Chern-Simons theory and U(1) Anomaly on Lattice

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DeMarco Wen, arXiv:1906.08270
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Simons Collaboration on
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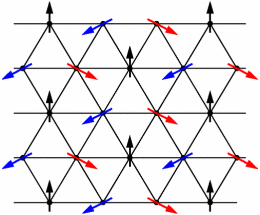
Put $U(1)$ Chern-Simons theory $S = 2\pi k \int_{M^3} a\mathrm{d}a$ on lattice

How to construct lattice models whose low energy effective field theory is a Chern-Simons theory?



- **Chiral spin liquid** on triangle lattice

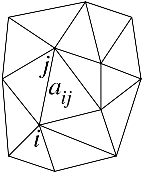
$$H = \sum_{(ij)} \boldsymbol{S_i} \cdot \boldsymbol{S_j} + \sum_{\triangle} \boldsymbol{S_i} \cdot (\boldsymbol{S_j} \times \boldsymbol{S_k})$$

$$\xrightarrow[\text{to confirm}]{\text{need numerics}} \mathrm{e}^{-S} = \mathrm{e}^{\mathrm{i}2\pi k \int_{M^3} a\mathrm{d}a}\Big|_{k=2}$$

Wen, Wilczek, Zee, PRB **39** 11413 (89)

- **Lattice cochain field theory**: (Give all vertices an order $i < j < k < l$)
  $\mathbb{R}$-valued 1-cochain: $a = \{a_{ij} \in \mathbb{R} \mid \text{all links } (ij)\}$.
  Derivative: 2-cochian $(\mathrm{d}a)_{ijk} = a_{jk} - a_{ik} + a_{ij}$ (value on triangle (ijk))
  Cup-product: 3-cochian $(a\mathrm{d}a)_{ijkl} = a_{ij}(\mathrm{d}a)_{jkl}$ (on tetrahedron $(ijkl)$)

$$\mathrm{e}^{-S} = \mathrm{e}^{\mathrm{i}2\pi k \int_{M^3} a\mathrm{d}a} = \mathrm{e}^{\mathrm{i}2\pi k \sum_{(ijkl)} (a\mathrm{d}a)_{ijkl}}$$



- $\mathbb{R}$-gauge invariance for closed M^3: $a \to a + \mathrm{d}f \to$
  $\int_{M^3} a\mathrm{d}a \to \int_{M^3} a\mathrm{d}a + \int_{M^3} \mathrm{d}f\,\mathrm{d}a = \int_{M^3} a\mathrm{d}a$
  Not a lattice $U(1)$ Chern-Simons theory, but a $\mathbb{R}$ CS theory

Put $U(1)$ Chern-Simons theory $S = 2\pi k \int_{M^3} a \mathrm{d} a$ on lattice

- Turn $\mathbb{R}$-valued cochain to $\mathbb{R}/\mathbb{Z}$-valued cochain via $\mathbb{Z}$-gauge invariance

$$a \sim a + m, \quad m = \mathbb{Z}\text{-value cochain}$$

Let $\lfloor x \rceil =$ nearest integer to x. $\mathbb{Z}$-gauge invariant combinations:

$$\bar{a} = a - \lfloor a \rceil, \quad \bar{b} = \mathrm{d}a - \lfloor \mathrm{d}a \rceil, \quad B = \lfloor \mathrm{d}a \rceil - \mathrm{d}\lfloor a \rceil.$$

- A lattice $U(1)$ Chern-Simons theory (?) $\qquad |b|^2 \equiv b \star b$

$$\mathrm{e}^{-S} = \underbrace{\mathrm{e}^{\mathrm{i}\,2\pi k \int_{M^3} (a - \lfloor a \rceil)(\mathrm{d}a - \lfloor \mathrm{d}a \rceil)}}_{= \mathrm{e}^{\mathrm{i}\,2\pi k \int_{M^3} \bar{a}\bar{b}}} \underbrace{\mathrm{e}^{-\int_{M^3} g^{-1} |\mathrm{d}a - \lfloor \mathrm{d}a \rceil|^2}}_{\text{lattice Maxwell term}}$$

- There is no $U(1)$-gauge invariance, but a $\mathbb{Z}$-gauge invariance. It is not even a $U(1)$ gauge theory.
- But this is OK, as long as a $U(1)$ Chern-Simons theory emerge at low energies in weak-field (small g) limit. This makes $\mathrm{d}a \approx$ int. Using $\mathbb{Z}$-gauge invariance, we further change $\mathrm{d}a \approx$ int to $\mathrm{d}a \approx 0 \to \lfloor \mathrm{d}a \rceil = 0$.

Put $U(1)$ Chern-Simons theory $S = 2\pi k \int_{M^3} a\mathrm{d}a$ on lattice

- But $\lfloor a \rceil$ is not small in weak-field limit, and $(a - \lfloor a \rceil)(\mathrm{d}a - \lfloor \mathrm{d}a \rceil)$ does not become Chern-Simons term even in weak-field limit

$$(a - \lfloor a \rceil)(\mathrm{d}a - \lfloor \mathrm{d}a \rceil) = \Big|_{\lfloor \mathrm{d}a \rceil = 0} a\mathrm{d}a - \underbrace{\lfloor a \rceil \mathrm{d}a}_{\text{extra}} \underbrace{\overset{\mathrm{d}}{=}}_{\text{up to } \mathrm{d}\#} a\mathrm{d}a - \underbrace{\mathrm{d}\lfloor a \rceil a}_{\text{important}}$$

- We can cancel the extra term by adding $\mathrm{d}\lfloor a \rceil a$. But the term $\mathrm{d}\lfloor a \rceil a$ is not $\mathbb{Z}$-gauge invariant. We can make it $\mathbb{Z}$-gauge invariant,

$$\mathrm{d}\lfloor a \rceil a \to (\mathrm{d}\lfloor a \rceil - \lfloor \mathrm{d}a \rceil)a \overset{1}{=} (\mathrm{d}\lfloor a \rceil - \lfloor \mathrm{d}a \rceil)(a - \lfloor a \rceil)$$

In weak-field limit $\lfloor \mathrm{d}a \rceil = 0$, the above term becomes $\lfloor a \rceil \mathrm{d}a$ up to d (requires M^D to be closed) and up to integers (requires $k \in \mathbb{Z}$)

- **Lattice $U(1)$ Chern-Simons theory** $\to \mathrm{e}^{\mathrm{i}2\pi k \int_{M^3} a\mathrm{d}a}$ in weak-field limit

$$\mathrm{e}^{-S} = \underbrace{\mathrm{e}^{\mathrm{i}2\pi k \int_{M^3} (a - \lfloor a \rceil)(\mathrm{d}a - \lfloor \mathrm{d}a \rceil) - (\lfloor \mathrm{d}a \rceil - \mathrm{d}\lfloor a \rceil)(a - \lfloor a \rceil)}}_{= \mathrm{e}^{\mathrm{i}2\pi k \int_{M^3} \bar{a}\bar{b} - B\bar{a}}, \quad k \in \mathbb{Z}} \underbrace{\mathrm{e}^{-\int_{M^3} g^{-1} |\mathrm{d}a - \lfloor \mathrm{d}a \rceil|^2}}_{\text{lattice Maxwell term}}$$

$$Z = \int D[a]\, \mathrm{e}^{-S} = \prod_{(ij)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}a_{ij}\, \mathrm{e}^{-S}$$

path integral is a well defined finite integral

Quantization of the level k for lattice Chern-Simons theory

- In the lattice $U(1)$ Chern-Simons theory

$$e^{-S} = \underbrace{e^{i2\pi k \int_{M^3} (a-\lfloor a \rceil)(da - \lfloor da \rceil) - (\lfloor da \rceil - d\lfloor a \rceil)(a - \lfloor a \rceil)}}_{= e^{i2\pi k \int_{M^3} \bar{a}\bar{b} - B\bar{a}}} \underbrace{e^{-\int_{M^3} g^{-1} |da - \lfloor da \rceil|^2}}_{\text{lattice Maxwell term}}$$

the level $k \in \mathbb{C}$ is not quantized. The action amplitude $e^{i2\pi k \int_{M^3} \bar{a}\bar{b} - B\bar{a}}$ is always $\mathbb{Z}$-gauge invariant (*ie* is well defined) for any k, g.

Only when $k \in \mathbb{Z}$ and in weak-field limit, $e^{i2\pi k \int_{M^3} \bar{a}\bar{b} - B\bar{a}}$ reduces to continuum Chern-Simons term $e^{i2\pi k \int_{M^3} a \wedge da}$ $\to$ $k \in \mathbb{Z}$ is a RG fixed point. Under RG flow, $k \to$ integer at low energies when g is small

- For flat $U(1)$ connection, $da \in \mathbb{Z}$, $\bar{b} = da - \lfloor da \rceil = 0$, we find

$$d(\bar{a}\bar{b} - B\bar{a}) = -d\big(\lfloor da \rceil - d\lfloor a \rceil)(a - \lfloor a \rceil)\big) = (\lfloor da \rceil - d\lfloor a \rceil)(da - d\lfloor a \rceil) \stackrel{1}{=} 0$$

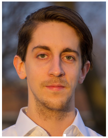
Thus $k(\bar{a}\bar{b} - B\bar{a}) = -kB\bar{a}$ is a $\mathbb{R}/\mathbb{Z}$-valued cocycle when $k \in \mathbb{Z}$.

In fact, the correction term $-kB\bar{a}$ is the $U(1)$ 3-cocycle

$$-kB\bar{a} = k(d\lfloor a \rceil - \lfloor da \rceil)a \in H^3(U(1)\, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}, \quad \text{for } k \in \mathbb{Z}.$$

Chern-Simons term is an extension of $U(1)$ 3-cocycle to non-flat connection

$U^{\kappa}(1)$ Chern-Simons (higher) gauge theory on lattice



- Constructed a model of rotors on **links** described by several $\mathbb{R}/\mathbb{Z}$-valued 1-cochain a_I fields in 3-**dimensional spacetime lattice**, that has the $\mathbb{Z}$-gauge invariance and reduces to quadratic continuum Maxwell Chern-Sinoms theory of type $\sum_{I\le J} k_{IJ} a_I \mathrm{d} a_J$ with several $U(1)$ gauge fields

DeMarco Wen, arXiv:1906.08270

$$Z = \int_{a_I^{\mathbb{R}/\mathbb{Z}}} \mathrm{e}^{\mathrm{i}\, 2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} \bar{a}_I \bar{b}_J - B_I \bar{a}_J} \mathrm{e}^{-\int_{\mathcal{M}^3} \frac{|\mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}} \rceil|^2}{g}}, \quad k_{IJ} \in \mathbb{Z}$$

- Constructed a model of rotors on n_I-**simplices** described by several $\mathbb{R}/\mathbb{Z}$-valued n_I-cochain a_I fields in D-**dimensional spacetime lattice**, that has the $\mathbb{Z}$-gauge invariance and reduces to quadratic continuum $\prod_I U^{(n_I)}(1)$ higher Chern-Simons theory of type $\sum_{I\le J} k_{IJ} a_I \mathrm{d} a_J$

$$\mathrm{e}^{-S} = \mathrm{e}^{\mathrm{i}\, 2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^D} \bar{a}_I \bar{b}_J + (-)^{n_I} B_I \bar{a}_J} \mathrm{e}^{-\int_{\mathcal{M}^D} \sum_I \frac{|\mathrm{d} a_I - \lfloor \mathrm{d} a_I \rceil|^2}{g}},$$

$$k_{IJ} \in \mathbb{Z}, \quad k_{IJ} = 0 \text{ if } n_I + n_J + 1 \neq D.$$

where $U^{(n)}(1)$ is $U(1)$ n-group for $(n-1)$-symmetry.

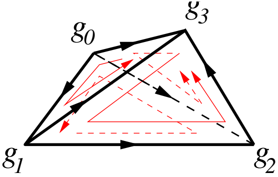
Framing anomaly in higher Chern-Simons theory $a_I \mathrm{d} a_J$

- Let a_I be $(2n+1)$-cochians, $n = 0, 1, \cdots$. For small g, we believe the path integral of higher-Chern-Simons theory to give rise to a **chiral topological order** with **gravitational Chern-Simons term** $\omega^{\text{grav CS}}_{4n+3}$ and **framing anomaly** (*ie* path integral depends on the choices of framing of the spacetime)

$$Z = \int D[a_I] \underbrace{\mathrm{e}^{\mathrm{i}2\pi \sum_{I \le J} k_{IJ} \int_{\mathcal{M}^{4n+3}} \bar{a}_I \bar{b}_J - B_I \bar{a}_J}}_{\sim \mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{M}^{4n+3}} \sum_{I,J} \frac{1}{2} K_{IJ} a_I \wedge \mathrm{d} a_J}} \mathrm{e}^{-\int_{\mathcal{M}^{4n+3}} \sum_I \frac{|\mathrm{d} a_I - \lfloor \mathrm{d} a_I \rceil|^2}{g}}$$

$$= \mathrm{e}^{-\epsilon V_{M^{4n+3}}} \mathrm{e}^{\mathrm{i}2\pi c_n \mathrm{sgn}(K) \int_{M^{4n+3}} \omega^{\text{grav CS}}_{4n+3}}.$$

$$K_{IJ} = K_{JI} = \begin{cases} 2k_{IJ}, & \text{if } I = J \\ k_{IJ}, & \text{if } I \neq J \end{cases}$$



But how can a spacetime lattice know about the framing and gravitational Chern-Simons term defined for a 3d smooth manifold?

- To define cup product $\smile$, we need to asign a branching structure to the spacetime trianglation. **Branching** $\rightarrow$ **Framing?**.

$\prod_I U^{(n_I)}(1)$ lattice higher Chern-Simons theory $a_I \mathrm{d}a_J \mathrm{d}a_L$

- We constructed a rotor model in D-**dimensional spacetime lattice** with several $\mathbb{R}/\mathbb{Z}$-valued n_I-**cochain** a_I fields, that has the $\mathbb{Z}$-gauge invariance and reduces to continuum $\prod_I U^{(n_I)}(1)$ higher Chern-Simons theory of type $\sum_{I\le J\le L} k_{IJL} a_I \mathrm{d}a_J \mathrm{d}a_L$

$$\mathrm{e}^{-S} = \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J\le L} k_{IJL} \int_{\mathcal{M}^D} \bar{a}_I \bar{b}_J \bar{b}_L + (-)^{n_I} B_I \bar{a}_J \bar{a}_L + (-)^{n_I+n_J} B_I B_J \bar{a}_L}$$

$$\mathrm{e}^{-\int_{\mathcal{M}^D} \sum_I \frac{|\mathrm{d}a_I - \lfloor \mathrm{d}a_I \rceil|^2}{g}},$$

$$k_{IJL} \in \mathbb{Z}, \quad k_{IJL} = 0 \text{ if } n_I + n_J + n_L + 2 \neq D.$$

Example: **Lattice regulation of compact axion QED**
Let $a_1 = f$ and $a_2 = a$ be $\mathbb{R}/\mathbb{Z}$-valued 0-cochain (axion phase-field) and 1-cochain ($U(1)$ gauge-field), respectively. Let $k_{122} = k$.

$$\mathrm{e}^{-S} = \mathrm{e}^{\mathrm{i}2\pi k \int_{\mathcal{M}^D} (f - \lfloor f \rceil)(\mathrm{d}a - \lfloor \mathrm{d}a \rceil)^2 + (\lfloor \mathrm{d}f \rceil - \mathrm{d}\lfloor f \rceil)(a - \lfloor a \rceil)(\mathrm{d}a - \lfloor \mathrm{d}a \rceil)}$$

$$\mathrm{e}^{-\mathrm{i}2\pi k \int_{\mathcal{M}^D} (\lfloor \mathrm{d}f \rceil - \mathrm{d}\lfloor f \rceil)(\lfloor \mathrm{d}a \rceil - \mathrm{d}\lfloor a \rceil)\mathrm{d}a} \mathrm{e}^{-\int_{\mathcal{M}^D} \frac{|\mathrm{d}f - \lfloor \mathrm{d}f \rceil|^2 + |\mathrm{d}a - \lfloor \mathrm{d}a \rceil|^2}{g}}$$

$$\to \Big|_{\text{small } g} \mathrm{e}^{\mathrm{i}2\pi k \int_{\mathcal{M}^D} f (\mathrm{d}a)^2} \mathrm{e}^{-\int_{\mathcal{M}^D} \frac{|\mathrm{d}f - \lfloor \mathrm{d}f \rceil|^2 + |\mathrm{d}a - \lfloor \mathrm{d}a \rceil|^2}{g}},$$

$\prod_I U^{(n_I)}(1)$ Chern-Simons theory $\to$ higher SPT state

- If we consider only **coboundary fluctuations** by setting $a_I = \mathrm{d}f_I$, the lattice $\prod_I U^{(n_I)}(1)$ higher Chern-Simons theory becomes an **exactly soluble** lattice rotor model realizing a $\prod_I U^{(n_I)}(1)$ **higher SPT state**.
- **Type-**$a_I \mathrm{d} a_J$ (for $k_{IJ} \in \mathbb{Z}, \quad k_{IJ} = 0$ if $n_I + n_J + 1 \neq D$)

$$\mathrm{e}^{-S} = \mathrm{e}^{\mathrm{i}\,2\pi \sum_{I\leq J} k_{IJ} \int_{\mathcal{M}^D} (-)^{n_I} B_I \bar{a}_J}\Big|_{a_I=\mathrm{d}f_I} = \underbrace{\mathrm{e}^{\mathrm{i}\,2\pi \sum_{I\leq J} k_{IJ} \int_{\mathcal{M}^D} (-)^{n_I} \mathrm{d}\lfloor \mathrm{d}f_I \rceil \mathrm{d}f_J}}_{=\mathrm{e}^{-\int_{\mathcal{M}^D} \mathcal{L}_D^{\text{SPT}}}}$$

- **Type-**$a_I \mathrm{d} a_J \mathrm{d} a_L$ (for $k_{IJL} \in \mathbb{Z}, \quad k_{IJL} = 0$ if $n_I + n_J + n_L + 2 \neq D$)

$$\mathrm{e}^{-S} = \mathrm{e}^{\mathrm{i}\,2\pi \sum_{I\leq J\leq L} k_{IJL} \int_{\mathcal{M}^D} (-)^{n_I+n_J} B_I B_J \bar{a}_L}\Big|_{a_I=\mathrm{d}f_I}$$
$$= \mathrm{e}^{\mathrm{i}\,2\pi \sum_{I\leq J\leq L} k_{IJL} \int_{\mathcal{M}^D} (-)^{n_I+n_J} \mathrm{d}\lfloor \mathrm{d}f_I \rceil \mathrm{d}\lfloor \mathrm{d}f_J \rceil \mathrm{d}f_L}$$

- The SPT rotor models have
  the $\prod_I U^{(n_I)}(1)$ $(n_I - 1)$**-form symmetries** $f_I \to f_I + \alpha_I, \quad \mathrm{d}\alpha_I = 0$,
  and $\mathbb{Z}$**-gauge invariance** $f_I \to f_I + m_I, \quad m_I \in \mathbb{Z}$.
- The models are **exactly soluble** since $\mathcal{L}_D^{\text{SPT}} = \mathrm{d}\mathcal{L}_{D-1}^{\text{ano}}$ and $\mathrm{e}^{-S} = 1$ for closed spacetime M^D. ($\mathrm{e}^{-S} \neq 1$ for open M^D $\to$ non-trivial boundary)

Quantization of k for lattice higher Chern-Simons theory

- Our lattice $\prod_I U^{(n_I)}(1)$ rotor model has two types of fluctuations:
  (1) $\mathrm{d}a_I$ **fluctuations** which are weak and semiclassical in small g limit
  (2) **coboundary fluctuations** $a_I = \bar{a}_I + \mathrm{d}f_I$ which are strong quantum and are gapped, for a weak-field background $\bar{a}_I$.

  The two types of fluctuations are **decoupled** provided that k **is an integer**. This is because in weak-field limit

  $$\mathrm{e}^{-S} \approx \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^D} a_I \wedge \mathrm{d}a_J} \quad \text{or} \quad \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J\le L} k_{IJL} \int_{\mathcal{M}^D} a_I \wedge \mathrm{d}a_J \wedge \mathrm{d}a_L}$$

  which is independent of lattice coboundary fluctuations $\mathrm{d}f_I$. The weak-field state is the equal weight superposition of all coboundary fluctuations:

  $$|\text{weak-field state}\rangle = \sum_{f_I} \mathrm{e}^{\mathrm{i}\phi(f_I, \bar{a}_I)} |\bar{a}_I + \mathrm{d}f_I\rangle$$

  Coboundary fluctuation = Pure gauge fluctuation

  Gapping of coboundary fluctuation = emergence of gauge invariance → Cocycle excitations = degenerate ground states

  All those happen when k is an integer.

$\prod_I U^{(n_I)}(1)$ higher SPT state $\to$ anomalous theory

- The SPT Lagrangian density $\mathcal{L}_D^{\text{SPT}}$ is a coboundary $\mathcal{L}_D^{\text{SPT}} = \mathrm{d}\mathcal{L}_{D-1}^{\text{ano}}$. This gives rise to a **Lagrangian density** $\mathcal{L}_{D-1}^{\text{ano}}$ **in one lower dimension** that describe a rotor model with an anomalous $\prod_I U^{(n_I)}(1)$ **higher-form symmetry**.
- Compute $\mathcal{L}_{D-1}^{\text{ano}}$ from $\mathcal{L}_D^{\text{SPT}} = \mathrm{d}\mathcal{L}_{D-1}^{\text{ano}}$ $\to$ system with anomalous $\prod_I U^{(n_I)}(1)$ higher-form symmetry

- **Type-**$a_I \mathrm{d} a_J$ (for $k_{IJ} \in \mathbb{Z}, \quad k_{IJ} = 0$ if $n_I + n_J + 1 \neq D$)

$$\mathrm{e}^{-\int_{B^{D-1}} \mathcal{L}_{D-1}^{\text{ano}}} = \mathrm{e}^{-\mathrm{i} 2\pi \sum_{I \leq J} k_{IJ} \int_{B^{D-1}} \mathrm{d}\lfloor \mathrm{d} f_I \rceil f_J}$$

- **Type-**$a_I \mathrm{d} a_J \mathrm{d} a_L$ (for $k_{IJL} \in \mathbb{Z}, \quad k_{IJL} = 0$ if $n_I + n_J + n_L + 2 \neq D$)

$$\mathrm{e}^{-\int_{B^{D-1}} \mathcal{L}_{D-1}^{\text{ano}}} = \mathrm{e}^{\mathrm{i} 2\pi \sum_{I \leq J \leq L} k_{IJL} \int_{B^{D-1}} \mathrm{d}\lfloor \mathrm{d} f_I \rceil \mathrm{d}\lfloor \mathrm{d} f_J \rceil f_L}$$

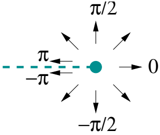
- $\mathcal{L}_{D-1}^{\text{ano}}$ is $\mathbb{Z}$-gauge invariant $\to$ well defined rotor model.
- $\mathcal{L}_{D-1}^{\text{ano}}$ changes by a coboundary under $\prod_I U^{(n_I)}(1)$ symmetry transformation $\to$ anomalous $\prod_I U^{(n_I)}(1)$ symmetry.
- Most general anomalous-$\prod_I U^{(n_I)}(1)$ system

$$\mathcal{L}_{D-1} = \mathcal{L}_{D-1}^{\text{ano}} + \mathcal{L}_{D-1}^{\text{free}}(\{\mathrm{d} f_I - \lfloor \mathrm{d} f_I \rceil\})$$

Example: anomalous $U(1)$ symmetry in 1+1D rotor model

- Consider a $1+1$D rotor model with a $U(1)$ symmetry, described by

  $e^{-S} = e^{-\int_{B^2} g^{-1}|\mathrm{d}f - \lfloor \mathrm{d}f \rceil|^2}$, $\quad f =$ 0-cochain (scaler field),

  where the value f_i of f on vertex-i corresponds to the rotor angle $2\pi f_i$ on that vertex. The $U(1)$ symmetry $f_i \to f_i + \alpha$ in the rotor model is anomaly-free.
- We can make the $U(1)$ symmetry anomalous by including a topological term $\quad e^{-S} = e^{-\int_{B^2} g^{-1}|\mathrm{d}f - \lfloor \mathrm{d}f \rceil|^2} e^{\mathrm{i}2\pi \int_{B^2} k \mathrm{d}\lfloor \mathrm{d}f \rceil f}$, $\quad k \in \mathbb{Z}$.

  The lagrangian changes by a coboundary under symmetry transformation
- Note that the Poincaré dual of $\lfloor \mathrm{d}f \rceil$ describes a 2π-domain-wall in spacetime, where the rotor angle changes between π and $-\pi$. The Poincaré dual of $\mathrm{d}\lfloor \mathrm{d}f \rceil$ is the end of 2π-domain-wall, which is a vortex. Thus the topological term $e^{\mathrm{i}2\pi \int_{L^2} k \mathrm{d}\lfloor \mathrm{d}f \rceil f}$ implies that **the $U(1)$ vortex is a source/drain of the $U(1)$ charge**.
- The perturbative anomalous $U(1)$ symmetry forces the system to be gapless. There is no gapped phase regardless $\mathcal{L}^{\text{free}}(\mathrm{d}f - \lfloor \mathrm{d}f \rceil)$.

Anomalous $U(1)$ symmetry in 1+1D lattice

- **Anomaly-free $U(1)$ symmetry**:
  Allow vortex fluctuations in 2D spacetime. Other combination of vortex-charge fluctuations are not allowed.

  Gapped phases are allowed

- **Anomalous $U(1)$ symmetry**:
  Allow vortex-k-charge fluctuations in 2D spacetime. Other combination of monople-charge fluctuations are not allowed.

  The perturbative anomalous $U(1)$ symmetry ($k \neq 0$) forces the system to be gapless. There is no gapped phase regardless how we tune the symmetry-preserving lattice interaction $\mathcal{L}^{\text{free}}(\mathrm{d}f - \lfloor \mathrm{d}f \rceil)$.

- The following rotor model must be gapless regardless $\mathcal{L}^{\text{free}}(\mathrm{d}f - \lfloor \mathrm{d}f \rceil)$:

$$\mathrm{e}^{-S} = \mathrm{e}^{-\int_{B^2} \mathcal{L}^{\text{free}}(\mathrm{d}f - \lfloor \mathrm{d}f \rceil)} \mathrm{e}^{\mathrm{i}2\pi \int_{B^4} k \mathrm{d}\lfloor \mathrm{d}f \rceil f}, \quad k \in \mathbb{Z}, \quad k \neq 0.$$

\- A general proof: The anomaly in a gapped state must have a finite order, while the $U(1)$ anomaly is order $\mathbb{Z}$.

Cordova Freed Teleman arXiv:2408.15148

Anomalous $U^{(2)}(1)$ 2-group (1-form) symm in 3+1D lattice

- Consider a $3+1$D rotor model with a $U^{(2)}(1)$ symmetry, described by
  $$\mathrm{e}^{-S} = \mathrm{e}^{-\int_{B^4} g^{-1}|\mathrm{d}a - \lfloor \mathrm{d}a \rceil|^2}, \quad a = \text{ 1-cochain (gauge vector potential)},$$
  where the value a_{ij} of a on link-ij corresponds to the rotor angle $2\pi a_{ij}$ on that link. The $U^{(2)}(1)$ symmetry in the rotor model is anomaly-free:
  $$a \to a + \alpha, \quad \alpha \in Z^1(B^4; \mathbb{R}/\mathbb{Z}), \quad \mathrm{d}\alpha = 0$$
  The rotor model is a lattice $U(1)$-gauge theory.
- Make the $U^{(2)}(1)$ symmetry anomalous by including a topological term
  $$\mathrm{e}^{-S} = \mathrm{e}^{-\int_{B^4} g^{-1}|\mathrm{d}a - \lfloor \mathrm{d}a \rceil|^2} \mathrm{e}^{\mathrm{i}2\pi \int_{B^4} k\,\mathrm{d}\lfloor \mathrm{d}a \rceil a}, \quad k \in \mathbb{Z}.$$
- We note that $\lfloor \mathrm{d}a \rceil$ is a $\mathbb{Z}$-valued 2-cochain dual to a Dirac worldsheet in 4D spacetime. $\mathrm{d}\lfloor \mathrm{d}a \rceil$ is a $\mathbb{Z}$-valued 3-cochain dual to a monople worldline in 4D spacetime.

  Thus the topological term $\mathrm{e}^{\mathrm{i}2\pi \int_{L^2} k\,\mathrm{d}\lfloor \mathrm{d}f \rceil f}$ implies that a $U(1)$ monople carries k-units of $U(1)$ gauge-charge.

Anomalous $U^{(2)}(1)$ 2-group (1-form) symm in 3+1D lattice

- **Anomaly-free $U^{(2)}(1)$ 2-group (1-form) symmetry**:
  Allow monople fluctuations. Other combination of monople-charge fluctuations are not allowed.

  Gapped phases are allowed

- **Anomalous $U^{(2)}(1)$ 2-group (1-form) symmetry**:
  Allow monople-k-charge fluctuations. Other combination of monople-charge fluctuations are not allowed.

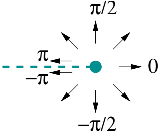
  The perturbative anomalous $U^{(2)}(1)$ 2-group (1-form) symmetry ($k \neq 0$) forces the system to be gapless. There is no gapped phase regardless how we tune the symmetry-preserving lattice interaction $\mathcal{L}^{\text{free}}(\mathrm{d}f - \lfloor \mathrm{d}f \rceil)$.

- The following rotor model must be gapless regardless $\mathcal{L}^{\text{free}}(\mathrm{d}a - \lfloor \mathrm{d}a \rceil)$:

$$\mathrm{e}^{-S} = \mathrm{e}^{-\int_{B^4} \mathcal{L}^{\text{free}}(\mathrm{d}a - \lfloor \mathrm{d}a \rceil)} \mathrm{e}^{\mathrm{i}2\pi \int_{B^4} k\, \mathrm{d}\lfloor \mathrm{d}a \rceil a}, \quad k \in \mathbb{Z}, \quad k \neq 0.$$

Anomalous $U(1)$ symmetry in 3+1D rotor model

- Consider a $3+1$D rotor model with a $U(1)$ symmetry, described by
  $\mathrm{e}^{-S} = \mathrm{e}^{-\int_{B^4} g^{-1}|\mathrm{d}f - \lfloor \mathrm{d}f \rceil|^2}, \quad f =$ 0-cochain (scaler field),

  where the value f_i of f on vertex-i corresponds to the rotor angle $2\pi f_i$ on that vertex. The $U(1)$ symmetry $f_i \to f_i + \alpha$ in the rotor model is anomaly-free.
- We can make the $U(1)$ symmetry anomalous by including a topological term $\quad \mathrm{e}^{-S} = \mathrm{e}^{-\int_{B^4} g^{-1}|\mathrm{d}f - \lfloor \mathrm{d}f \rceil|^2} \mathrm{e}^{\mathrm{i}2\pi \int_{B^4} k(\mathrm{d}\lfloor \mathrm{d}f \rceil)^2 f}, \quad k \in \mathbb{Z}.$
- We note that the Poincaré dual of 1-cochain $\lfloor \mathrm{d}f \rceil$ describes a 2π-domain-wall in spacetime, where the rotor angle changes between π and $-\pi$. The Poincaré dual of 4-cochain $(\mathrm{d}\lfloor \mathrm{d}f \rceil)^2$ describes self-intersection points of 2π-domain-wall, in 4D spacetime.
  Thus the topological term $\mathrm{e}^{\mathrm{i}2\pi \int_{L^2} k(\mathrm{d}\lfloor \mathrm{d}f \rceil)^2 f}$ implies that a self-intersection point are source/drain of the $U(1)$ charge.
- The following rotor model must be gapless regardless $\mathcal{L}^{\text{free}}(\mathrm{d}f - \lfloor \mathrm{d}f \rceil)$:
  $$\mathrm{e}^{-S} = \mathrm{e}^{-\int_{B^4} \mathcal{L}^{\text{free}}(\mathrm{d}f - \lfloor \mathrm{d}f \rceil)} \mathrm{e}^{\mathrm{i}2\pi \int_{B^4} k(\mathrm{d}\lfloor \mathrm{d}f \rceil)^2 f}, \quad k \in \mathbb{Z}, \quad k \neq 0.$$

Summary

- Put $U(1) \times U(1) \times \cdots$ higher Chern-Simons theory of type $a_I \mathrm{d} a_J$ and $a_I \mathrm{d} a_J \mathrm{d} a_L$ on spacetime lattice as rotor models.
  So the theory is mathematically well defined at microscopic level

- Constructed exactly soluble rotor models to realize $U(1) \times U(1) \times \cdots$ higher SPT phases in any dimensions.
  Exactly soluble $U(1)$ SPT Lagrangian has infinite dimensional Hilbert space per site, and has discontiuities for large $U(1)$ phases.

- Constructed lattice rotor models with $U(1) \times U(1) \times \cdots$ higher symmetry that must be gapless (*ie* no gapped phases).
  A lattice model without gapped phases is novel