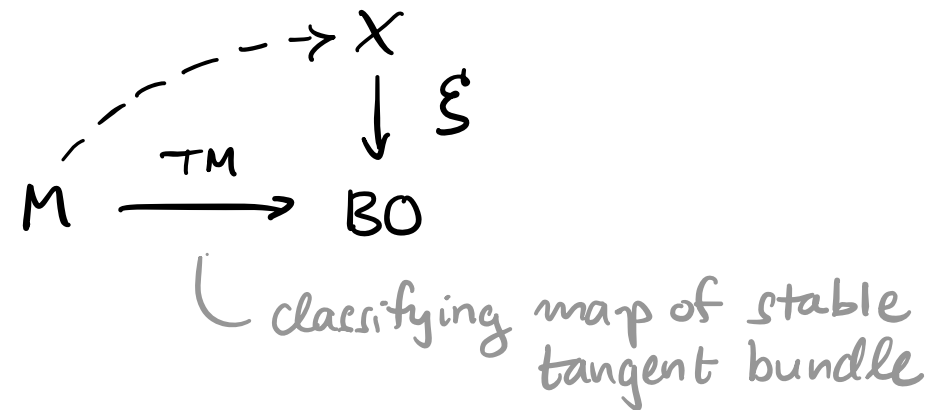
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$X = BPin^+, BPin^- \rightsquigarrow$ $pin^\pm$ structures $\sim$ unoriented versions of spin structures

Modeling interacting theories II: bordism

defn. The bordism group of d -manifolds w/ ξ -structure is

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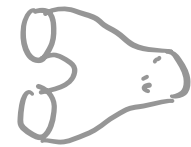
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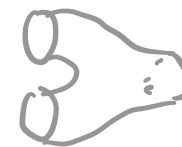
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example. In $d=2$ with $\xi: B\text{Pin}^- \rightarrow BO$, the group is

$$\Omega_2^{\text{Pin}^-} \cong \mathbb{Z}/8 \quad \text{generated by } [\mathbb{R}P^2]$$

Modeling interacting theories III : invertible field theories

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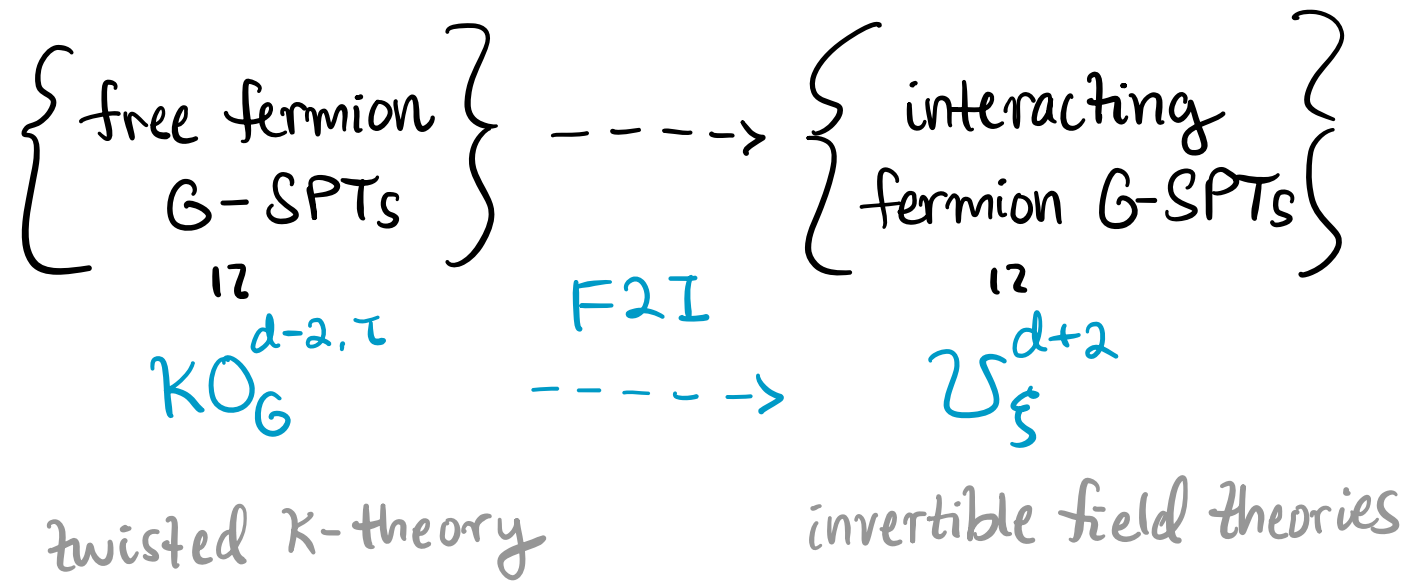
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↖ bordism groups
↖ group of IFTs

Connecting the classifications

To construct the map



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$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{free fermion} \\ G\text{-SPTs} \end{array} \right\} & \dashrightarrow & \left\{ \begin{array}{c} \text{interacting} \\ \text{fermion } G\text{-SPTs} \end{array} \right\} \\ \begin{array}{c} \text{IZ} \\ KO_G^{d-2, \tau} \end{array} & \begin{array}{c} \text{F2I} \\ \dashrightarrow \end{array} & \begin{array}{c} \text{IZ} \\ \mathcal{U}_\xi^{d+2} \end{array} \\ \text{twisted K-theory} & & \text{invertible field theories} \end{array}$$

we need to match up G, τ and $\xi \dots$

Fermionic Symmetry Groups & Algebras I

defn. [Benson, '88] A fermionic group is...

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$\downarrow$
 $(-1)^F$ fermion parity

Fermionic Symmetry Groups & Algebras I

defn. [Benson, '88] A fermionic group is...

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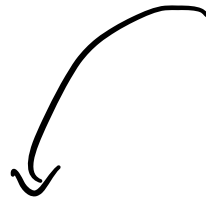
- * a grading $\theta: G_f \rightarrow \mathbb{Z}/2$ $\rightsquigarrow$ classified by $\theta \in H^1(BG_b; \mathbb{Z}/2)$
 $\theta^{-1}(+1)$ unitary, $\theta^{-1}(-1)$ antiunitary

Fermionic Symmetry Groups & Algebras II

fermionic group
 (G_b, θ, ω)

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assume G_b
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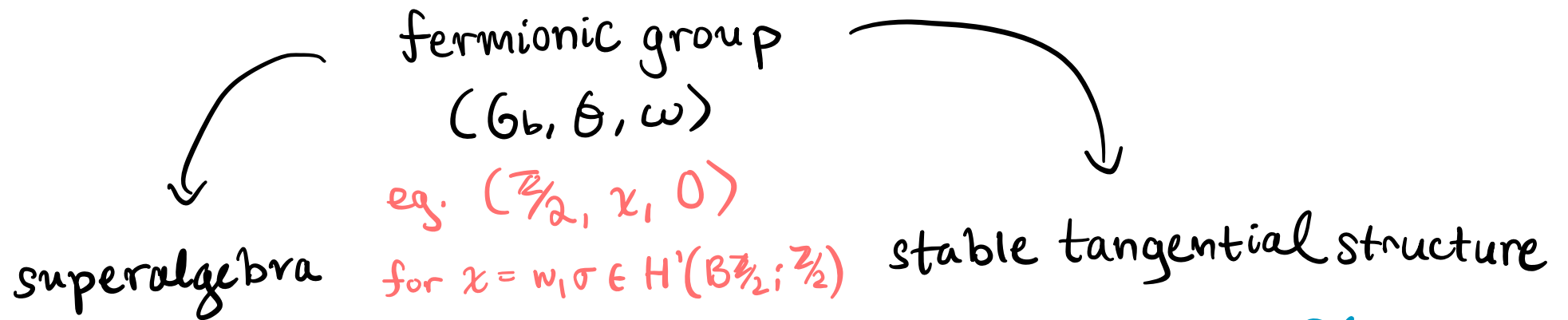
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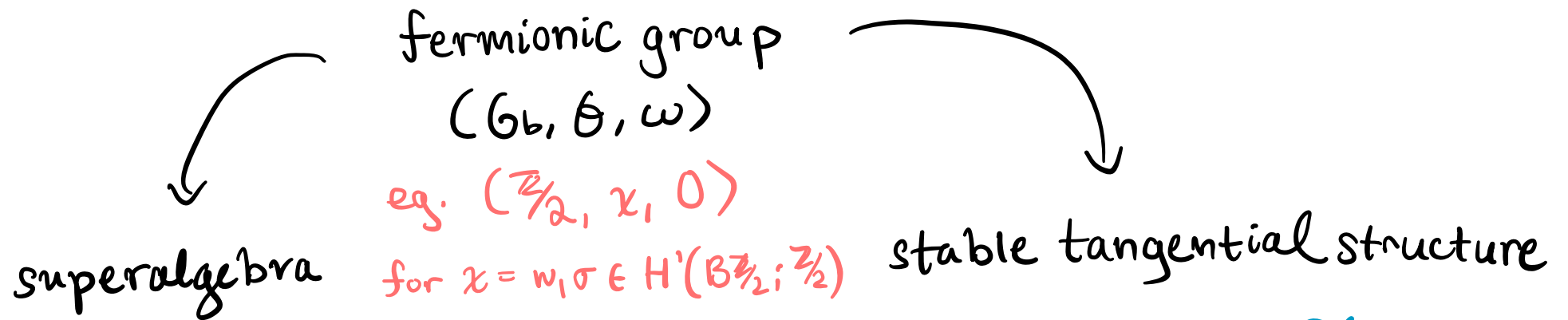
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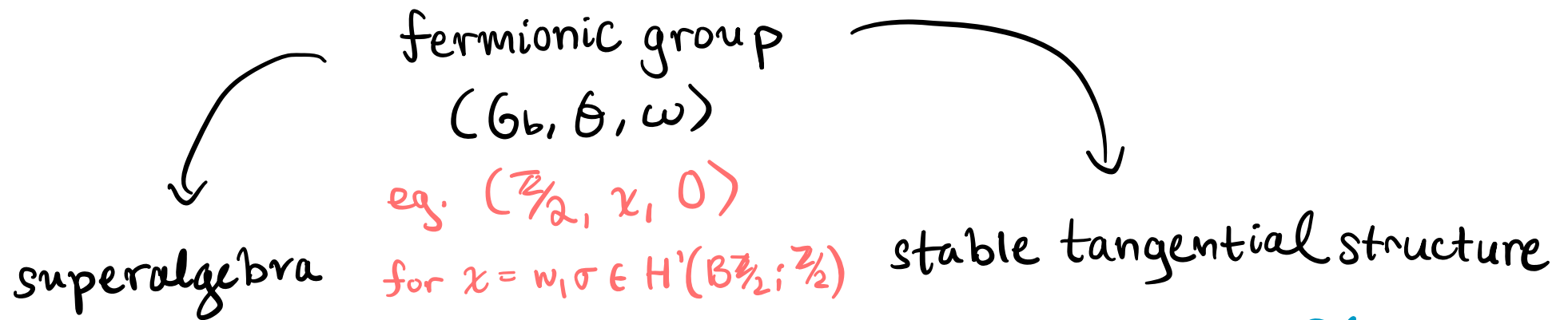
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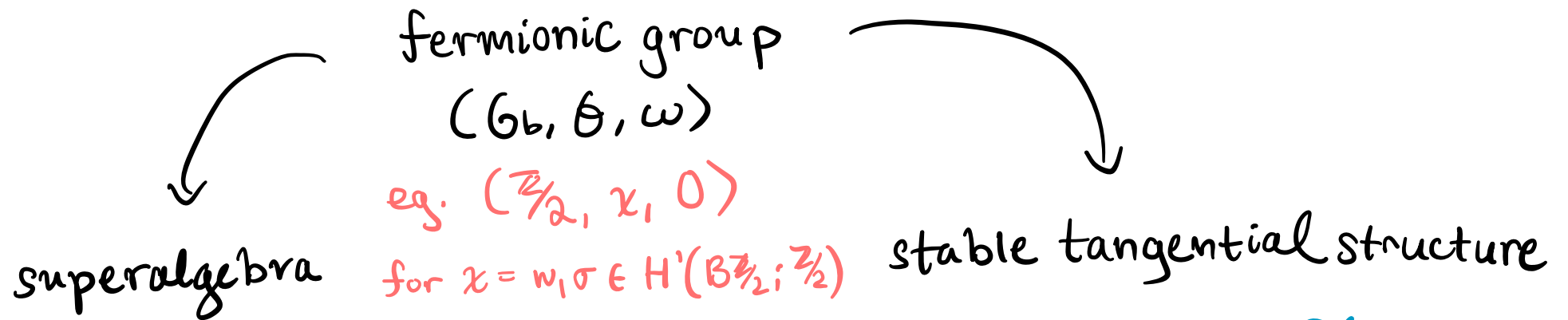
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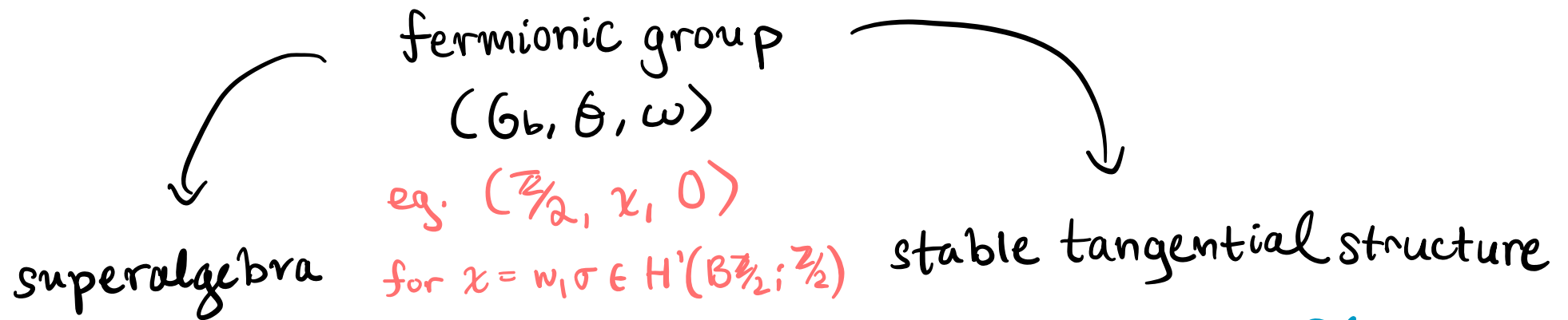
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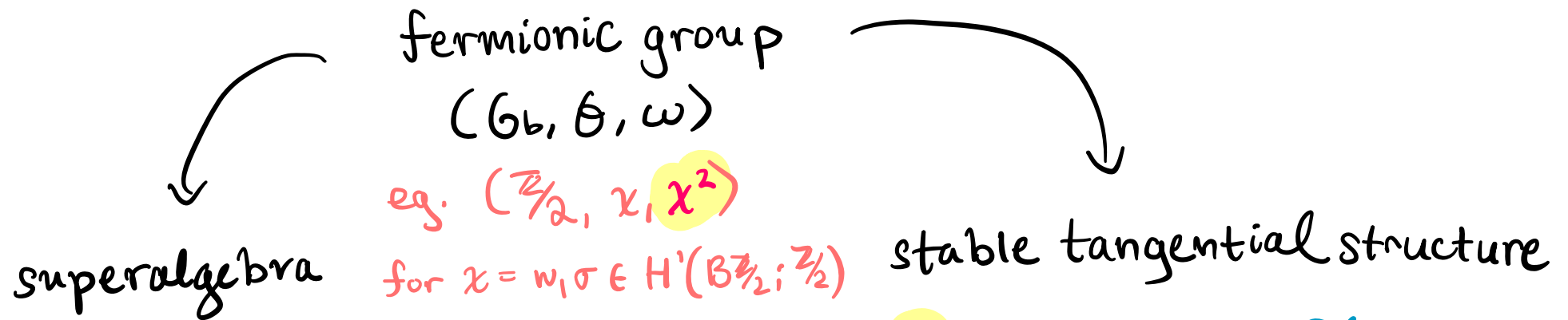
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Fermionic Symmetry Groups & Algebras II



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say $d=3$

free SPTs

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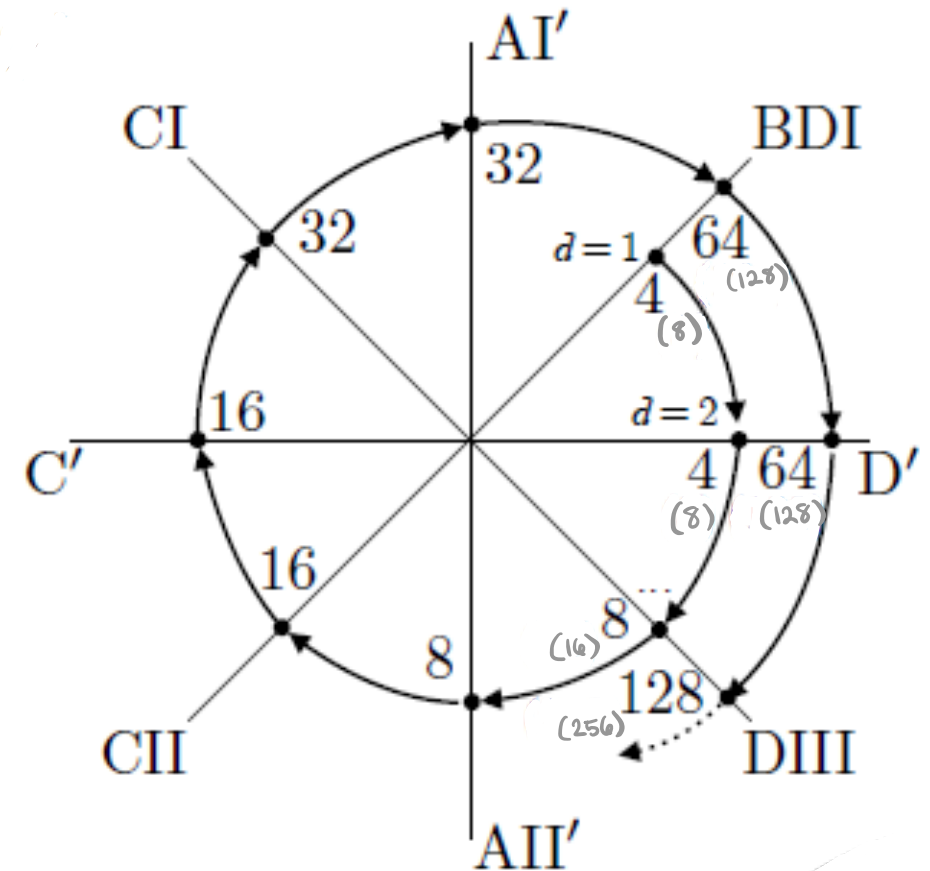
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Symmetries in the Bott spiral

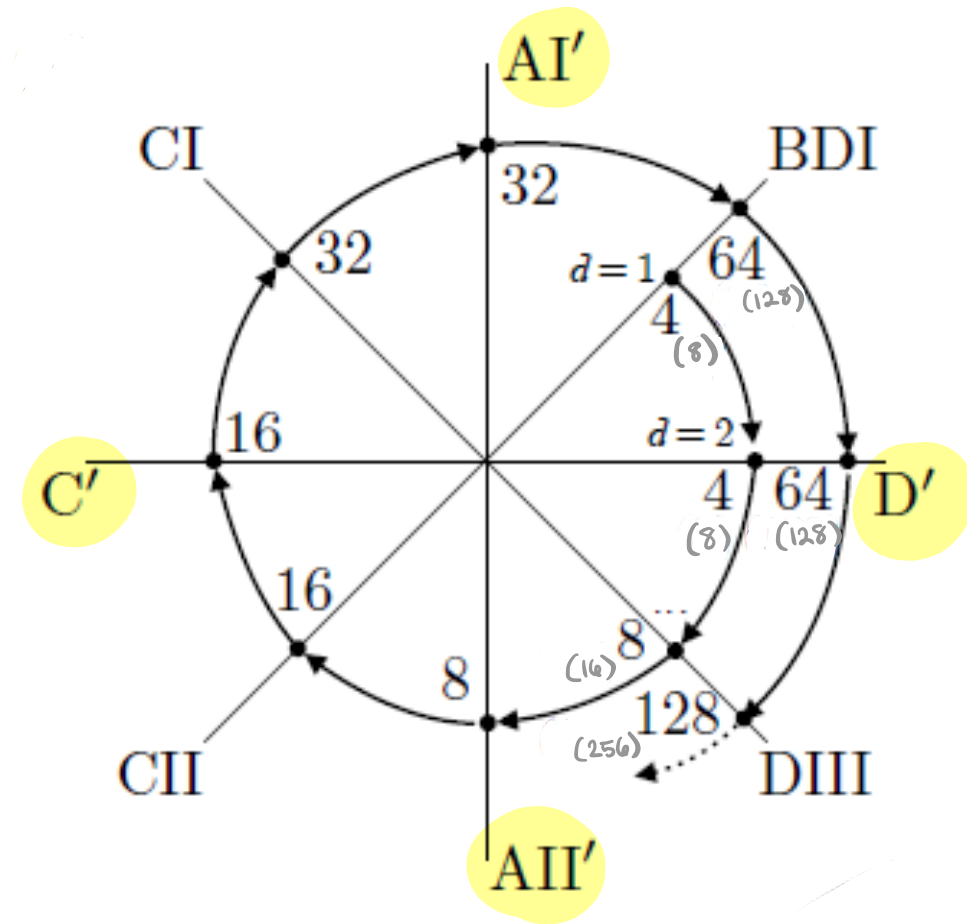
The choice of fermionic group is subtle...



Symmetries in the Bott spiral

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half the types are **primed** AZ classes:
[Yao-Ryu, '13, Qi, '13]

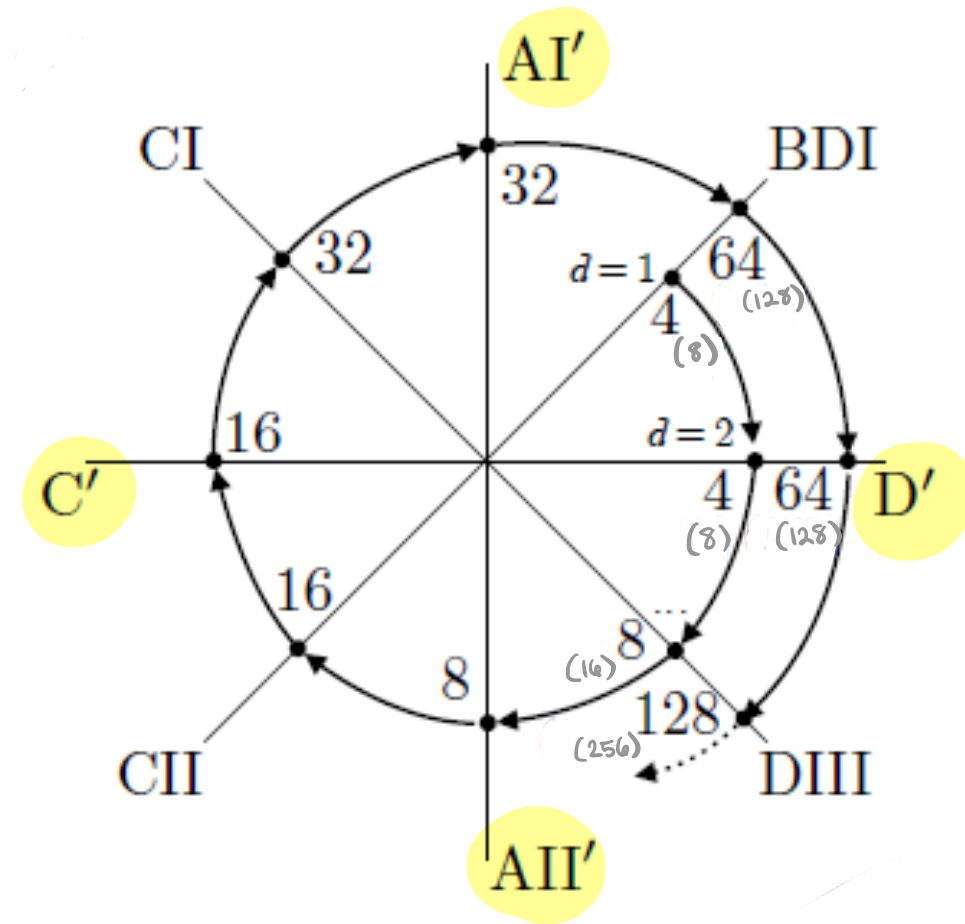
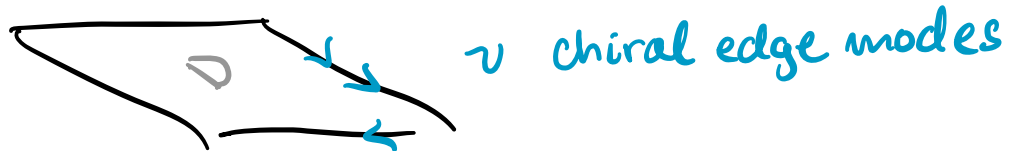


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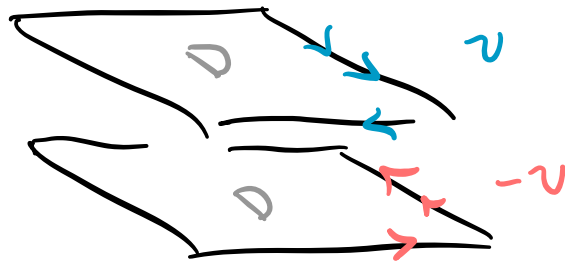
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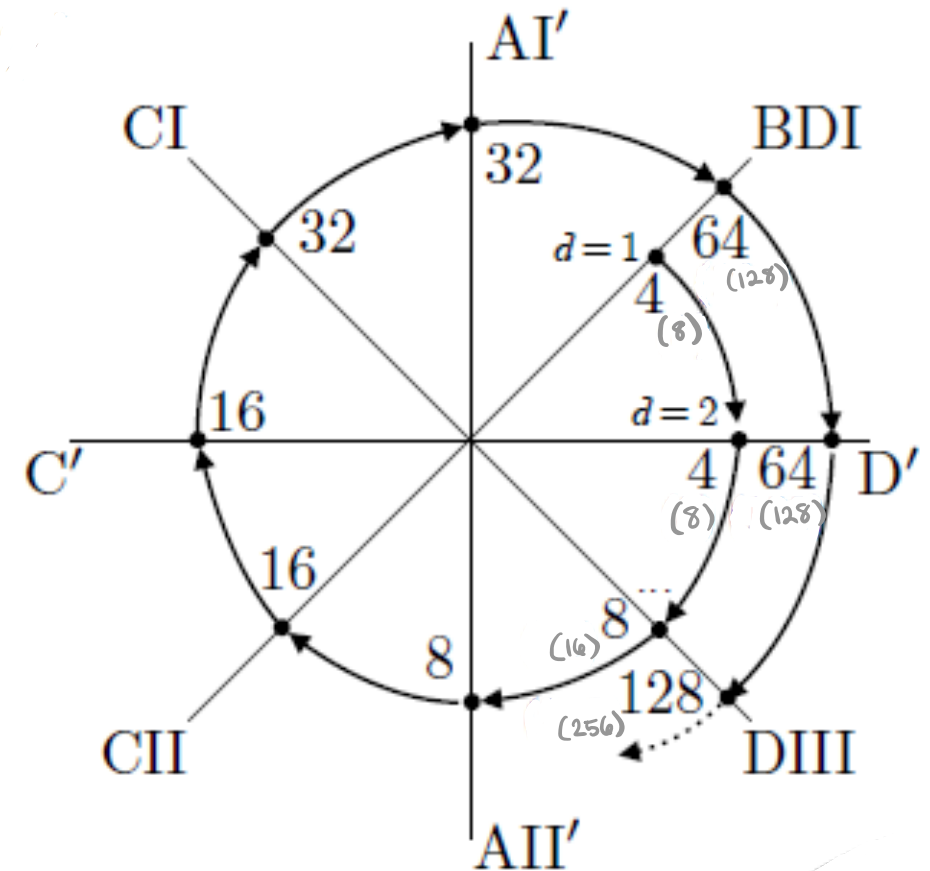
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Stack

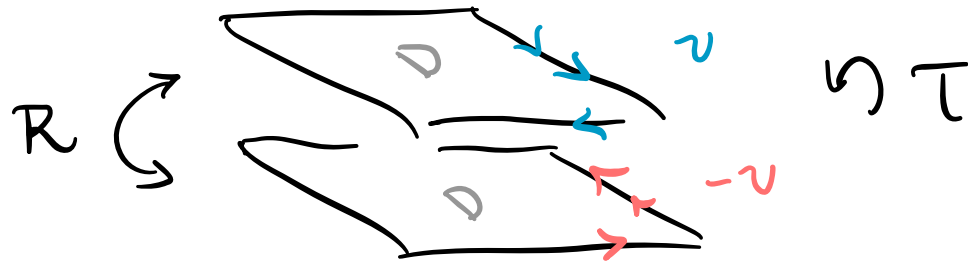


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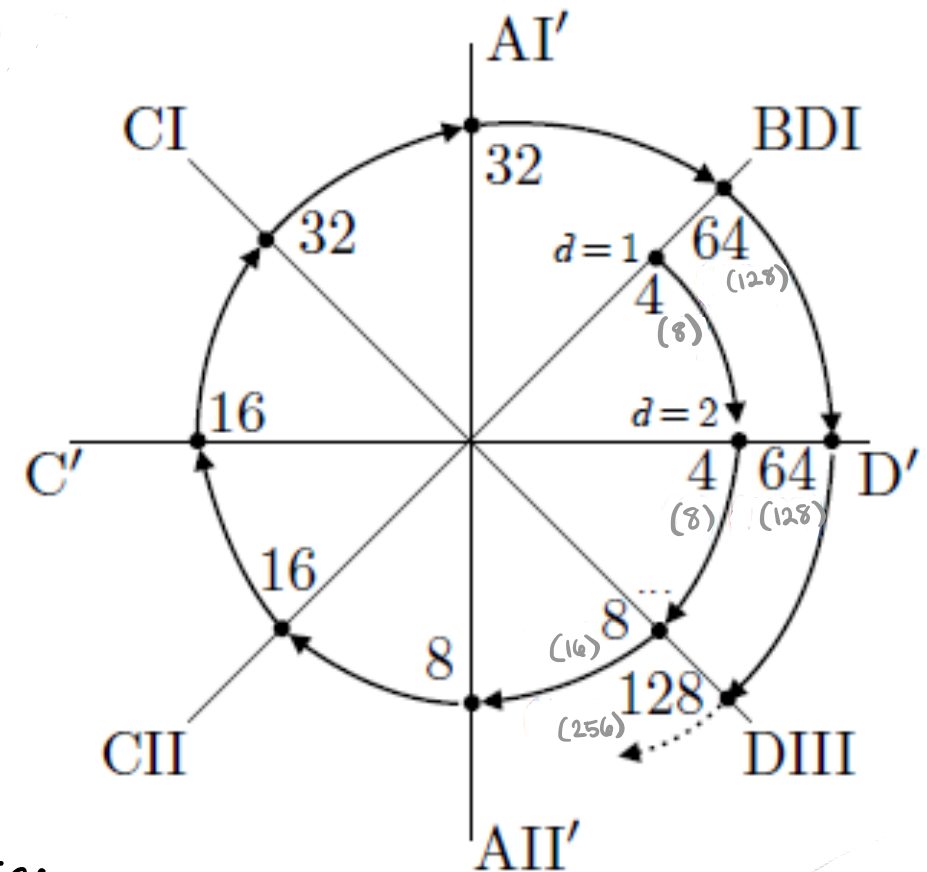
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enforce symmetries
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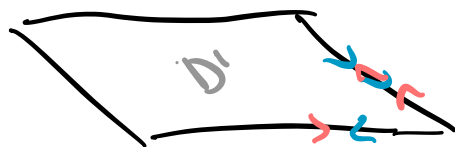


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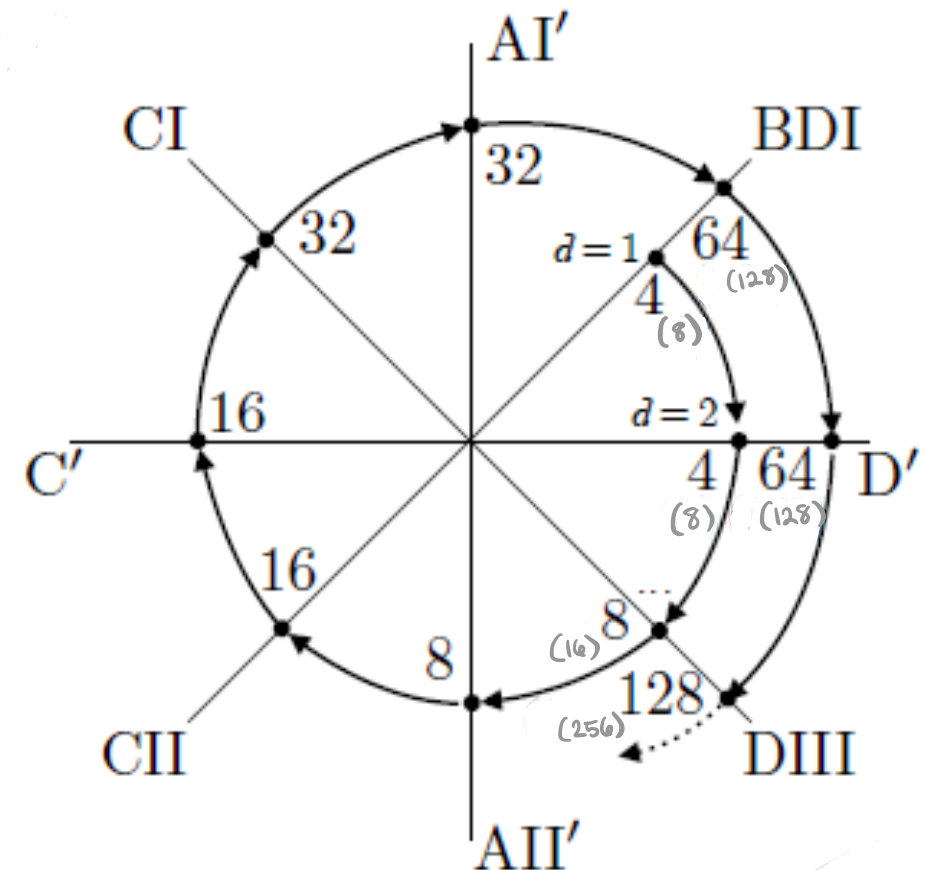
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$$v \in \mathbb{Z}$$

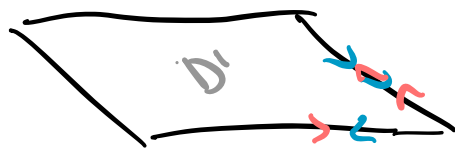


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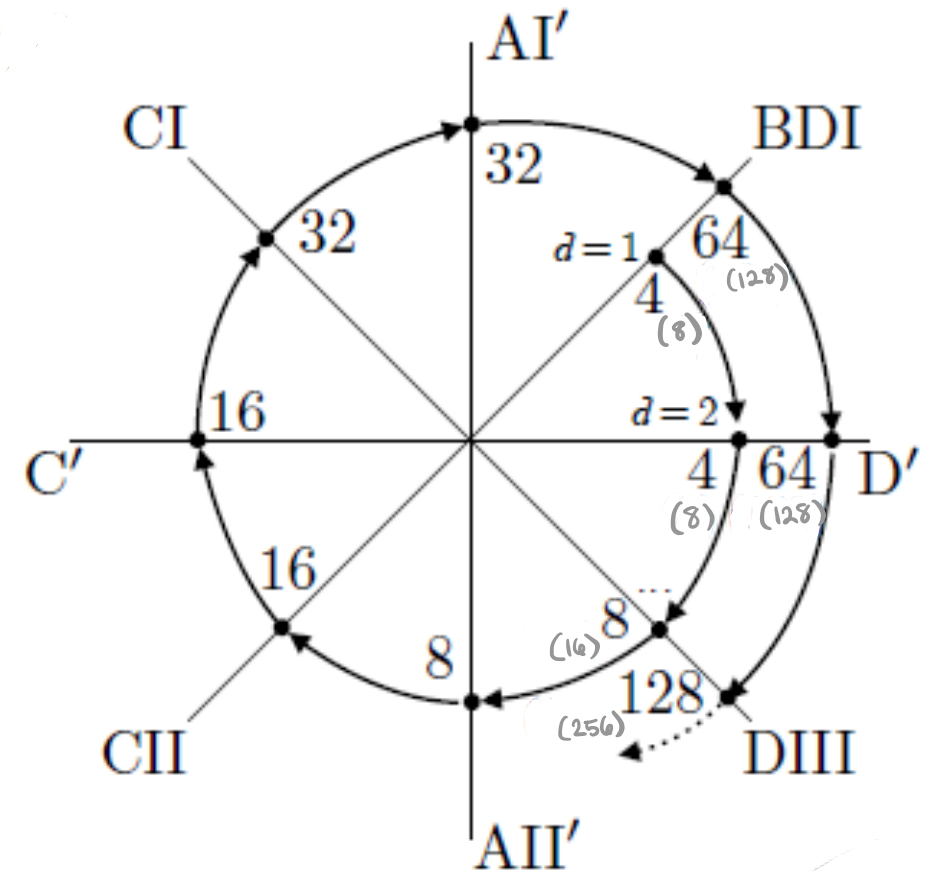
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defn. [DKS] If G_f corresponds to a symmetry type,

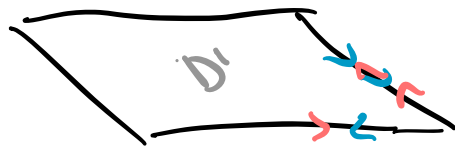
the primed type is given by $G_f' := G_f \hat{\times} \text{Pin}^+(1) \hat{\times} \text{Pin}^-(1)$

Symmetries in the Bott spiral

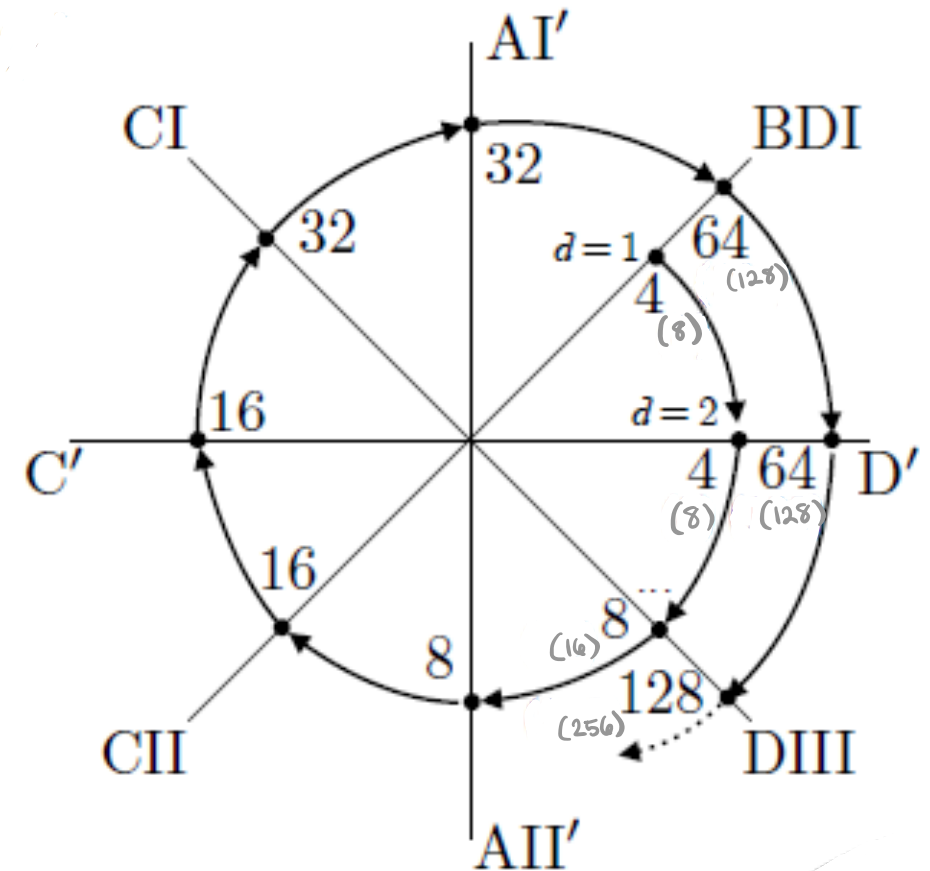
The choice of fermionic group is subtle...

half the types are **primed** AZ classes:
[Yao-Ryu, '13, Qi, '13]

eg. to form D' in $d=2 \dots$



$$v \in \mathbb{Z}$$



defn. [DKS] If G_f corresponds to a symmetry type,

the primed type is given by $G_f' := G_f \hat{\times} \text{Pin}^+(1) \hat{\times} \text{Pin}^-(1)$

corresponds to $\mathcal{C}_{l+1}$

fermionic tensor product

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'Morita variance'

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	(class D) $\int_d^{\text{Spin}}$	$\neq$	(class D') $\int_d^{\text{"Spin-(1,D)"}}$
$d=0$	$\mathbb{Z}$		$\mathbb{Z}/2$
1	$\mathbb{Z}/2$		$\mathbb{Z}/2$
2	$\mathbb{Z}/2$		$(\mathbb{Z}/2)^2$
3	0		$\mathbb{Z}/8$
4	$\mathbb{Z}$		$(\mathbb{Z}/2)^2$

Defining the free-to-interacting map

The desired map is I_2 -dual to an ABS-type map [Freed-Hopkins, '21]

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But we only have **twisted** spin structures...

so we generalize from the twisted ABS maps

for Pin^+ and Pin^- defined by [Freed-Hopkins, '21]

Twisted ABS maps for $\text{spin}-(l,k)$ manifolds

defn. [DKS] For class D' , the **twisted ABS map** is

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cut out a submanifold
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ABS_{1,1}

* We actually define $\text{ABS}_{l,k}$ for **$\text{spin}-(l,k)$** manifolds:

manifolds M s.t. $TM + \underbrace{L_1 + \dots + L_l}_{l \text{ real line bundles}} - \underbrace{L_{l+1} - \dots - L_{l+k}}_{k \text{ virtual real line bundles}}$ is spin

Twisted ABS maps for $\text{spin}-(l,k)$ manifolds

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 \Omega_n^{\text{Spin}-(l,k)} \cong \Omega_n^{\text{Spin}} \left((B\mathbb{Z}_2^{\sigma-1})^{\wedge l} (B\mathbb{Z}_2^{1-\sigma})^{\wedge k} \right) \\
 \downarrow \text{forget a } \mathbb{Z}_2 \text{ bundle} \\
 \Omega_{n-l}^{\text{Spin}}(B\mathbb{Z}_2^{\sigma-1}) \xrightarrow{\text{ABS}} KO_{n-l}((B\mathbb{Z}_2^{\sigma-1})^{\wedge k}) \\
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 KO_{n-l+k}
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cut out a submanifold

forget a $\mathbb{Z}_2$ bundle

classical ABS

difference bundle
KO-class of $\sigma \rightarrow B\mathbb{Z}_2$

$\text{ABS}_{l,k}$

correspond to
 $\mathbb{R}^f[G_f] = \mathcal{C}_{l,k}$

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Computational results.

Via $ABS_{\ell,k}$, some auxiliary maps, and some Adams spectral sequences...

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theorem [Debray-K.-Stehouwer]

Let $\tilde{m} := \lfloor (d-l+k+i+1)/8 \rfloor$, where $i \equiv l \pmod{4}$. For d large enough...

$$F2I_{l,k} : KO^{d+l-k-2} \rightarrow \mathcal{U}_{Spin-(l,k)}^{d+2} \quad \text{has image}$$

$$\begin{cases} \mathbb{Z}/2^{4+4\tilde{m}-i} \\ \mathbb{Z}/2^{5+4\tilde{m}-i} \\ \mathbb{Z}/2 \end{cases}$$

$$d \equiv l+k-2i+2 \pmod{8}$$

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* $\tilde{m}$ measures how many times around the spiral $\sim \times 16$ each transit

* results for our model match [Queiroz-Khalaf-Stern, '16]!

Outline

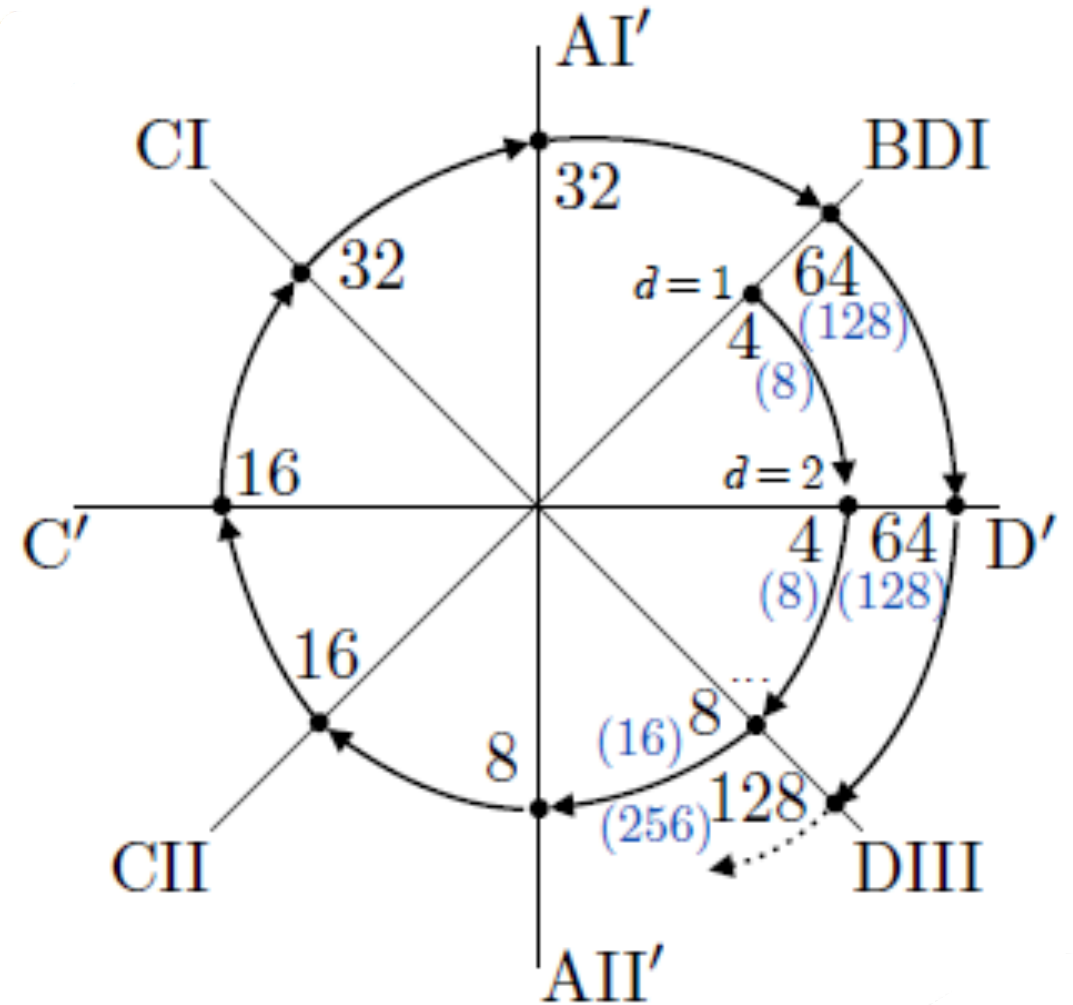
1. The Problem: free versus interacting SPTs
in the 'Bott spiral' [Queiroz-Khalaf-Stern, '16]

2. How well can we model w/ invertible field theories?

3. What new things do we learn?

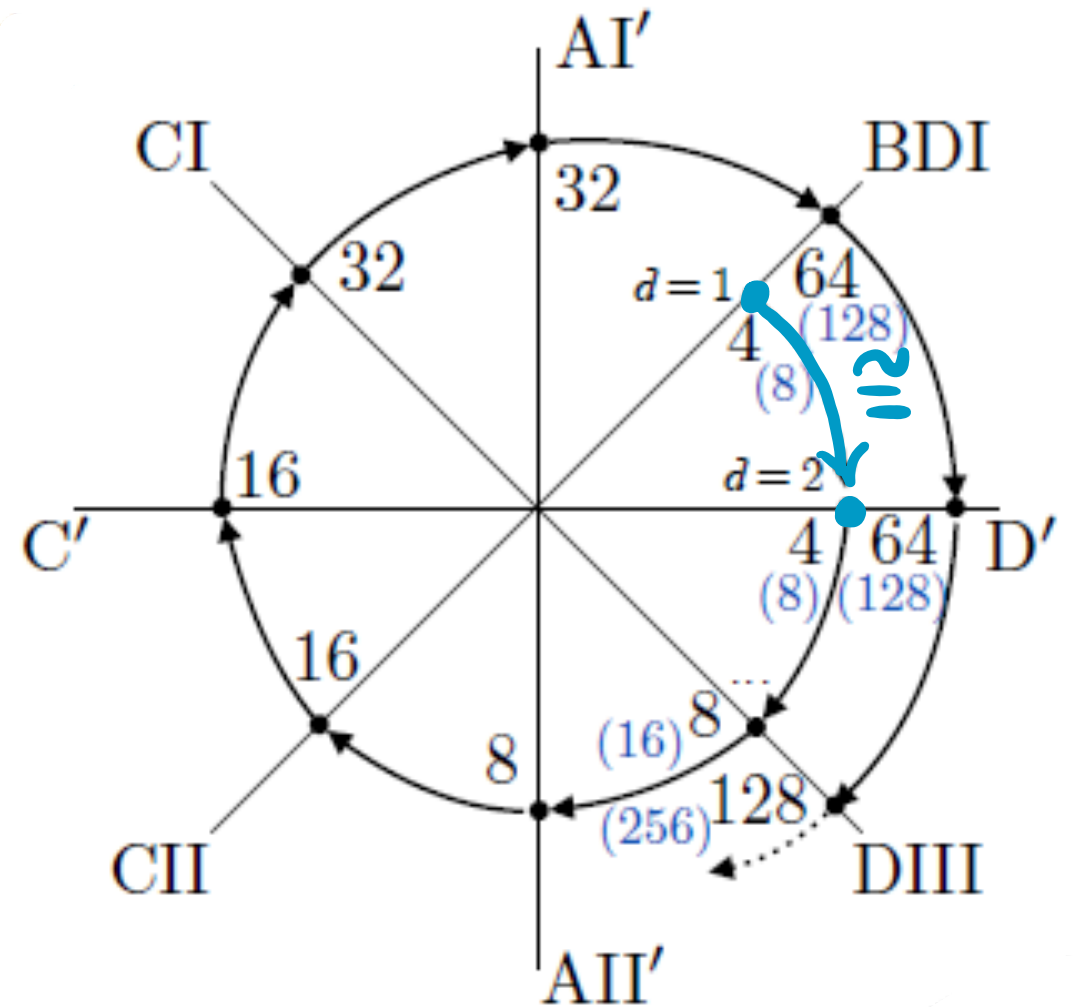
Dimensional reduction I

In addition to defining F2I, we also wanted to model dimensional reduction using IFTs...



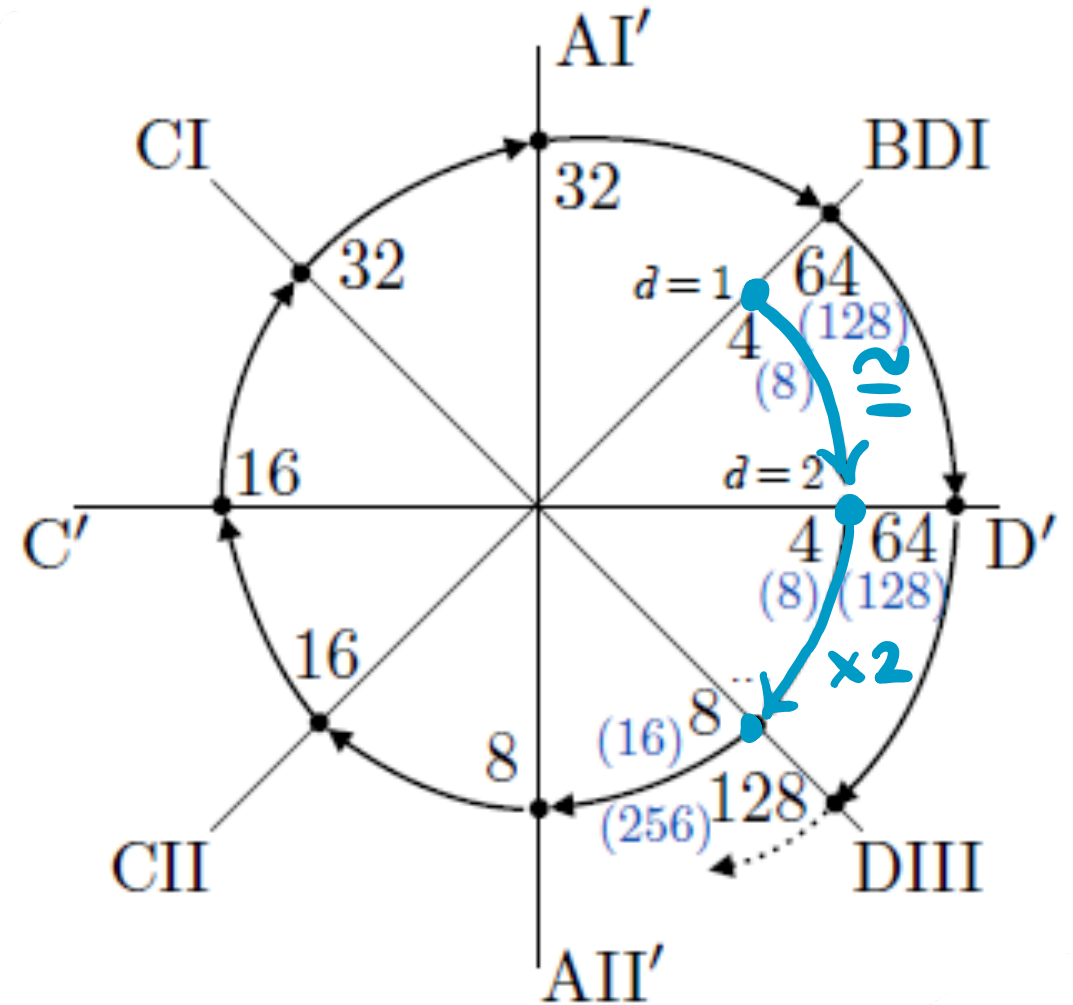
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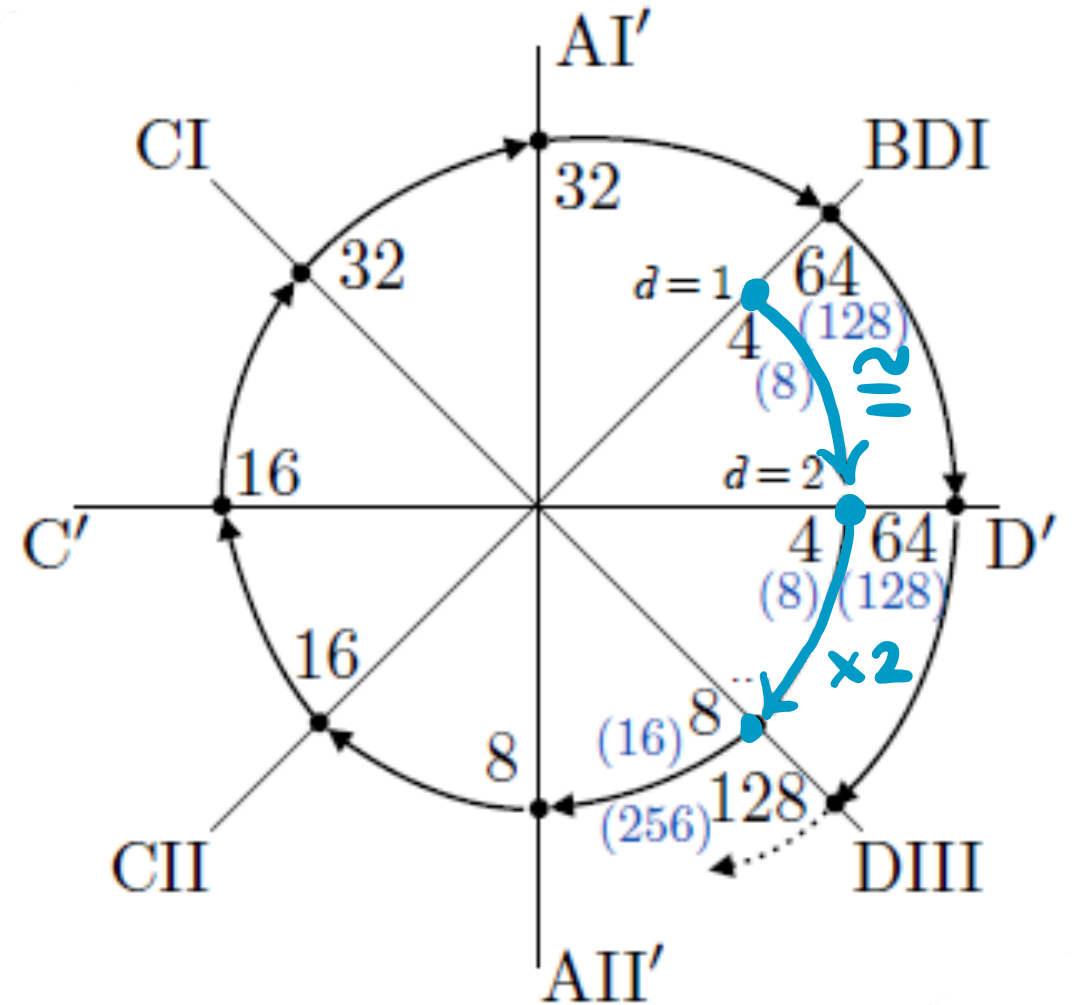
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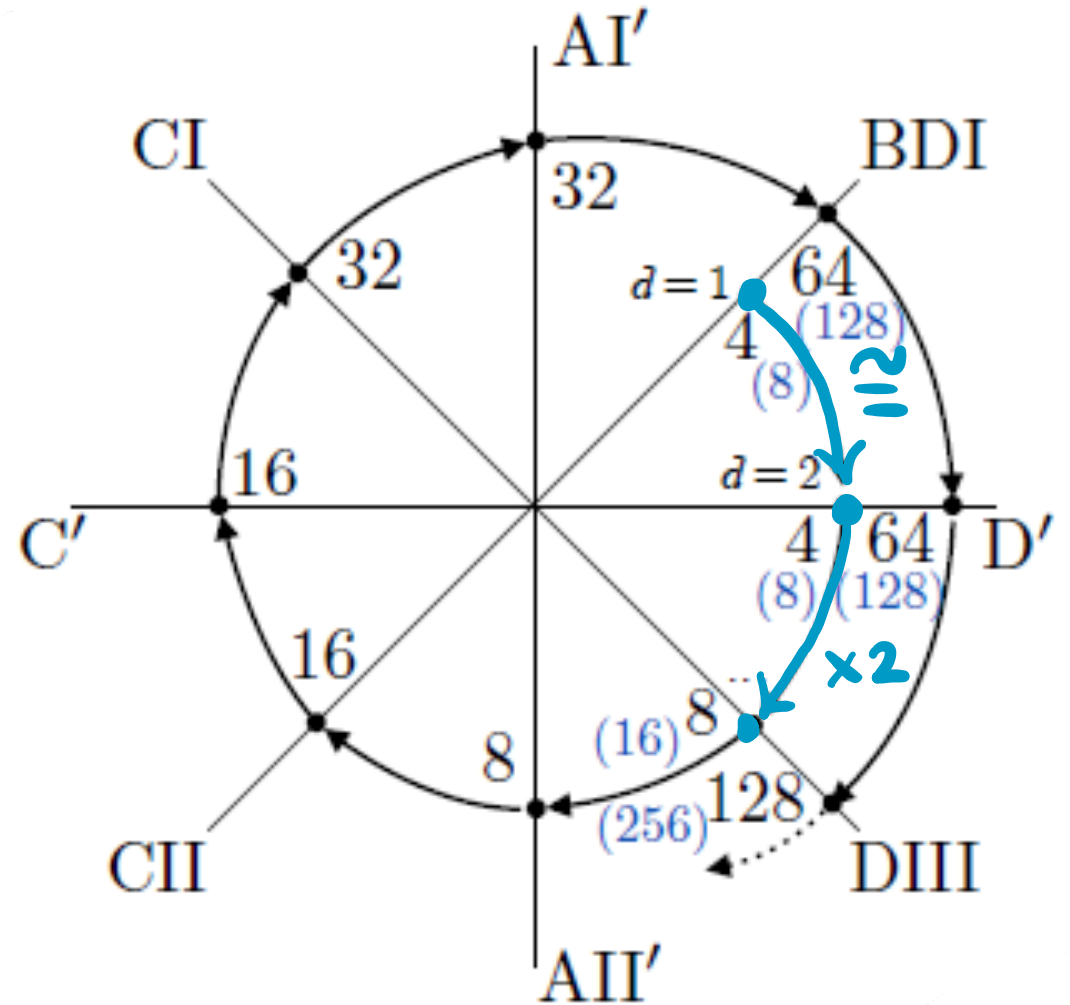


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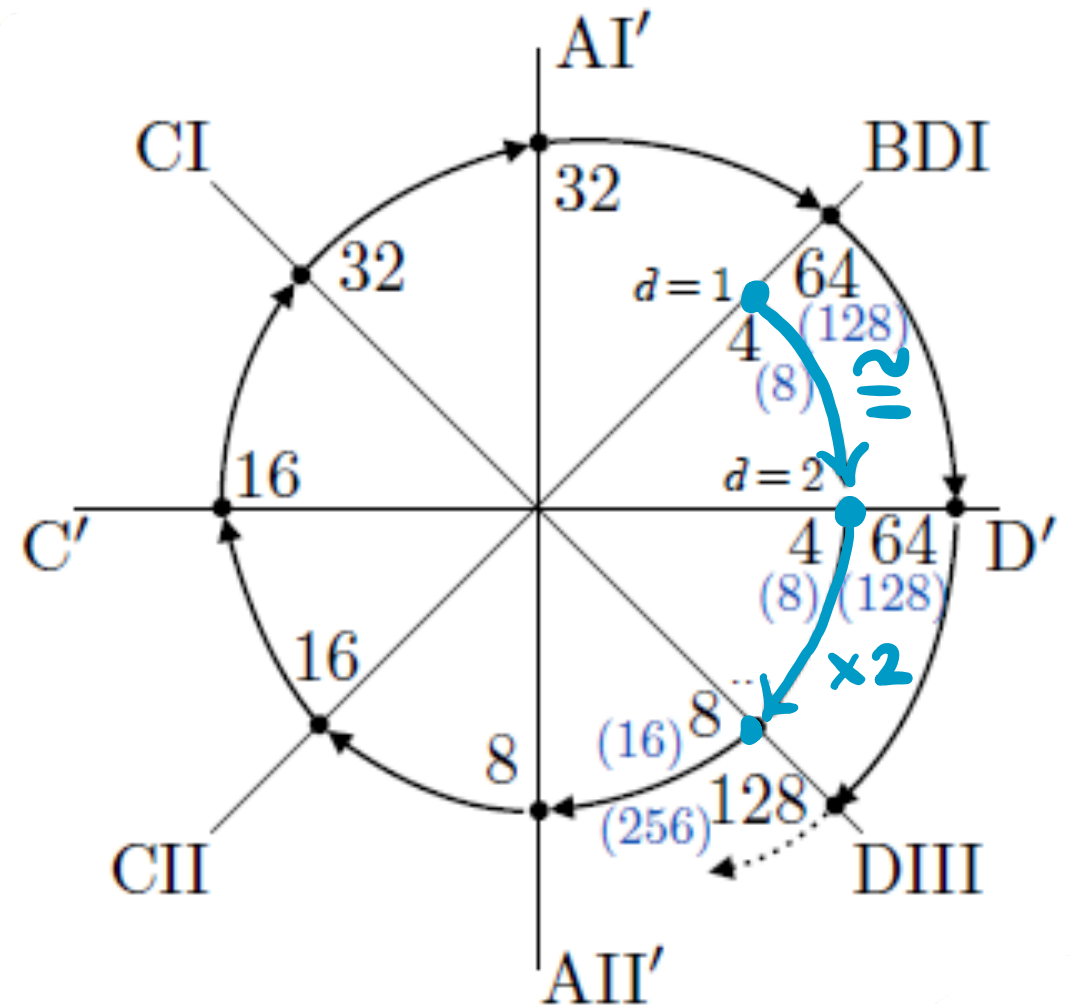
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$$\begin{array}{ccc} \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

submanifold w/ normal bundle $\cong \sigma$



see also e.g.

[Cordova-Ohmori-Shao-Yan, '20]

[Debray-Devalapurkar-K.-Liu-

Pacheco-Tallaj, Thorngren, '25 + '26]

Dimensional reduction $\mathbb{I}$

*abbreviate $\Omega_n^{l,k} := \Omega_n^{\text{Spin}-(l,k)}$

We defined two kinds of 'spiral maps', which are I_z -dual to . . .

$\phi_{l,k}$

$\psi_{l,k}$

Dimensional reduction Π

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Smith
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take codim 1
submfld
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$\downarrow \text{forget}$

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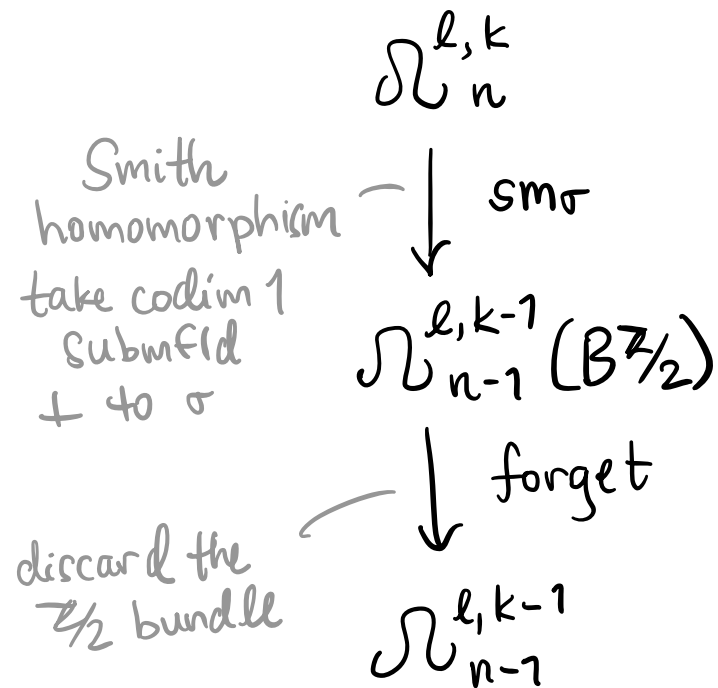
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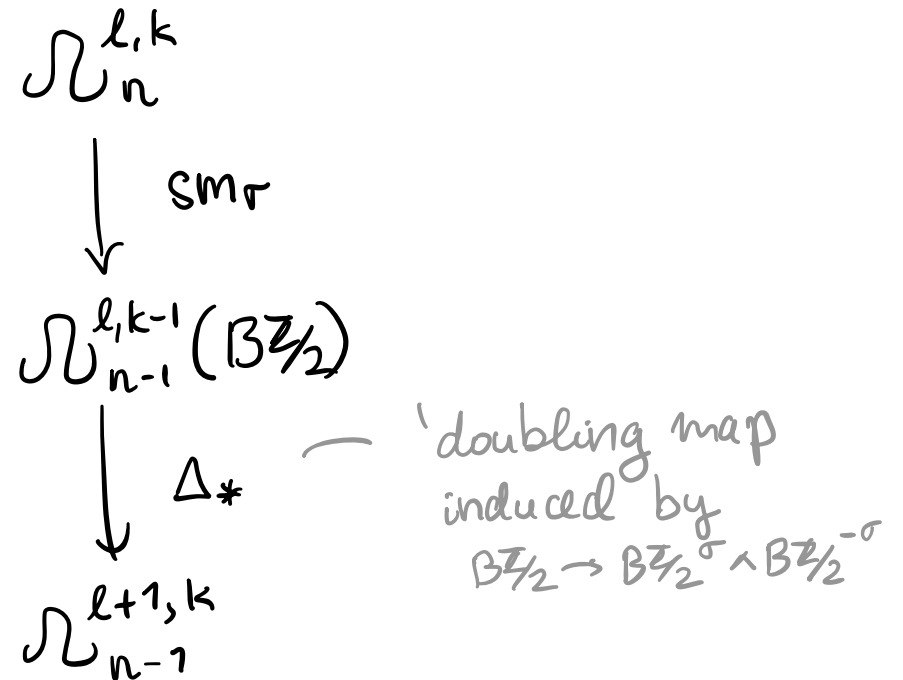
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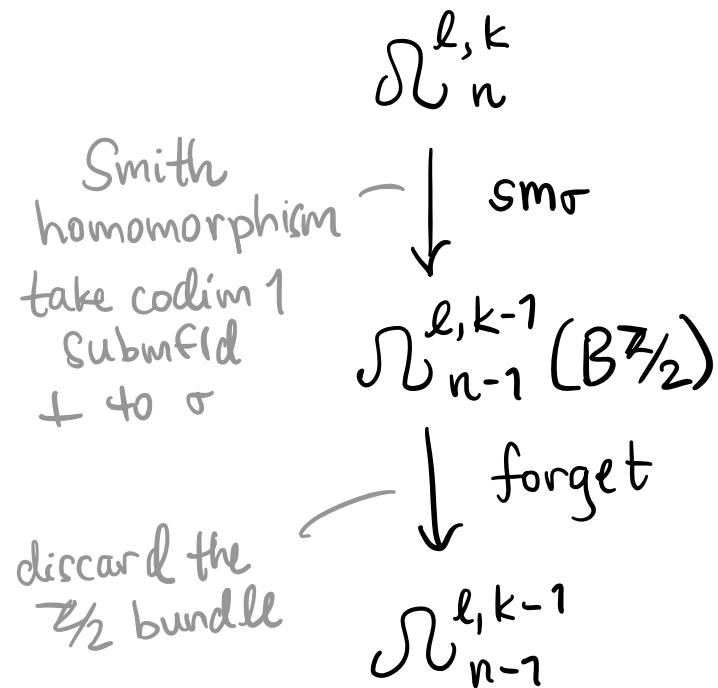
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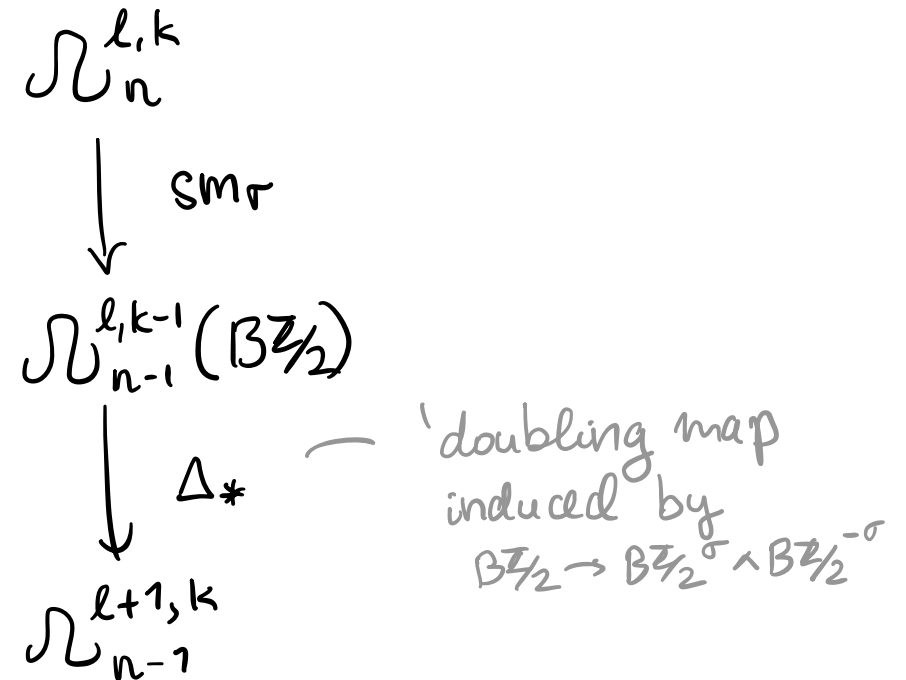
$\phi_{l,k}$

example. $\phi_{1,1}$

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Dimensional reduction Π

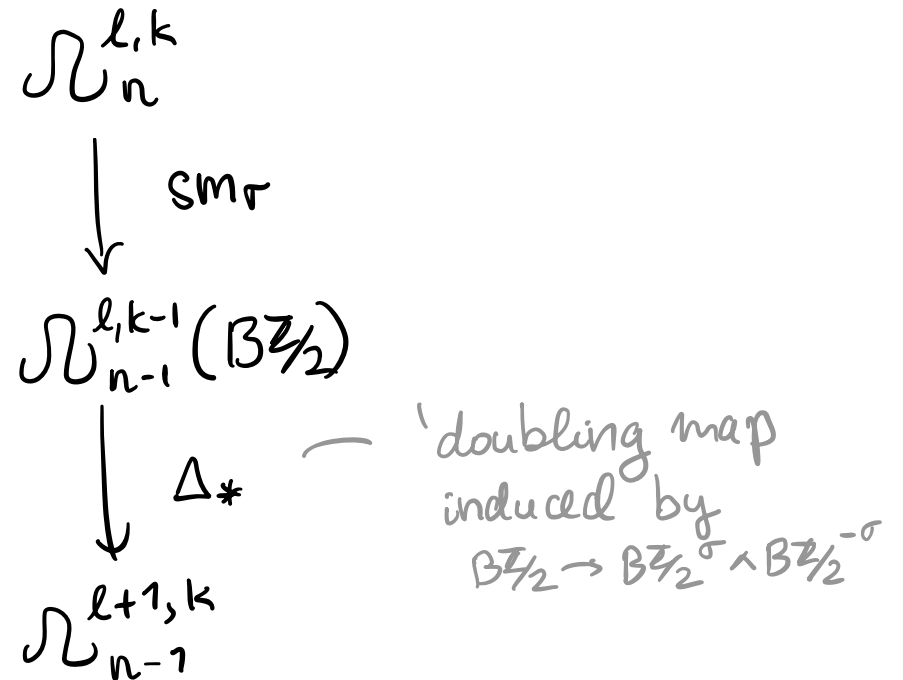
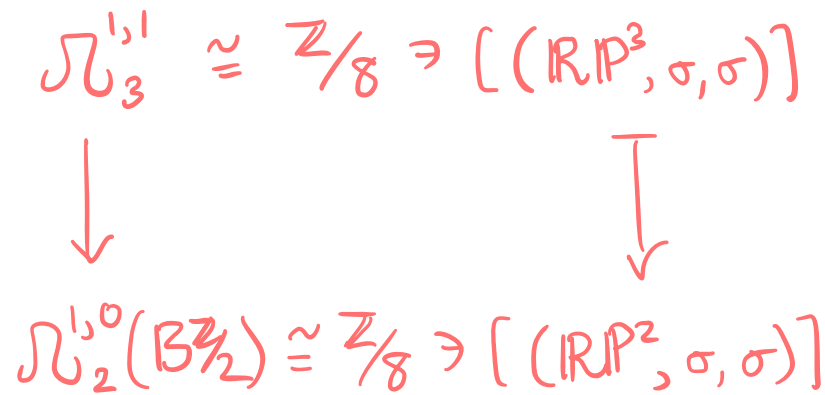
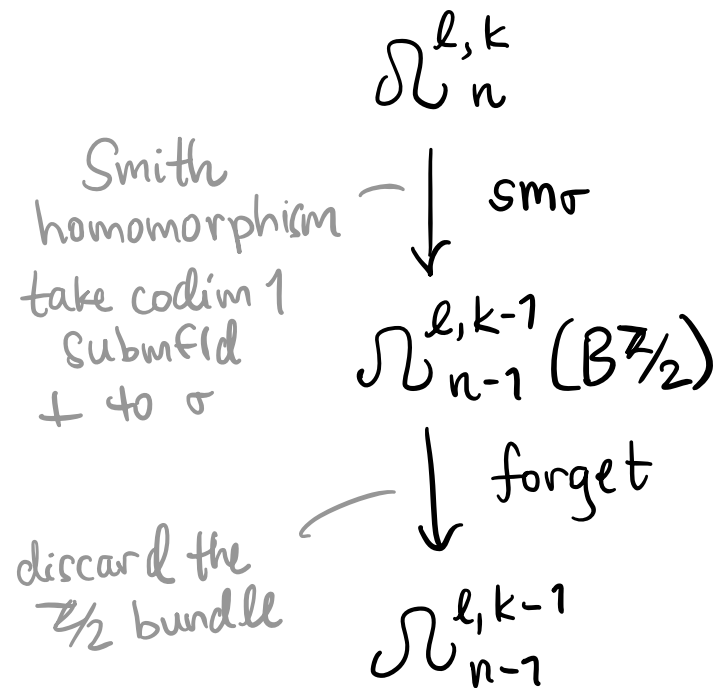
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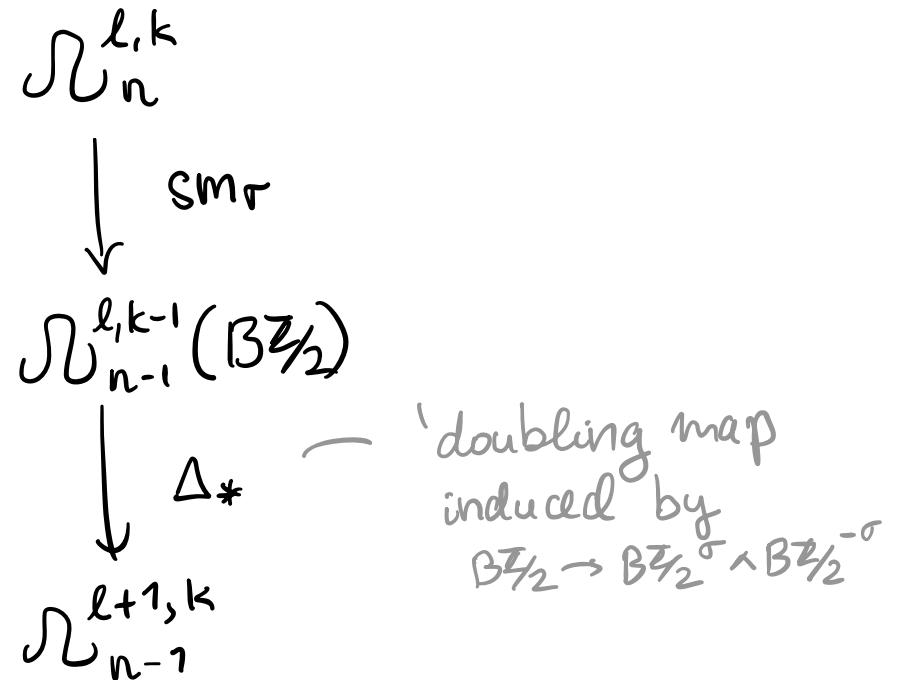
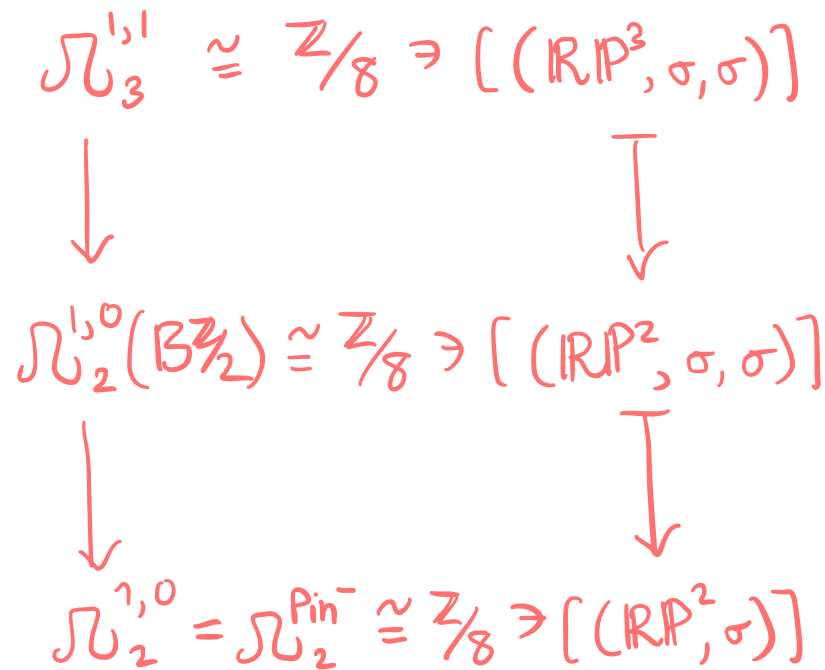
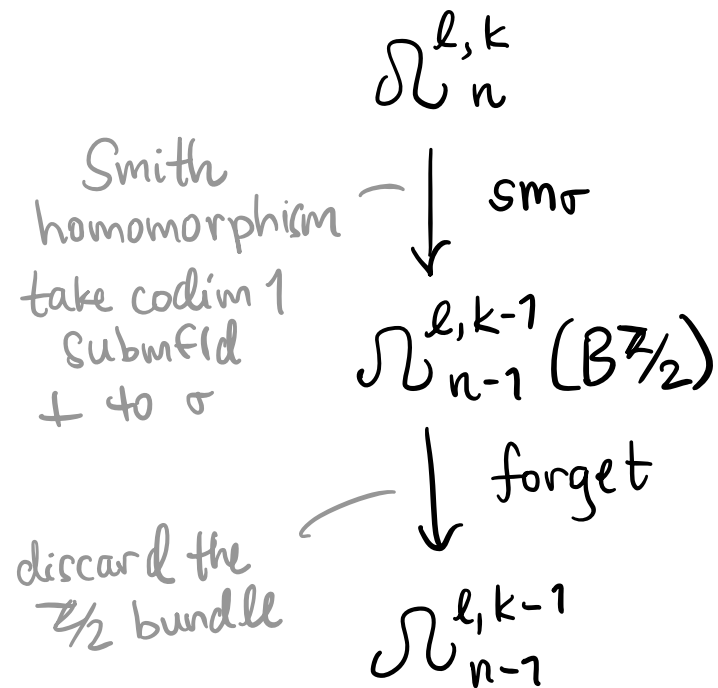
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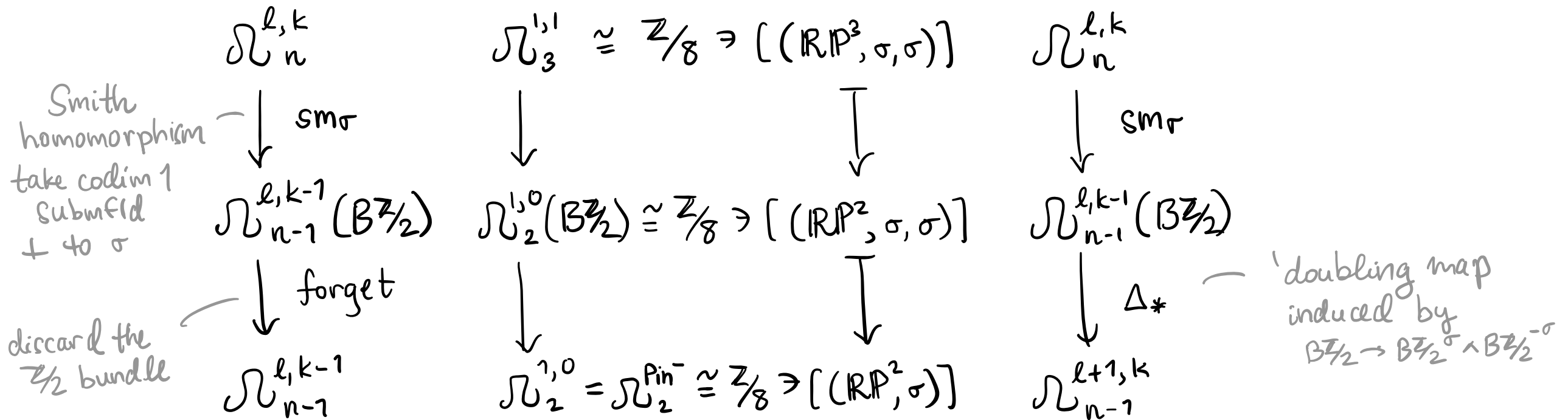
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$\uparrow$ dual to 2d $D' \rightarrow 1d$ BDI
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$$\downarrow$$

$$\Omega_2^{1,0}(B\mathbb{Z}/2) \cong \mathbb{Z}/8 \ni [(\mathbb{R}P^2, \sigma, \sigma)]$$

$$\downarrow$$

$$\Omega_2^{1,0} = \Omega_2^{\text{Pin}^-} \cong \mathbb{Z}/8 \ni [(\mathbb{R}P^2, \sigma)]$$

$$\downarrow$$

$$\downarrow$$

$$\Omega_n^{l,k}$$

$$\downarrow \text{sm}\sigma$$

$$\Omega_{n-1}^{l,k-1}(B\mathbb{Z}/2)$$

$$\downarrow \Delta^*$$

$$\Omega_{n-1}^{l+1,k}$$

these both
require
 $k > 0$!

'doubling map
induced by
 $B\mathbb{Z}/2 \rightarrow B\mathbb{Z}/2^{\sigma} \wedge B\mathbb{Z}/2^{-\sigma}$

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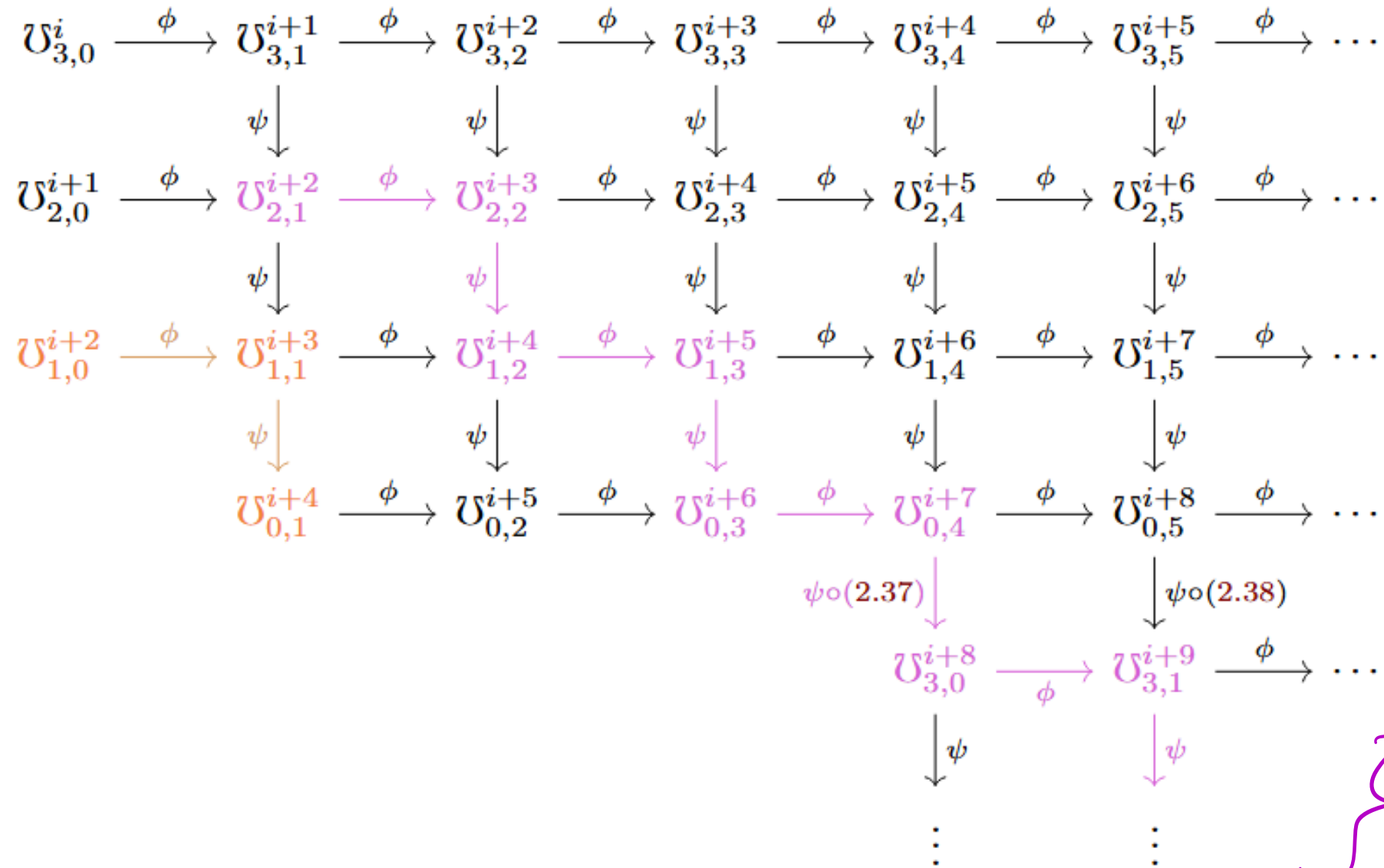
Dimensional reduction III

*abbreviate $\mathcal{U}_{\ell,k}^i := \mathcal{U}_{\text{spin}-(\ell,k)}^i = \{\text{IFTs on spin}-(\ell,k) \text{ manifolds}\}$

$$\begin{array}{ccccccccccc}
 \mathcal{U}_{3,0}^i & \xrightarrow{\phi} & \mathcal{U}_{3,1}^{i+1} & \xrightarrow{\phi} & \mathcal{U}_{3,2}^{i+2} & \xrightarrow{\phi} & \mathcal{U}_{3,3}^{i+3} & \xrightarrow{\phi} & \mathcal{U}_{3,4}^{i+4} & \xrightarrow{\phi} & \mathcal{U}_{3,5}^{i+5} & \xrightarrow{\phi} & \dots \\
 & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \\
 \mathcal{U}_{2,0}^{i+1} & \xrightarrow{\phi} & \mathcal{U}_{2,1}^{i+2} & \xrightarrow{\phi} & \mathcal{U}_{2,2}^{i+3} & \xrightarrow{\phi} & \mathcal{U}_{2,3}^{i+4} & \xrightarrow{\phi} & \mathcal{U}_{2,4}^{i+5} & \xrightarrow{\phi} & \mathcal{U}_{2,5}^{i+6} & \xrightarrow{\phi} & \dots \\
 & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \\
 \mathcal{U}_{1,0}^{i+2} & \xrightarrow{\phi} & \mathcal{U}_{1,1}^{i+3} & \xrightarrow{\phi} & \mathcal{U}_{1,2}^{i+4} & \xrightarrow{\phi} & \mathcal{U}_{1,3}^{i+5} & \xrightarrow{\phi} & \mathcal{U}_{1,4}^{i+6} & \xrightarrow{\phi} & \mathcal{U}_{1,5}^{i+7} & \xrightarrow{\phi} & \dots \\
 & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \\
 & & \mathcal{U}_{0,1}^{i+4} & \xrightarrow{\phi} & \mathcal{U}_{0,2}^{i+5} & \xrightarrow{\phi} & \mathcal{U}_{0,3}^{i+6} & \xrightarrow{\phi} & \mathcal{U}_{0,4}^{i+7} & \xrightarrow{\phi} & \mathcal{U}_{0,5}^{i+8} & \xrightarrow{\phi} & \dots \\
 & & & & & & \downarrow \psi \circ (2.37) & & \downarrow \psi \circ (2.38) & & & & \\
 & & & & & & \mathcal{U}_{3,0}^{i+8} & \xrightarrow{\phi} & \mathcal{U}_{3,1}^{i+9} & \xrightarrow{\phi} & \dots & & \\
 & & & & & & \downarrow \psi & & \downarrow \psi & & & & \\
 & & & & & & \vdots & & \vdots & & & &
 \end{array}$$

Dimensional reduction III

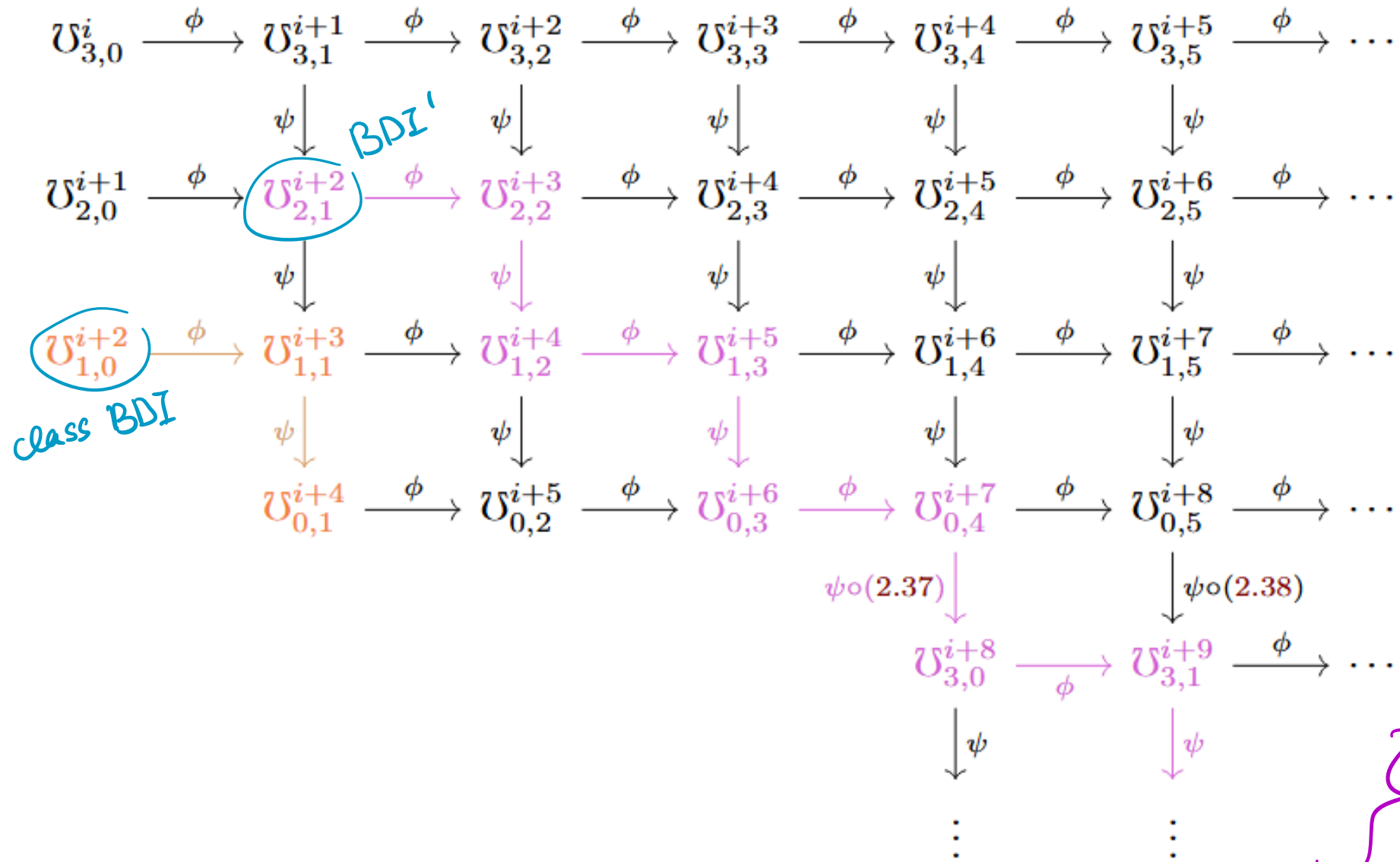
*abbreviate $\mathcal{U}_{\ell,k}^i := \mathcal{U}_{\text{spin}-(\ell,k)}^i = \{ \text{IFTs on spin}-(\ell,k) \text{ manifolds} \}$



} Bott spiral
starting from
class BDI'

Dimensional reduction III

*abbreviate $\mathcal{U}_{\ell,k}^i := \mathcal{U}_{\text{spin}-(\ell,k)}^i = \{ \text{IFTs on spin}-(\ell,k) \text{ manifolds} \}$

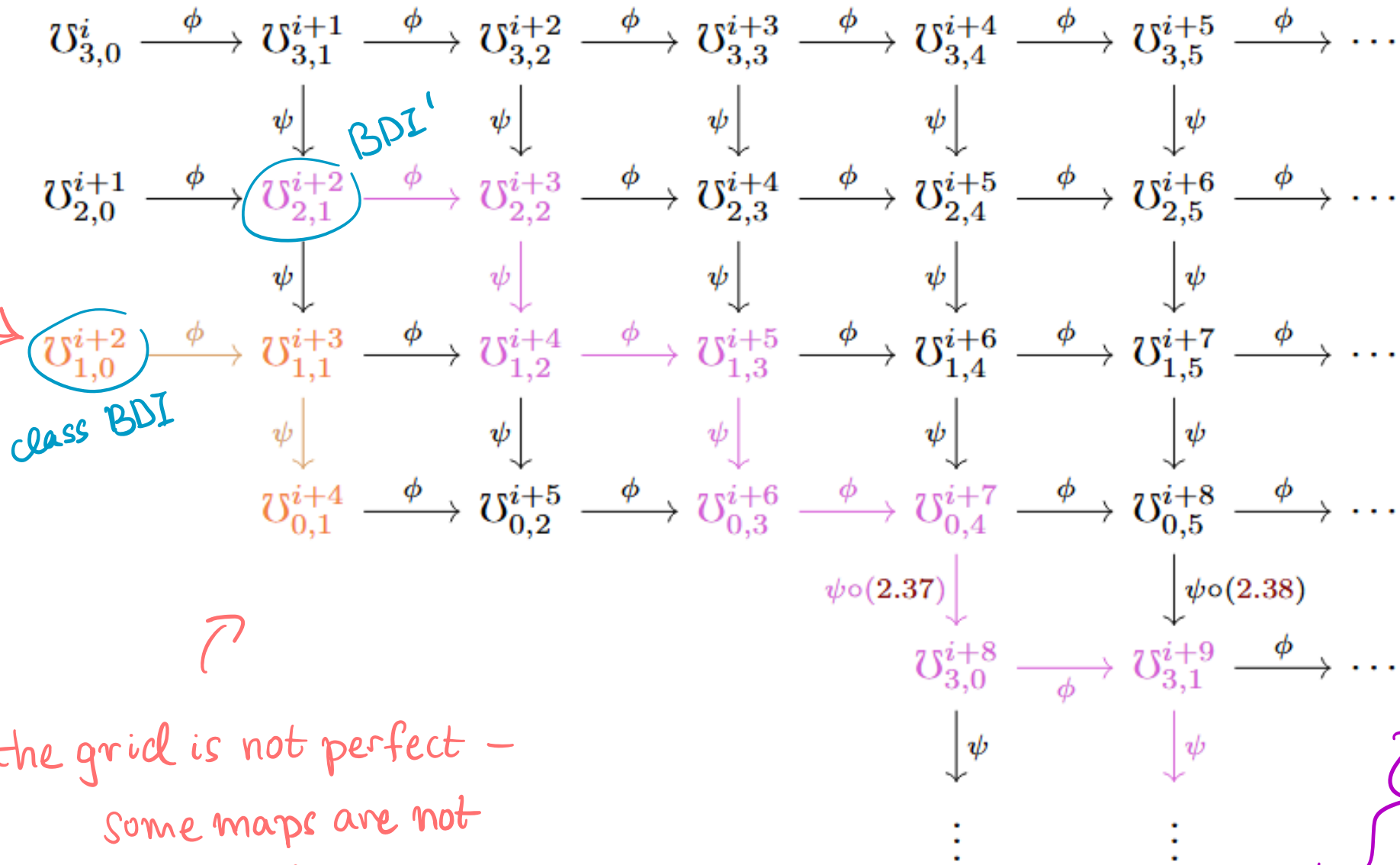


} Bott spiral starting from class BDI'

Dimensional reduction III

*abbreviate $\mathcal{U}_{l,k}^i := \mathcal{U}_{\text{spin}-(l,k)}^i = \{ \text{IFTs on spin}-(l,k) \text{ manifolds} \}$

we can't spiral 8 steps from here!



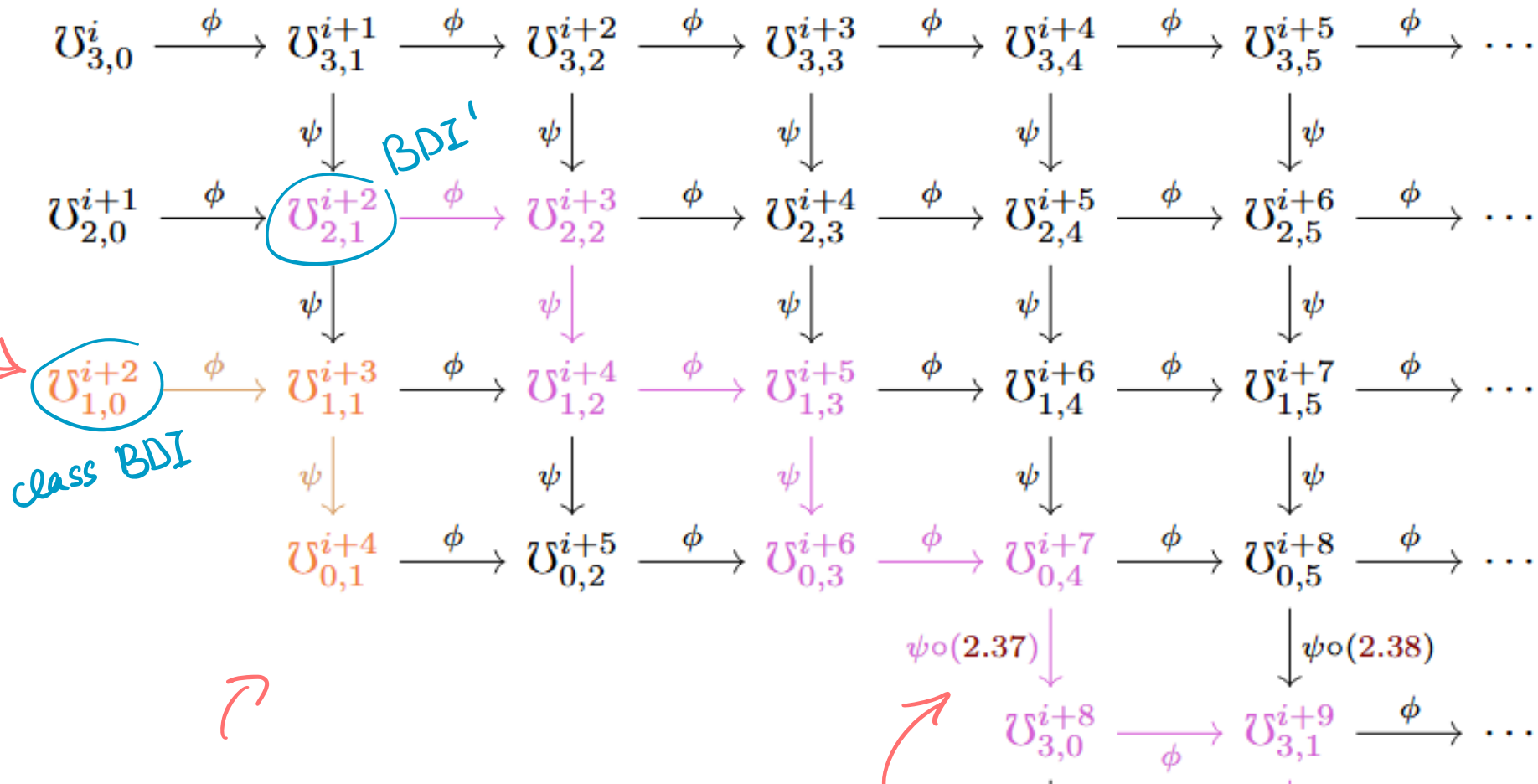
the grid is not perfect -
some maps are not
defined!

Bott spiral
starting from
class BDI'

Dimensional reduction III

*abbreviate $\mathcal{U}_{l,k}^i := \mathcal{U}_{\text{spin}-(l,k)}^i = \{ \text{IFTs on spin}-(l,k) \text{ manifolds} \}$

we can't spiral 8 steps from here!



the grid is not perfect -
some maps are not
defined!

need a trick:

$$\mathcal{U}_{0,4}^i \approx \mathcal{U}_{4,0}^i$$

Bott spiral
starting from
class BDI'

Surprising findings

- * We really needed $\text{spin}-(l,k)$:
'continuous' models for G_f don't admit spiral maps!
- * We needed to use multiply-primed phases
- * Eightfold dimensional reduction, or
'interacting Bott periodicity'
relies on an isomorphism of fermionic groups

$$E_{0,4} \cong E_{4,0}$$

'Discrete' vs 'continuous' models for symmetry types I

Our choices of fermionic groups for the tenfold way (+primes) are **finite**:

defn. Let $E_{l,k}$ be the fermionic group defined by

$$G_b = (\mathbb{Z}/2)^{l+k}, \quad \theta = \sum_{i=1}^{l+k} x_i, \quad \omega = \sum_{i=l+1}^{l+k} x_i^2 + \sum_{1 \leq i < j \leq l+k} x_i x_j$$

$x_i \in H^1((B\mathbb{Z}/2)^{\wedge l+k}; \mathbb{Z}/2)$
dual to standard
basis of $\mathbb{Z}/2^{l+k}$

equivalently, $E_{l,k} := p^* \left((\mathbb{Z}/2)^{l+k} \underset{\text{diagonal}}{\subset} O_{l,k} \right)$ for $p: \text{Pin}_{l,k} \twoheadrightarrow O_{l,k}$

$\rightsquigarrow$ superalgebra $\mathcal{R}^f(E_{l,k}) \cong \mathcal{C}_{l,k}$ see e.g. [Albuquerque-Majid, '02]

$\rightsquigarrow$ tangential structure $H(E_{l,k}) = \text{Spin}-(l,k)$

'Discrete' vs 'continuous' models for symmetry types II

example. Model class AI with $E_{2,0}$, encoding T_1, T_2 .

If the unitary symmetry T_1, T_2 were supposed to encode charge $\in U(1)$, we should instead use $\text{Pin}^+(2)$.

* $E_{2,0} \subset \text{Pin}^+(2)$ both correspond to $\text{Cl}_{2,0}$

* $H(E_{2,0}) = \text{Spin}^-(2,0)$, $H(\text{Pin}^+(2)) = \text{Pin}^{\tilde{-}} := \text{Pin}^- \rtimes_{\pm 1} U(1)$ a.k.a. G^- [Freed-Hopkins, '16]

corollary. [Debray-K.-Stehouwer]

$$\Omega_2^{\text{Spin}^-(2,0)} \cong (\mathbb{Z}/2)^{\text{some factor}} \not\cong \mathbb{Z} \cong \Omega_2^{\text{Pin}^{\tilde{-}}}$$

(and similar for the other relevant classes)

Computations I

computing $F2I_{\ell,k}$ boils down to computing

$$ABS_{\ell,k} : MTSpin \wedge \underbrace{(B\mathbb{Z}/2^{\sigma-1})^{\wedge \ell} \wedge (B\mathbb{Z}/2^{1-\sigma})^{\wedge k}}_{=: ME_{\ell,k}} \rightarrow \Sigma^{k-\ell} KO$$

to do this, we study the connective version,

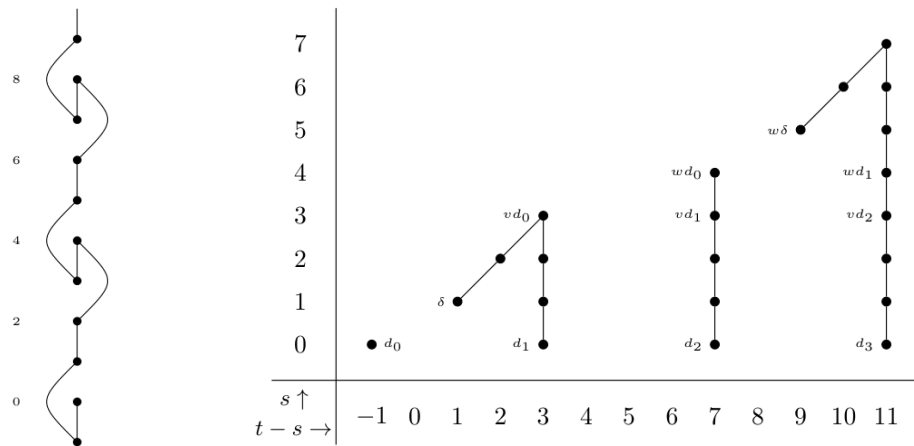
$$KO_*(ME_{\ell,k})$$

using Adams spectral sequences

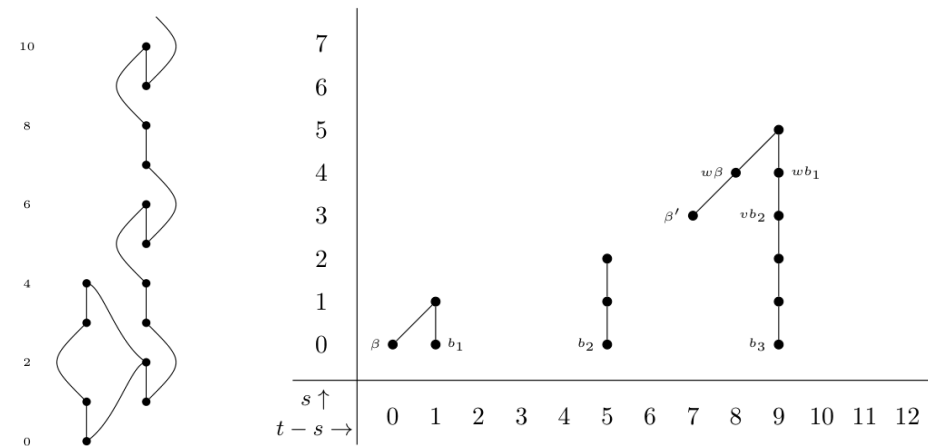
Computations II input for the spectral sequence

$H^*(ME_{4k}; \mathbb{Z}/2)$ has a nice expression in terms of the $A(1)$ -modules ...

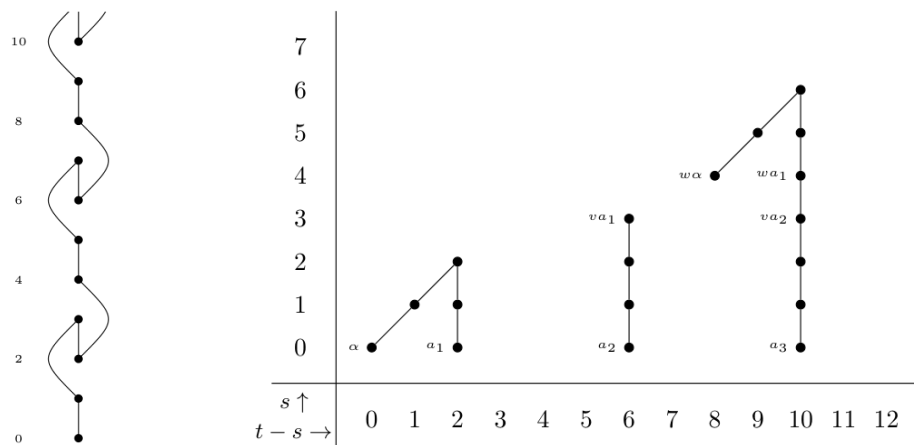
$$N_0 = \Sigma^{-1} H^*(B\mathbb{Z}/2^{\sigma^{-1}}) \quad \text{Ext}(N_0)$$



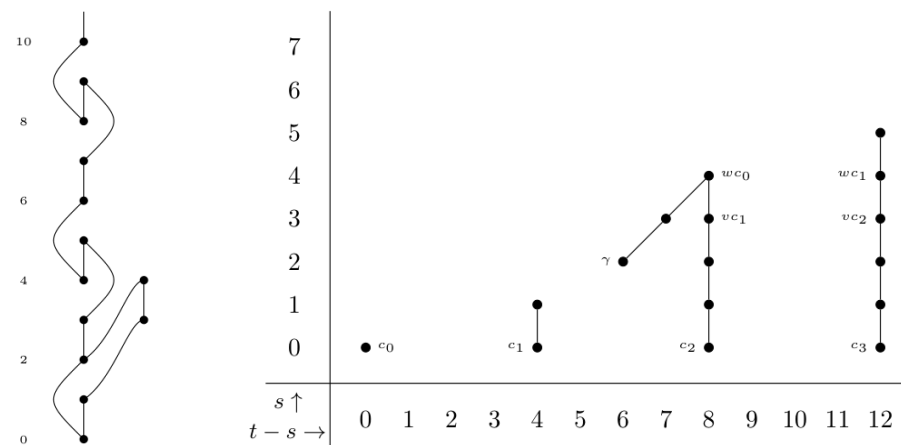
$$N_2 \subset \Sigma^{-2} \tilde{H}^*(BA_4) \quad \text{Ext}(N_2)$$



$$N_1 = H^*(B\mathbb{Z}/2^{\sigma^{-1}}) \quad \text{Ext}(N_1)$$



$$N_3 \subset \Sigma^{-3} \tilde{H}^*(BA_4 \wedge B\mathbb{Z}/2) \quad \text{Ext}(N_3)$$



Computations III

Lemma. [Yu, '95] Up to suspensions, there are stable isos

$$N_i \otimes N_j \cong N_{i+j \bmod 4}$$

proposition [Debray - K. - Stehouwer]

Let $i = l \bmod 4$ and $j = L^k/4$.

If at least one of l, k is positive, then

$$H^*(ME_{l,k}, \mathbb{Z}/2) \cong \sum^{4j+k} N_i.$$