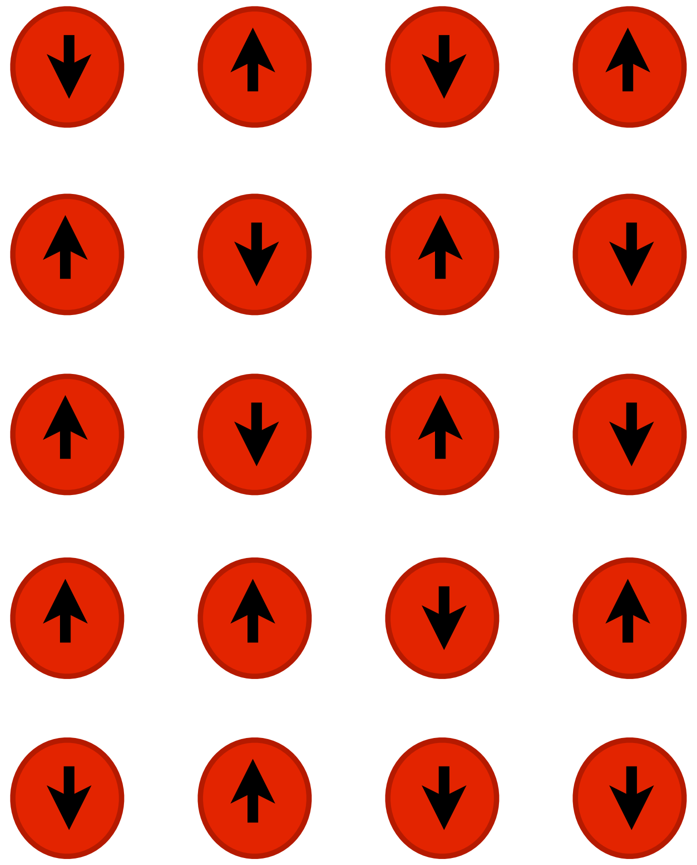
Chiral Lattice Gauge Theories from Symmetry Disentanglers

Lukasz Fidkowski

(with Ryan Thorngren and John Preskill; based on work with Cenke Xu, Carolyn Zhang)

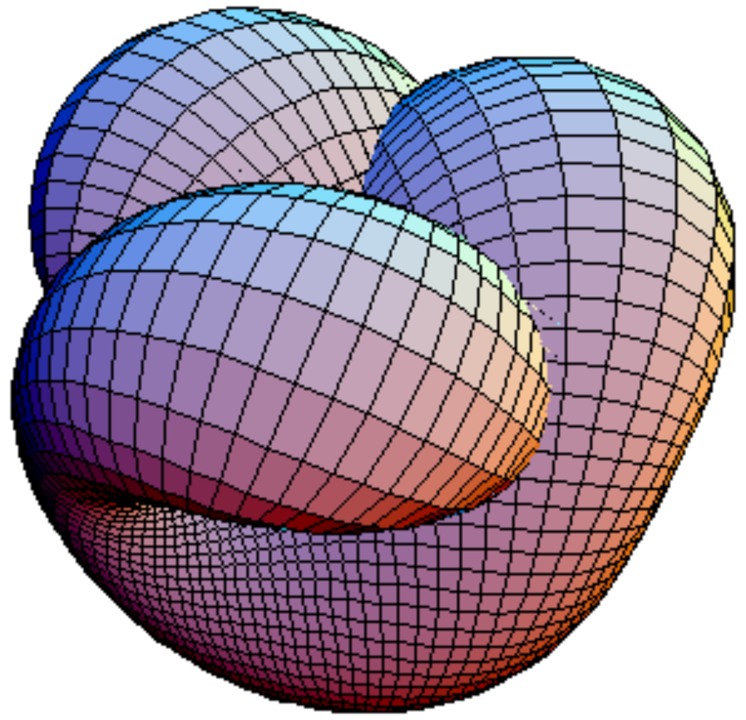
Condensed Matter:

High energy:



a great gulf

← ?????? →



real projective plane RP^2

$$\mathcal{H} = \otimes_{\text{sites i}} \mathcal{H}_i$$

$$H = \sum_{\alpha} H_\alpha$$

(M, B) -> exp[i S(M,B)]

spacetime manifold

principal G-bundle

exponentiated action

- different perspectives on same problem: e.g. classification of topological or symmetry protected phases of quantum matter

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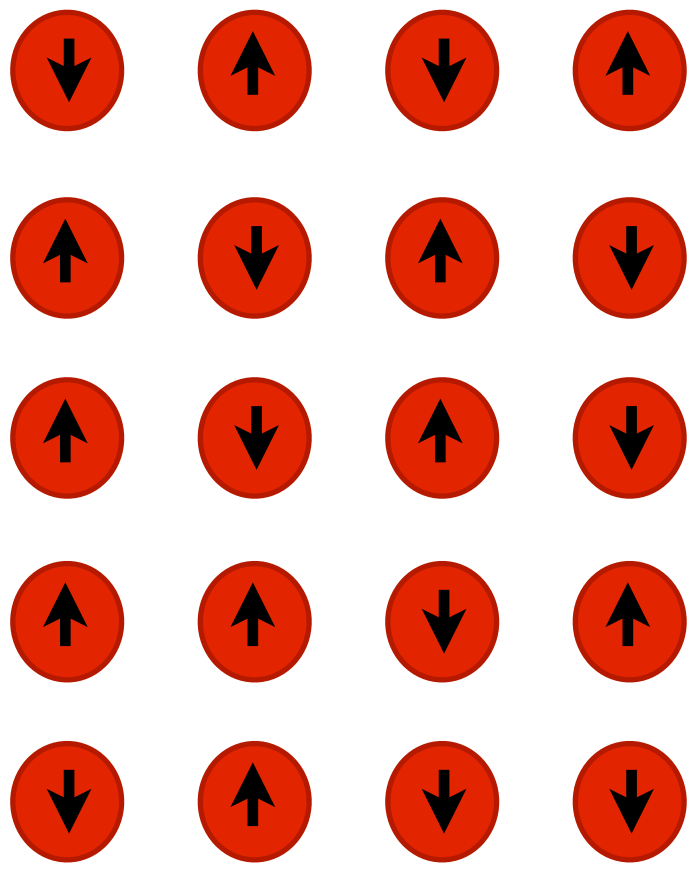

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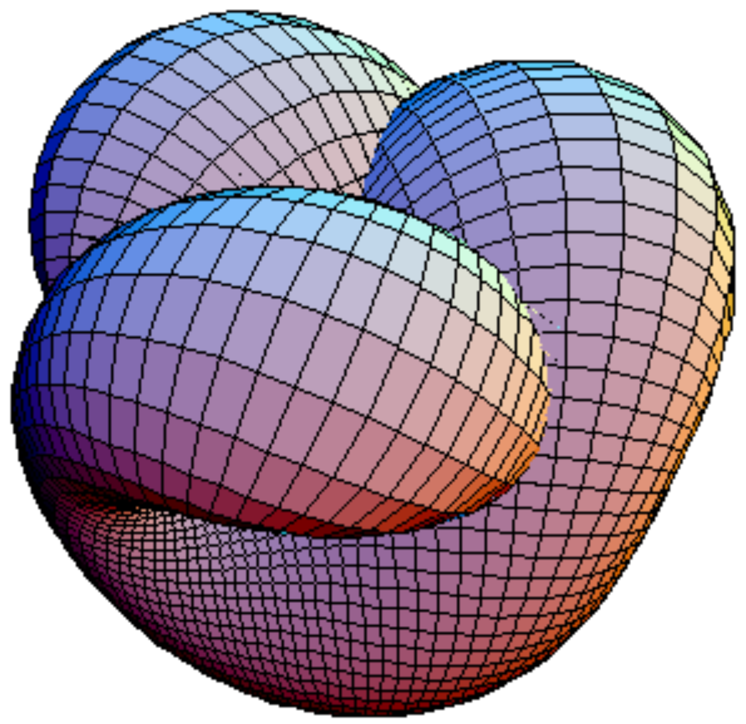

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- different perspectives on same problem: e.g. classification of topological or symmetry protected phases of quantum matter

- chiral lattice gauge theories

Plan

1) Overview: chiral theories on the spatial lattice

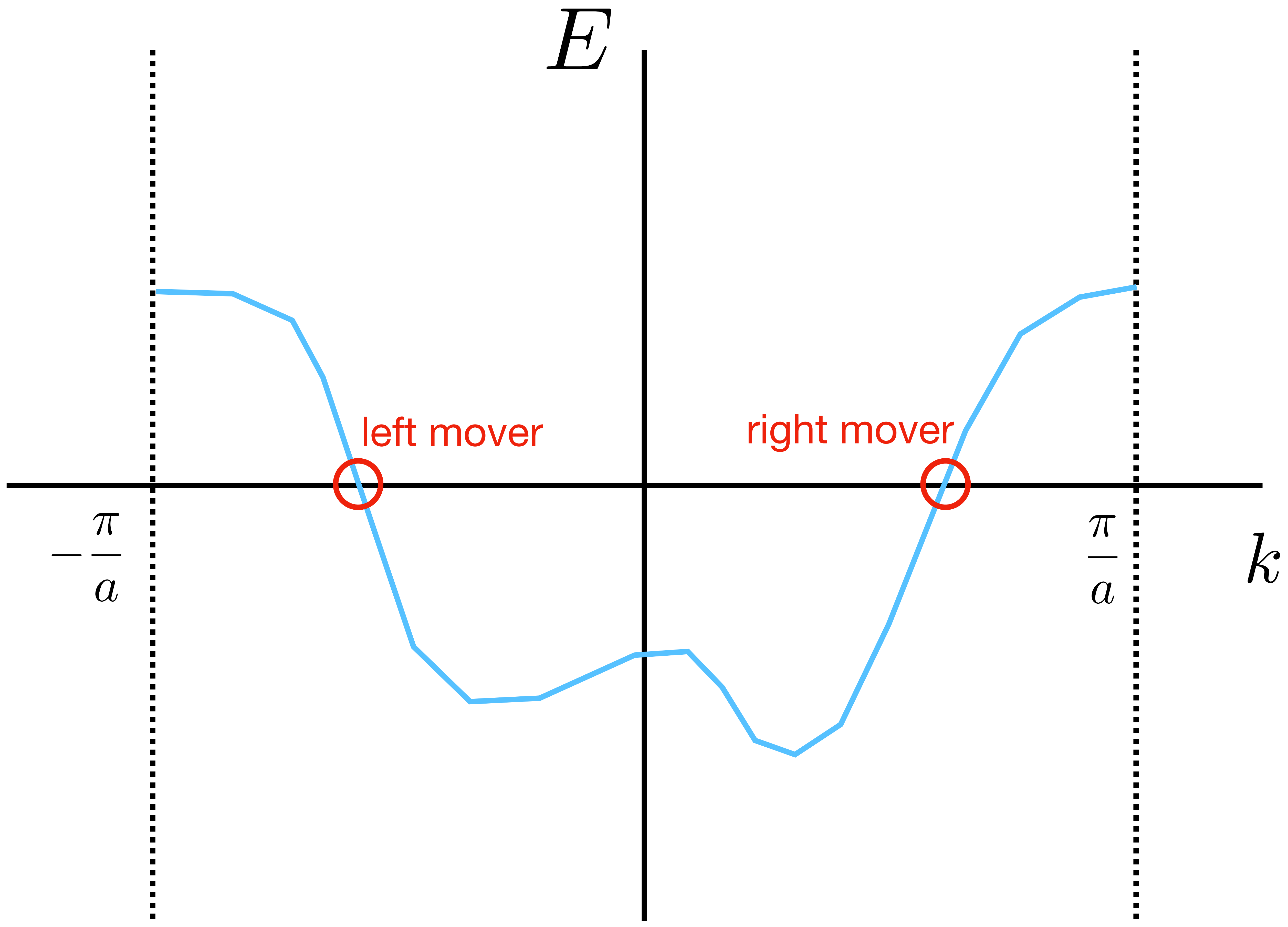
2) Strategy for disentangling anomaly-free chiral symmetries

3) Exactly solved Hamiltonian for 3-4-5-0 theory in 1+1d

4) Anomaly-free chiral symmetries in 3+1d

5) Conclusions

1+1 d fermion



Brillouin Zone

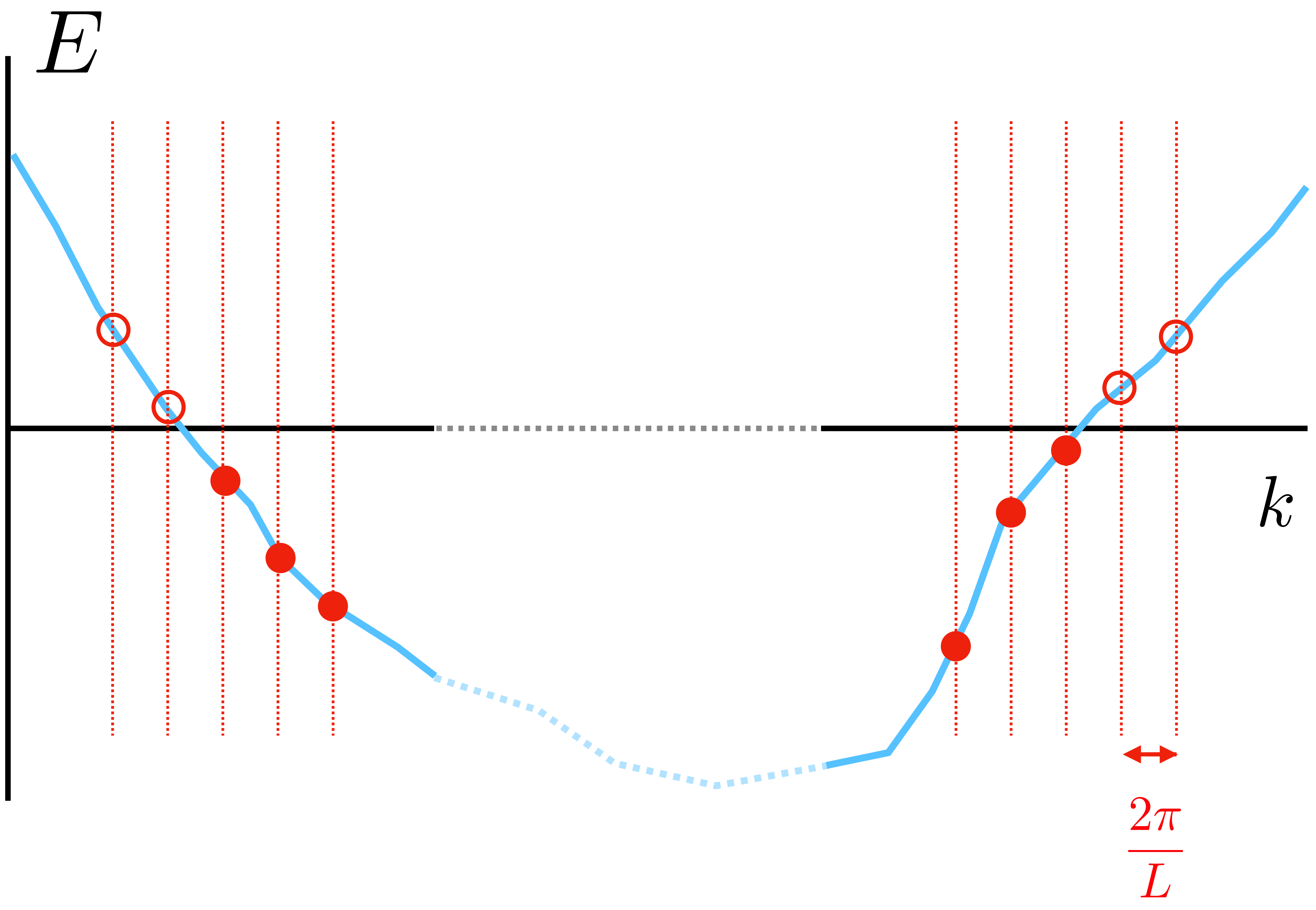
microscopic (UV) symmetry $U(1)_V$:

$$Q_V = N_L + N_R$$

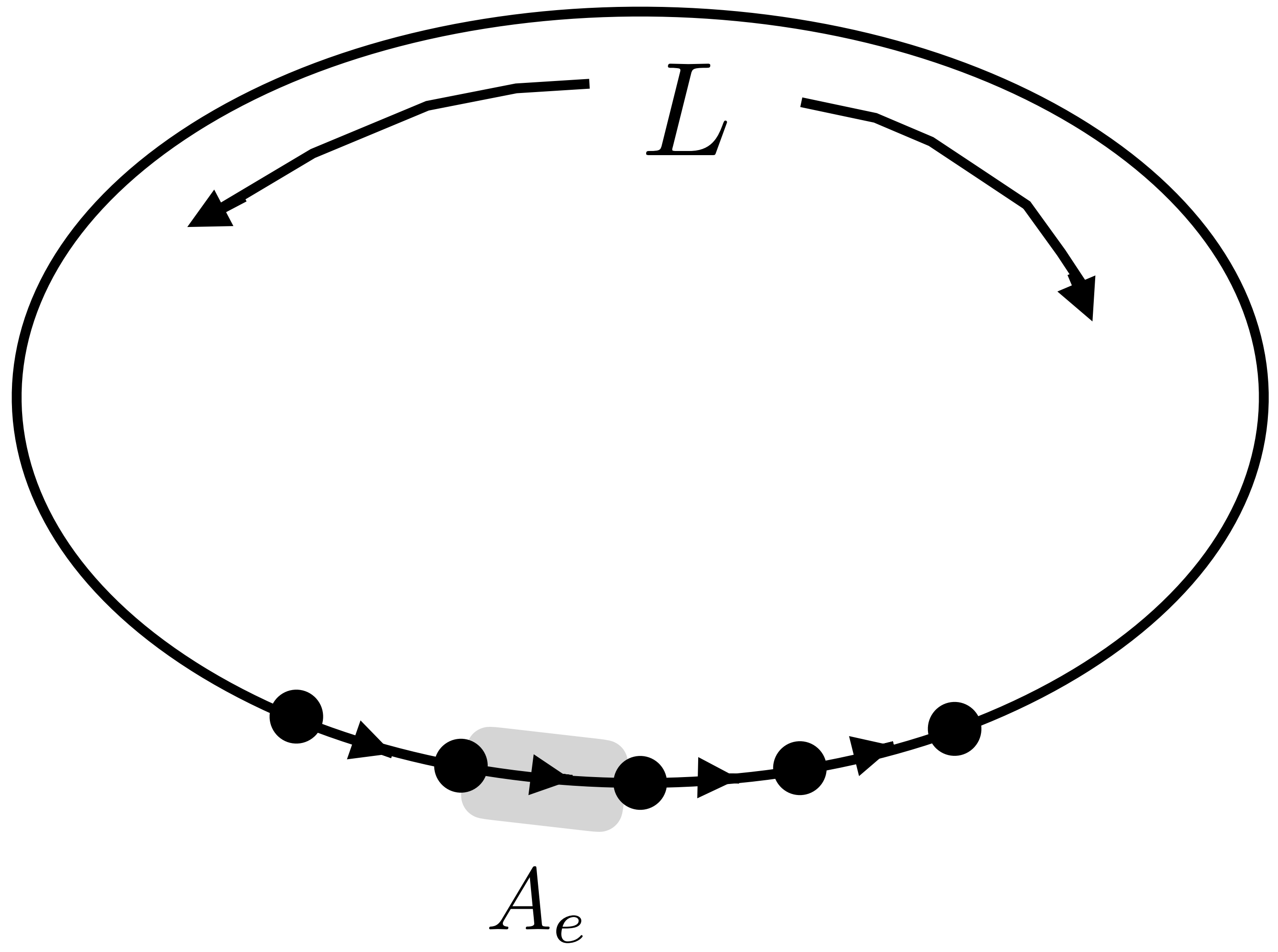
emergent symmetry $U(1)_A$:

$$Q_A = N_L - N_R$$

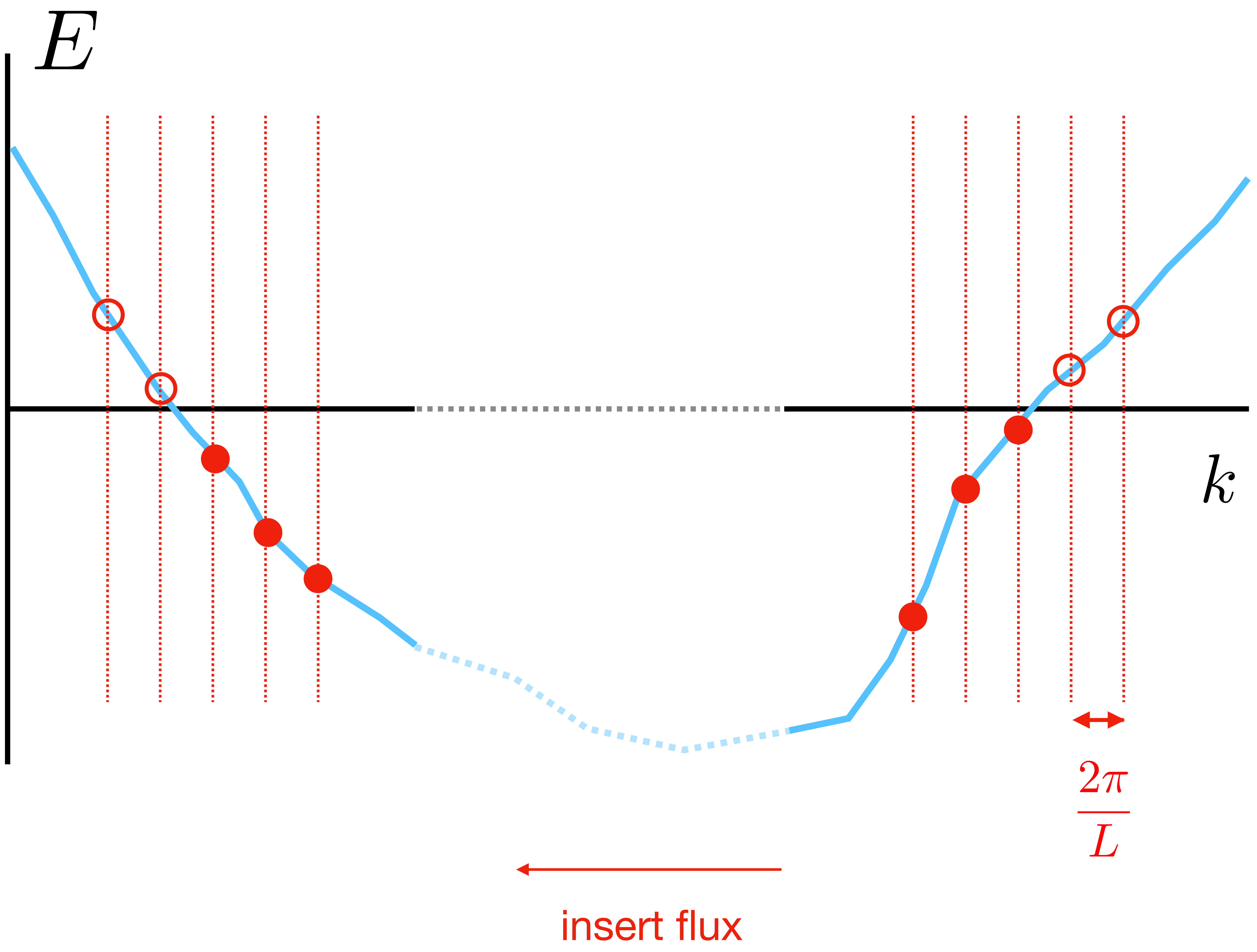
Mixed anomaly between $U(1)_V$ and $U(1)_A$



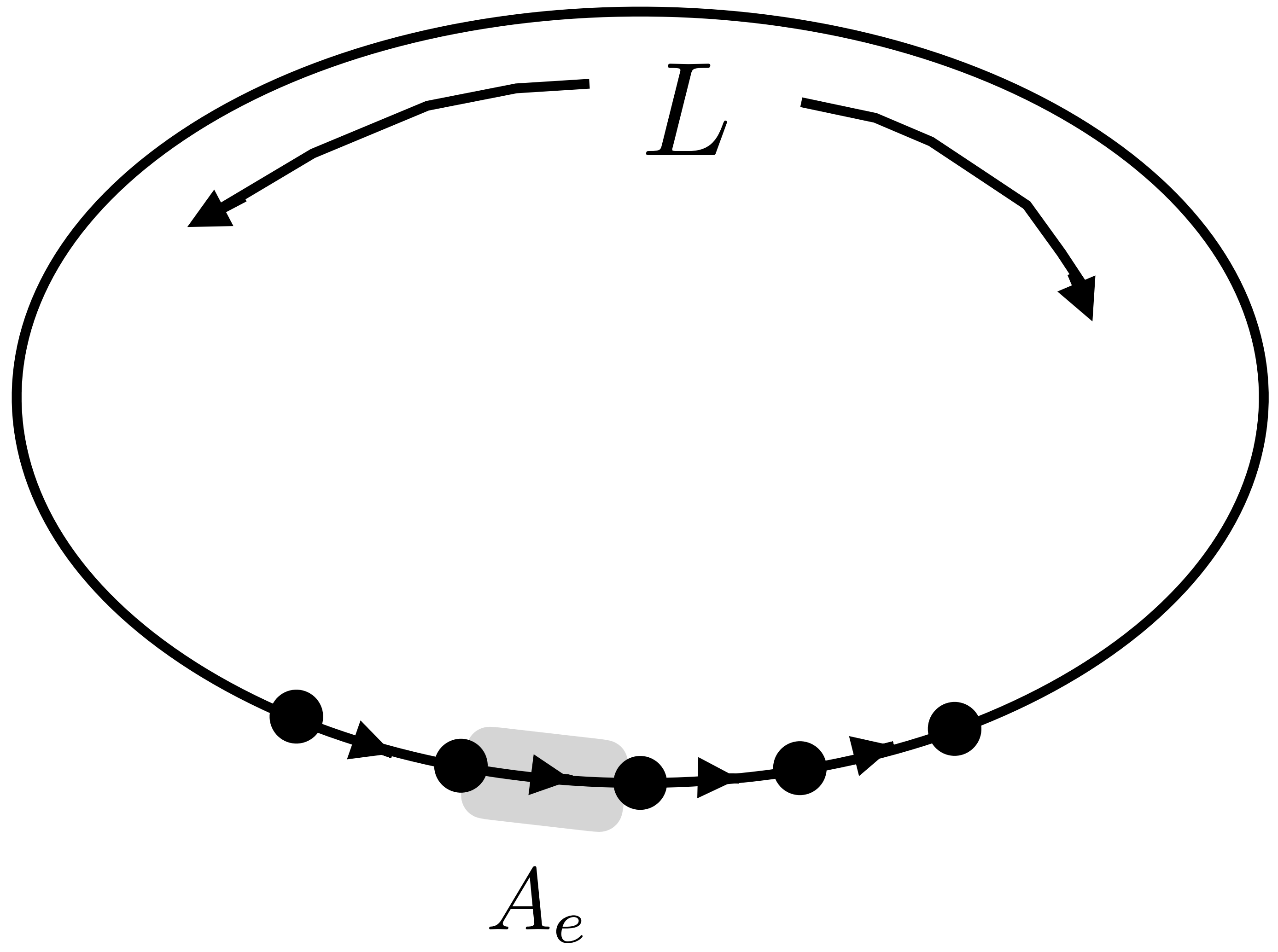
- gauge $U(1)_V$:



Mixed anomaly between $U(1)_V$ and $U(1)_A$



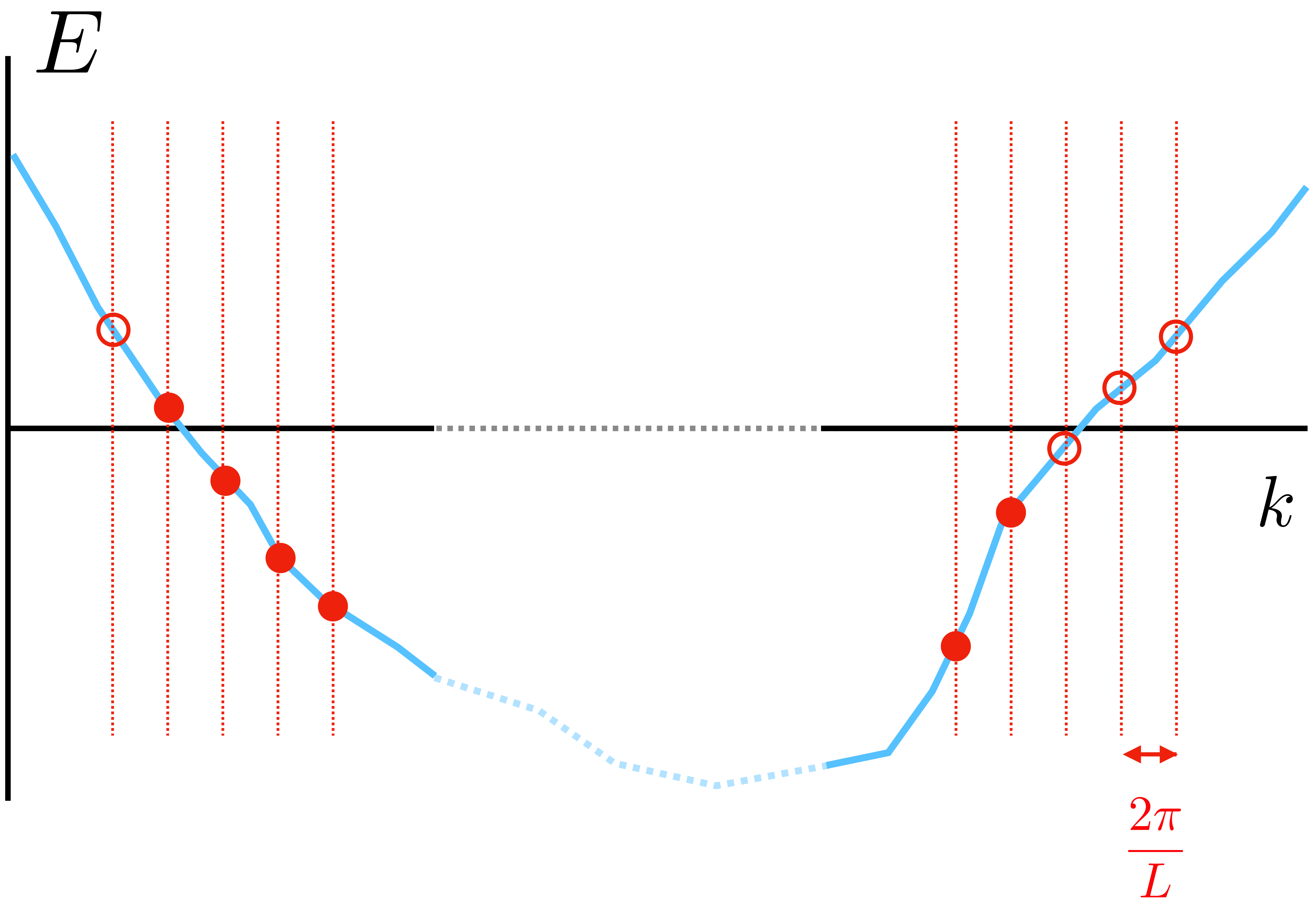
- gauge $U(1)_V$:



- inserting flux $\Phi = \sum_e A_e$ is equivalent to twisting boundary conditions by Φ

$$k \to k - \frac{\Phi}{L}$$

Mixed anomaly between $U(1)_V$ and $U(1)_A$

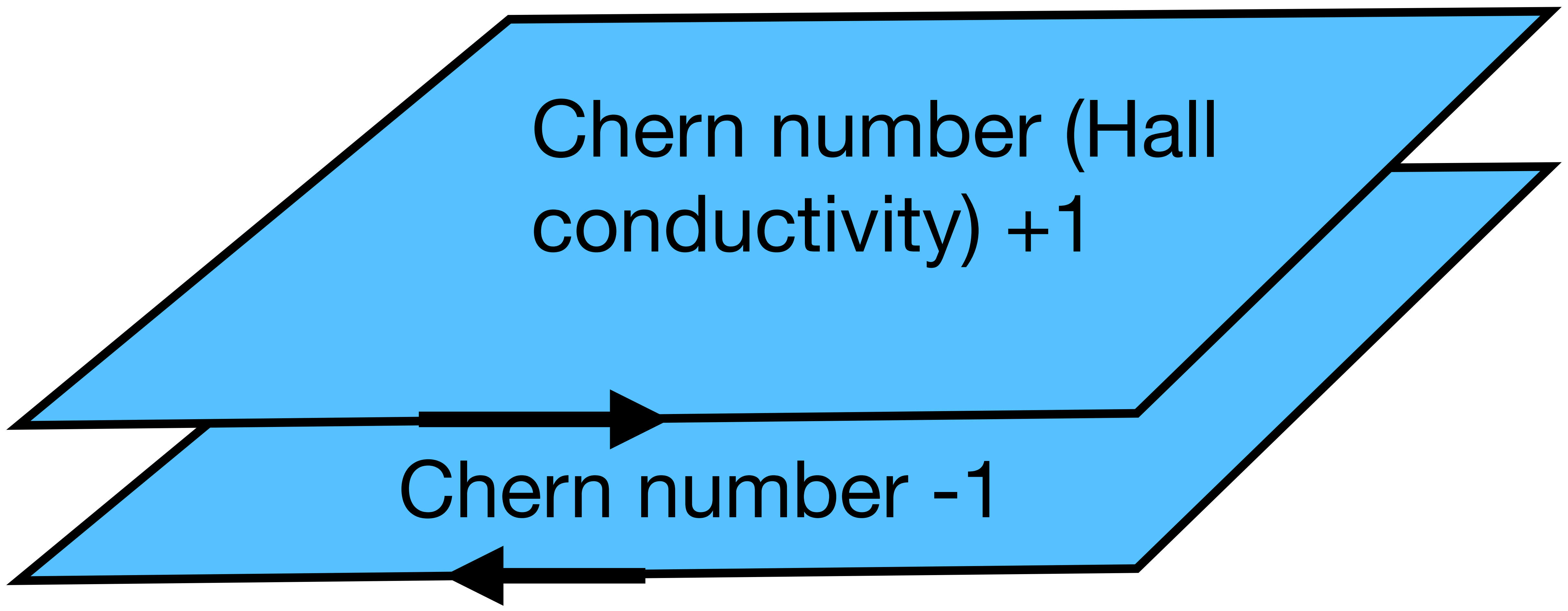


- adiabatically inserting $\Phi = 2\pi$ flux pumps a unit of charge from the right to the left Fermi point

$$Q_A \to Q_A + 2$$

- impossible to simultaneously realize $U(1)_V$ and $U(1)_A$ as microscopic onsite symmetries

Anomaly inflow perspective



- realize $U(1)_V$ and $U(1)_A$ as microscopic onsite symmetries of 2d gapped Symmetry Protected Topological (SPT) phase

$$S_{\text{inflow}}[A_L, A_R] = \frac{1}{4\pi}\int_{M_3} (A_L \wedge dA_L - A_R \wedge dA_R)$$

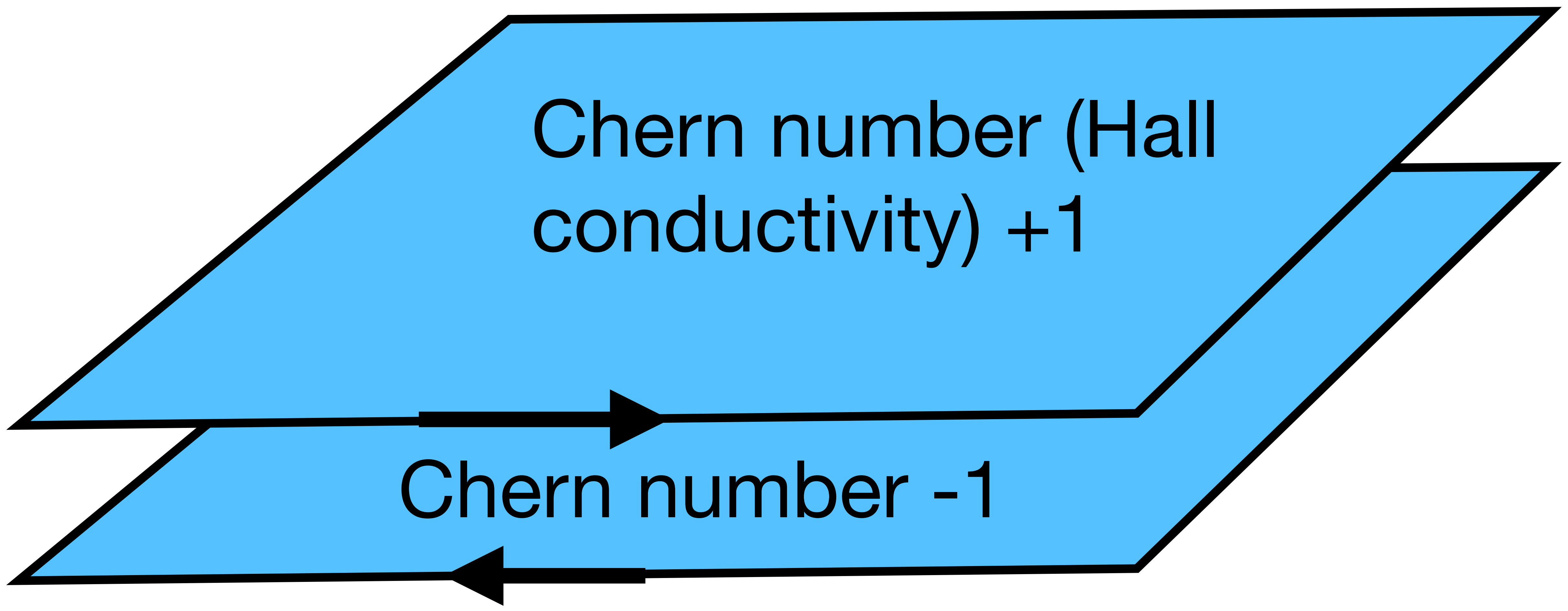
$$= \frac{1}{\pi}\int_{M^3} (A_V \wedge dA_A)$$

$$\begin{pmatrix} A_L = A_V + A_A \\ A_R = A_V - A_A \end{pmatrix}$$

(modulo boundary terms)

domain wall fermions: Kaplan 1992

Anomaly inflow perspective



- general U(1) subgroup

$$Q = mQ_L + nQ_R$$

$$(A_L = mA, A_R = nA)$$

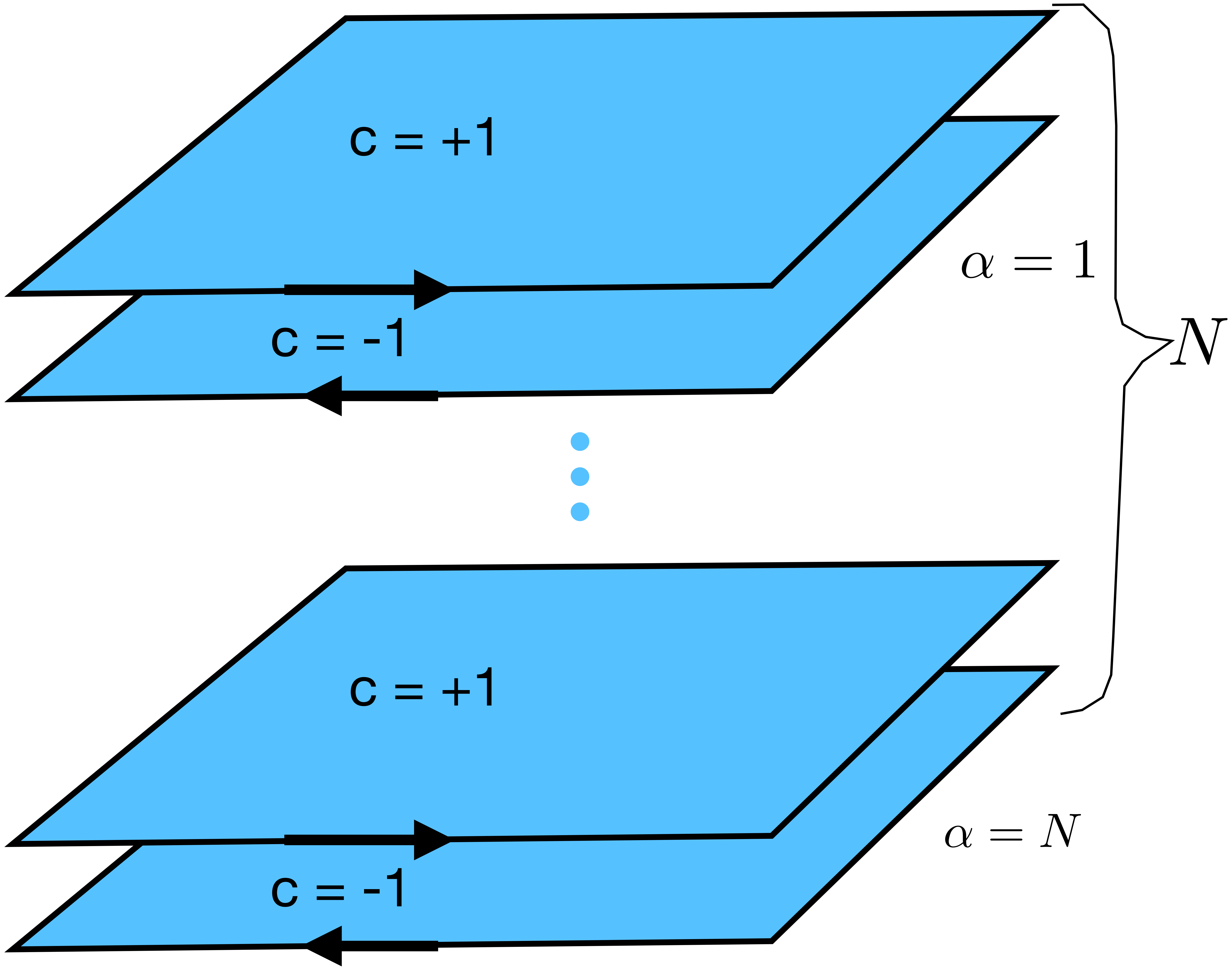
$$S_{\text{inflow}}[A] = \frac{m^2 - n^2}{4\pi} \int_{M_3} A \wedge dA$$

anomaly-free precisely when $m = \pm n$, in which case Hall conductivity vanishes and one edge of slab geometry can be gapped out with back-scattering or pairing terms, leading to quasi 1-d realization

Realizing theory = gapping `doublers' (mirror fermions); symmetric mass generation

Eichten, Preskill 1986

Anomaly inflow perspective: N copies

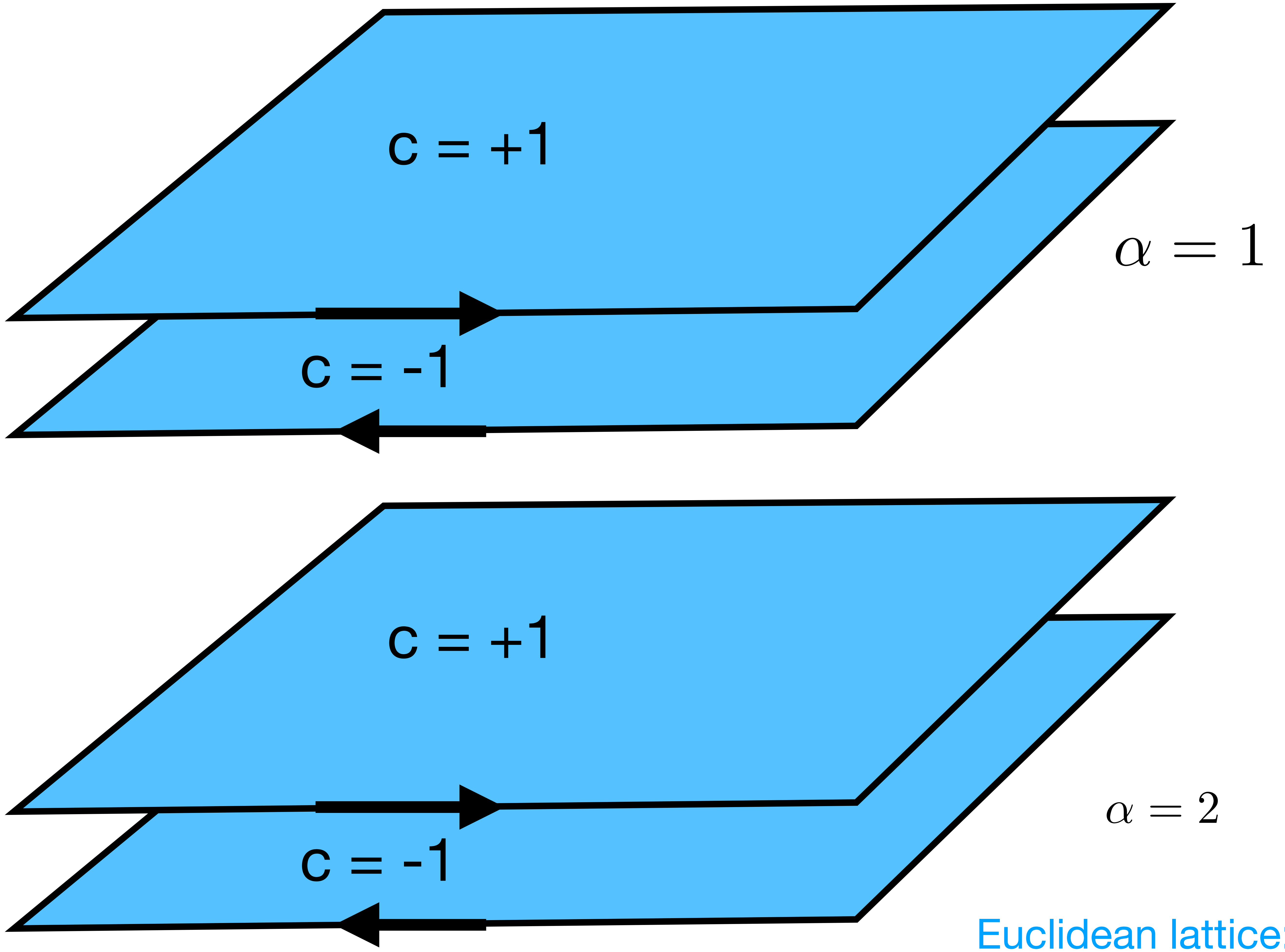


$$Q = \sum_{\alpha=1}^{N} \left(m_\alpha Q_L^\alpha + n_\alpha Q_R^\alpha\right)$$

conjecture: Q can be realized as a microscopic onsite symmetry precisely when net Hall conductivity vanishes:

$$\sum_{\alpha=1}^{N} \left(m_\alpha^2 - n_\alpha^2\right) = 0$$

Example: 3-4-5-0 theory



$$Q = \sum_{\alpha=1}^{2} (m_\alpha Q_L^\alpha + n_\alpha Q_R^\alpha)$$

$$m_1 = 3 \qquad n_1 = 0$$
$$m_2 = 4 \qquad n_2 = 5$$

question: How to realize the 3-4-5-0 theory in a 1+1 d microscopic lattice theory with Q acting onsite?

(note that there are no Q - symmetric relevant perturbations, since all fermion bilinears are forbidden)

Euclidean lattice: Jacobson et al 2024; Suzuki et al. 2024

numerics: Wang, Wen 2019; Y.-Z. You et al 2022

supersymmetry: Tong 2021

Seifnashri 2026

Anomaly cancellation in 3+1 d:

- N left handed Weyl fermions $\psi_\alpha \;\; (\alpha = 1, \ldots, N)$ with charges q_α satisfying

$$\sum_{\alpha=1}^{N} q_\alpha = 0$$

(mixed U(1) - gravitational anomaly vanishes)

$$\sum_{\alpha=1}^{N} q_\alpha^3 = 0$$

(self U(1) anomaly vanishes)

- **Conjecture**: such a theory can be realized in a lattice model with Q acting onsite (and therefore straightforwardly be gauged, giving a non-perturbative definition of a `chiral' gauge theory)

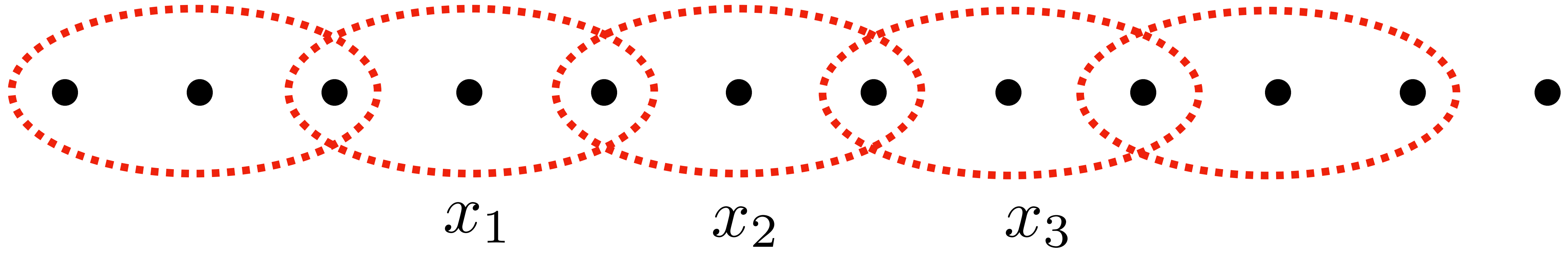
- example: U(1) hypercharge in one generation of the Standard Model

Strategy

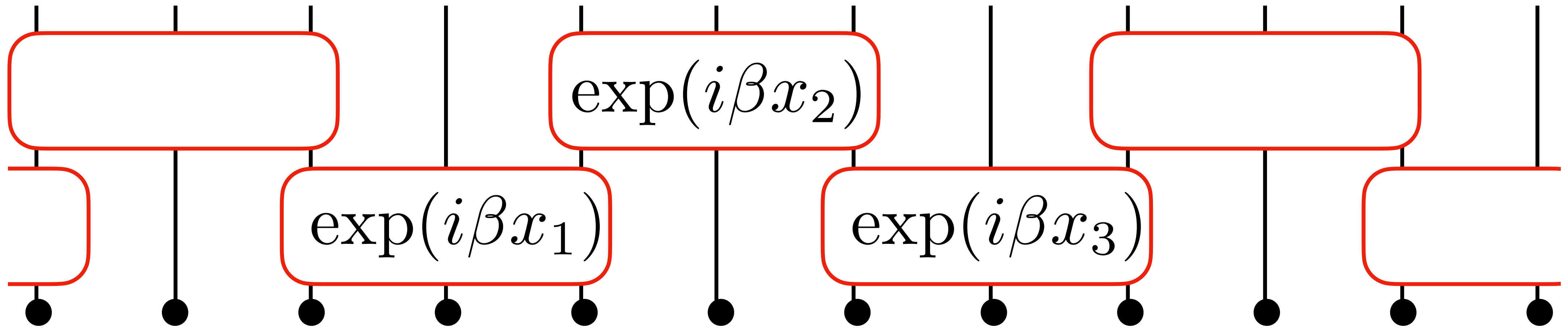
1) (one layer) realize anomalous $U(1)_V \times U(1)_A$ on the lattice using *not-on-site* operators

$$X = \sum_j x_j$$

with $[x_i, x_j] = 0$



$$\exp(i\beta X) =$$



motivated by SPT work of Else, Nayak (2014)

- lattice anomaly matches that of desired low energy theory

Strategy

2) (one layer) Construct a Hamiltonian H' commuting with $U(1)_V \times U(1)_A$ and realizing the desired low energy theory:

1+1d:

(rigorous construction)

	ψ_L	ψ_R
$U(1)_V$	1	1
$U(1)_A$	1	-1

Dirac fermion

3+1d:

(heuristic)

	ψ_1	ψ_2	ψ_3	ψ_4
$U(1)_V$	1	-1	0	0
$U(1)_A$	1	1	-1	-1

4 left-handed Weyls

=

2 Dirac fermions

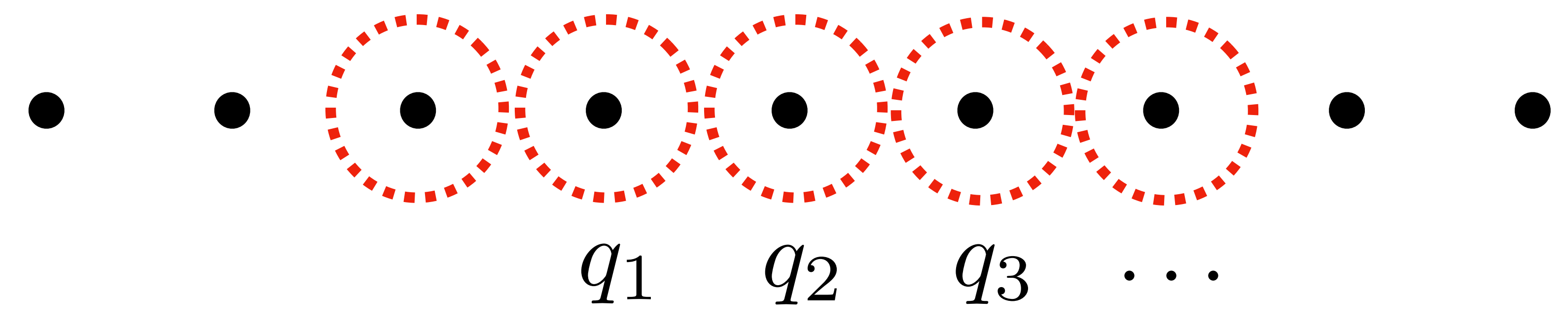
Strategy

3) Tensor product of N copies ('stack'): not-on-site $(U(1)_V \times U(1)_A)^N$

Given an anomaly-free subgroup $U(1) \subset (U(1)_V \times U(1)_A)^N$, construct a finite depth circuit of local unitaries ('*symmetry disentangler*') W such that

$$W^\dagger Q W = Q_{\text{onsite}} = \sum_j q_j$$

anomaly-free
U(1) generator



constructed in discrete case by Seifnashri, Shirley (2025)

Define:

$$H = W^\dagger \left(\sum_{\alpha=1}^{N} H'_\alpha \right) W$$

N identical
copies of H'

$$[H, Q_{\text{onsite}}] = 0$$

Caveats

1) In 1+1d we realize the bosonized version of the 3-4-5-0 theory

in 3+1d we have an intrinsically fermionic Hilbert space

2) Hilbert spaces are constructed from tensor products of *rotors* instead of finite dimensional spins

rotor Hilbert space:

$$L^2(S^1) : \{\psi(\phi)\}, \phi \sim \phi + 1$$

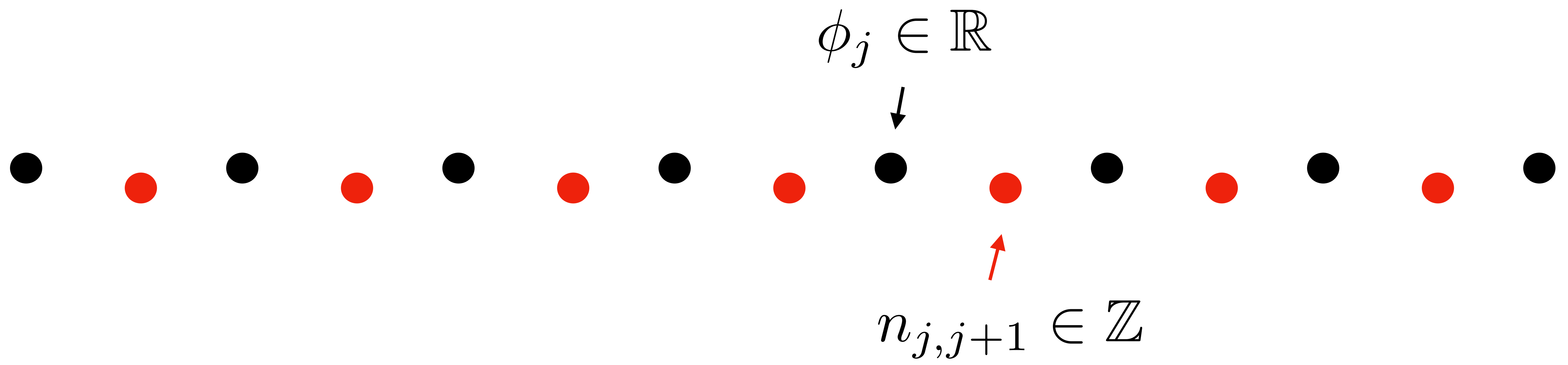
(countably) infinite orthonormal basis:

$$\{|m\rangle \ = \exp(2\pi i m \phi)\}, m \in \mathbb{Z}$$

3) Our disentanglers will have unitaries that multiply by *discontinuous* functions of ϕ

- in particular, they do not preserve the domain of the (unbounded) Hamiltonian operator

1+1d: Villain model



- Villain constraint: $\Psi(\{\phi_j + m_j, n_{j,j+1} + m_{j+1} - m_j\}) = \Psi(\{\phi_j, n_{j,j+1}\})$

- $U(1)_V$ and $U(1)_A$ symmetries:

$$Q_V^{\text{Vil}} = \frac{i}{2\pi}\sum_j \frac{d}{d\phi_j}$$

$$Q_A^{\text{Vil}} = \sum_j (\phi_{j+1} - \phi_j - n_{j,j+1}) = -\sum_j n_{j,j+1}$$

(for periodic boundary conditions)

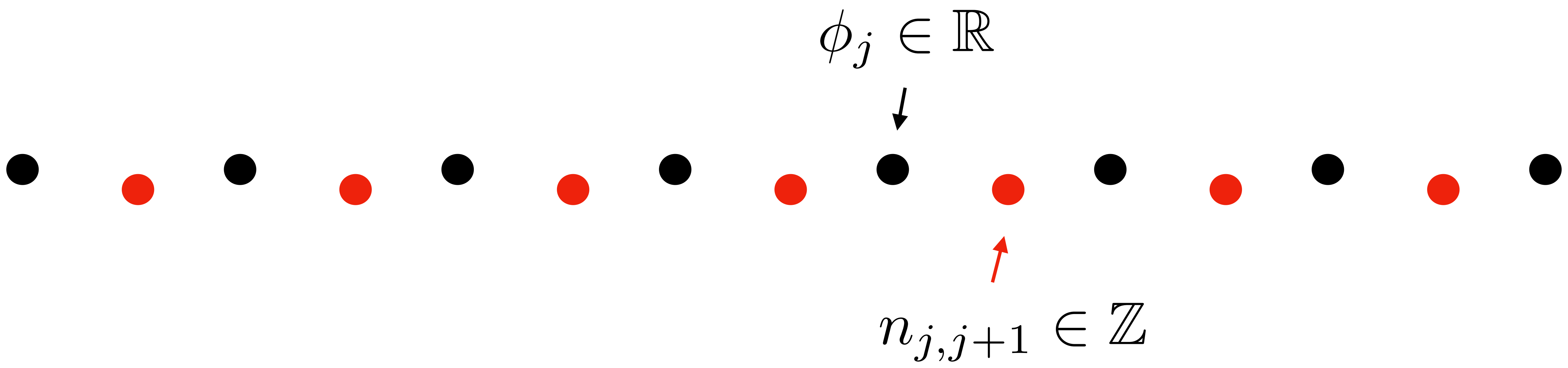
compact cochain notation:

$$\Psi(\phi + m, n + dm) = \Psi(\phi, n)$$

$$Q_V^{\text{Vil}} = \frac{i}{2\pi}\int \frac{d}{d\phi}$$

$$Q_A^{\text{Vil}} = \int (d\phi - n) = -\int n$$

1+1d: Villain model



- bosonic Hamiltonian commuting with $U(1)_V$ and $U(1)_A$

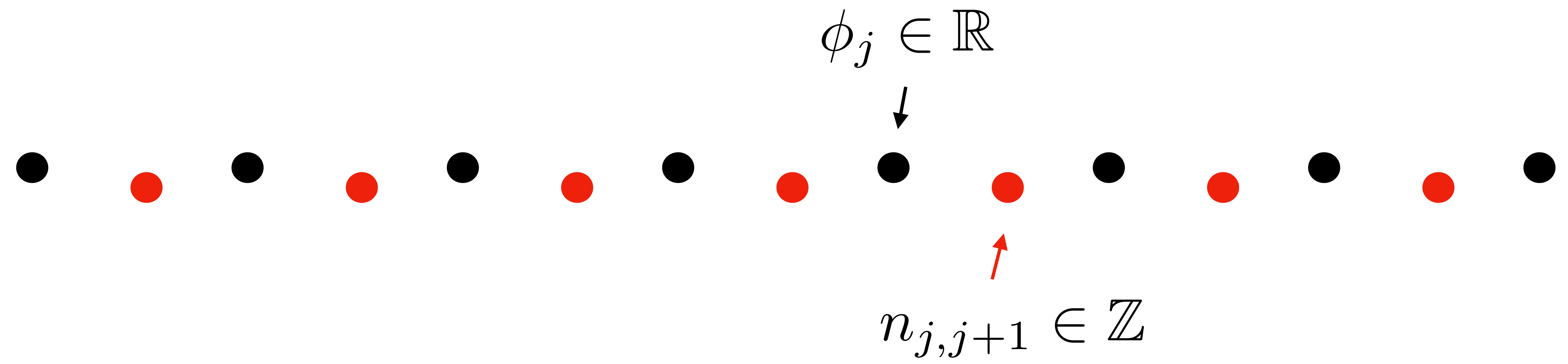
$$H^{\mathrm{Vil}} = -\frac{U}{2}\sum_j \frac{d^2}{d\phi_j^2} + \frac{J}{2}\sum_j (\phi_{j+1} - \phi_j - n_{j,j+1})^2$$

Cheng, Seiberg 2022

- decoupled harmonic oscillators: Luttinger liquid (compact boson) with Luttinger parameter determined by U/J

- $U(1)_V$ and $U(1)_A$ act as compact momentum and winding respectively

1+1d: Disentangling the Villain Model



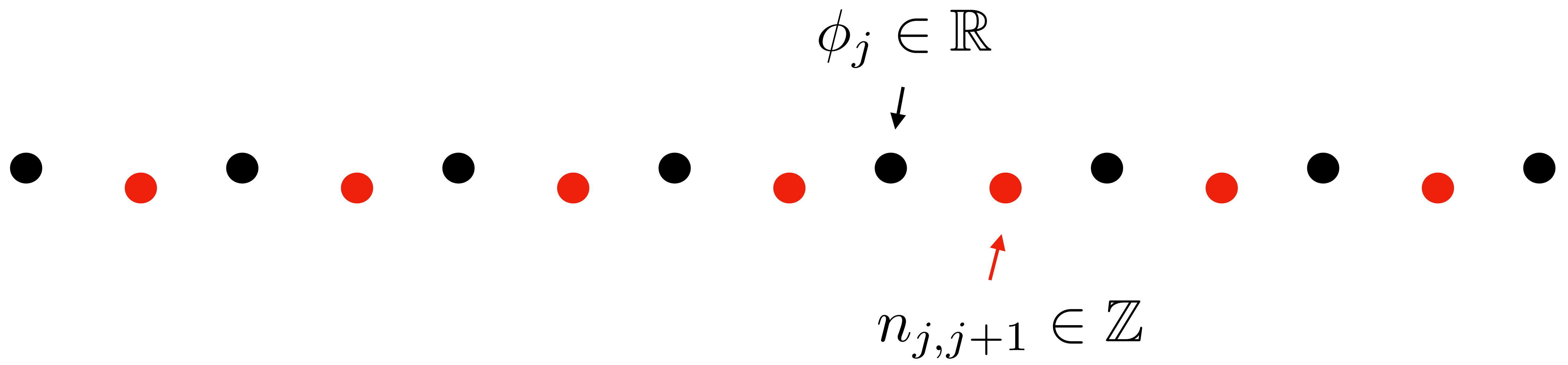
- Villain disentangler:

$$C = \prod_j \exp\left(i\lfloor \phi_j - \phi_{j+1} \rceil \chi_{j,j+1}\right) \qquad \left(e^{i\chi_{j,j+1}}|n_{j,j+1}\rangle = |n_{j,j+1}+1\rangle\right)$$

$$C|\{\phi_j, n_{j,j+1}\}\rangle = |\{\phi_j, n_{j,j+1} + \lfloor \phi_j - \phi_{j+1} \rceil\}\rangle \qquad \lceil x \rfloor = \text{integer closest to } x$$

- maps wavefunctions satisfying Villain condition to arbitrary wavefunctions of periodic (rotor) variables ϕ_j and $n_{j,j+1}$

1+1d: Disentangling the Villain Model



$$Q_V = C Q_V^{\mathrm{Vil}} C^{-1} = \frac{i}{2\pi} \sum_j \frac{d}{d\phi_j}$$

$$Q_A = C Q_A^{\mathrm{Vil}} C^{-1} = \sum_j \left(\phi_{j+1} - \phi_j - \lfloor \phi_{j+1} - \phi_j \rceil - n_{j,j+1}\right)$$

$$H = C H^{\mathrm{Vil}} C^{-1}$$

exactly solvable, equivalent to free compact boson (Luttinger liquid), commutes with Q_V and Q_A, which generate compact momentum and winding respectively

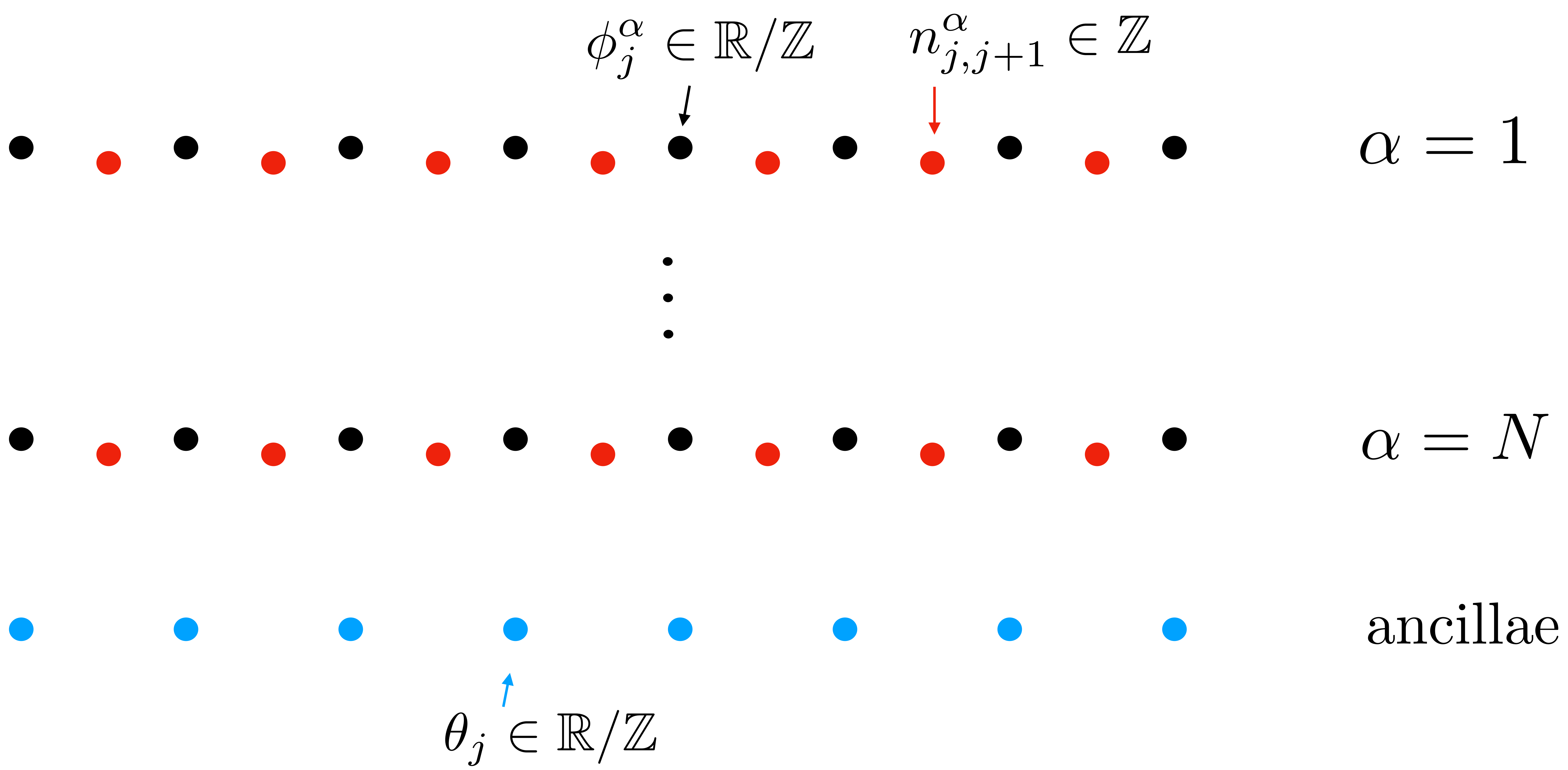
compact cochain notation:

$$Q_V = \frac{i}{2\pi} \int \frac{d}{d\phi}$$

$$Q_A = \int (d\phi - \lfloor d\phi \rceil - n)$$

see also DeMarco, Wen 2020, 2021

1+1d: Stacked Hilbert space



1+1d: Stacked Hilbert space

- anomaly-free $U(1) \subset (U(1)_V \times U(1)_A)^N$:

$$Q = \sum_{\alpha=1}^{N} (q_V^\alpha Q_V^\alpha + q_A^\alpha Q_A^\alpha) + \frac{i}{2\pi} \sum_j \frac{d}{d\theta_j}$$

ancillae carry charge 1

with $$\sum_{\alpha=1}^{N} \left(\left(\frac{q_V^\alpha + q_A^\alpha}{2} \right)^2 - \left(\frac{q_V^\alpha - q_A^\alpha}{2} \right)^2 \right) \sim \sum_{\alpha=1}^{L} q_V^\alpha q_A^\alpha = 0$$

anomaly-free condition

$$\text{e.g. } N = 2, q_V^1 = 3, q_A^1 = 3, q_V^2 = 9, q_A^2 = -1 \text{ for } 3-4-5-0 \text{ theory}$$

- goal: disentangle Q into an onsite operator

1+1d: Symmetry disentangler

- define $W = \Phi \cdot S$ with

$$S = \exp\left(\sum_{j,\alpha} q_V^\alpha \theta_j \frac{d}{d\phi_j^\alpha}\right)$$

$$\Phi = \exp\left(-i \sum_j \theta_j \left(\sum_\alpha q_A^\alpha \lfloor \phi_{j+1}^\alpha - \phi_j^\alpha \rceil - \lfloor \sum_\alpha q_A^\alpha \phi_{j+1}^\alpha \rceil + \lfloor \sum_\alpha q_A^\alpha \phi_j^\alpha \rceil \right)\right)$$

compact cochain notation:

$$S = \exp\left(\int \theta \sum_\alpha q_V^\alpha \frac{d}{d\phi^\alpha}\right)$$

$$\Phi = \exp\left(-i \int \theta \left(\sum_\alpha q_A^\alpha \lfloor d\phi^\alpha \rceil - d\lfloor \sum_\alpha q_A^\alpha \phi^\alpha \rceil \right)\right)$$

- **claim:** W is a finite depth circuit of local unitaries and

$$W^\dagger Q W = \frac{i}{2\pi} \sum_j \frac{d}{d\theta_j} - \sum_{j,\alpha} q_A^\alpha n_{j,j+1}^\alpha \equiv Q_{\text{onsite}}$$

$$= \frac{i}{2\pi} \int \left(\frac{d}{d\theta} - \sum_\alpha q_A^\alpha n^\alpha\right)$$

(see also Seifnashri, Shirley 2025 for disentangling anomaly-free discrete symmetries)

1+1d: Symmetry disentangler

- **Proof**: direct computation.

$$W^\dagger \exp(i\beta Q) W = \Phi' \cdot \exp(i\beta Q_{\text{onsite}})$$

where

$$\Phi' = \exp \int \theta \left(d\lfloor \sum_\alpha (q_A^\alpha \phi^\alpha + q_A^\alpha q_V^\alpha \beta) \rceil - d\lfloor \sum_\alpha q_A^\alpha \phi^\alpha \rceil \right) = 1$$

by anomaly cancellation $\left(\sum_\alpha q_A^\alpha q_V^\alpha = 0\right)$.

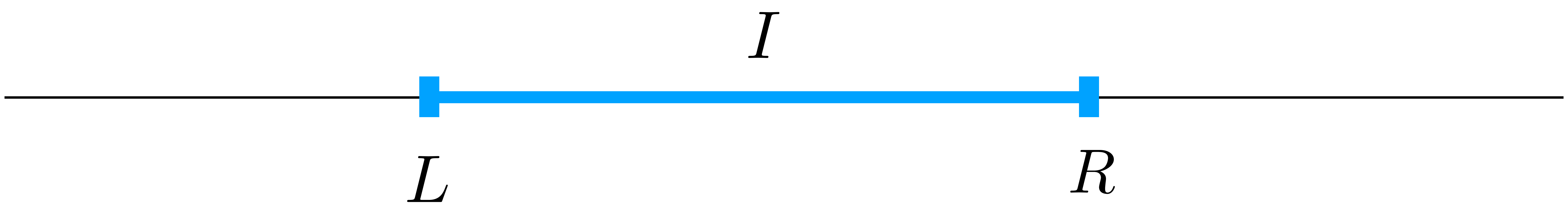
1+1d: Symmetry disentangler

- once a symmetry disentangler exists, symmetry can be made onsite

- once the symmetry is onsite, it can be consistently gauged, on the spatial lattice, by minimal coupling to lattice U(1) gauge field

- this gives a **non-perturbative** construction of a chiral gauge theory

1+1d: lattice diagnostic of anomaly



$$Q_A = \sum_j \left(\phi_{j+1} - \phi_j - \lfloor \phi_{j+1} - \phi_j \rceil - n_{j,j+1}\right)$$

$\big\downarrow$ truncate to I

$$Q_A^I = \sum_{j \in I} \left(\phi_{j+1} - \phi_j - \lfloor \phi_{j+1} - \phi_j \rceil - n_{j,j+1}\right)$$

$$\exp(2\pi i Q_A^I) = \exp\left(2\pi i \phi_R\right) \exp\left(-2\pi i \phi_L\right) \longleftarrow$$

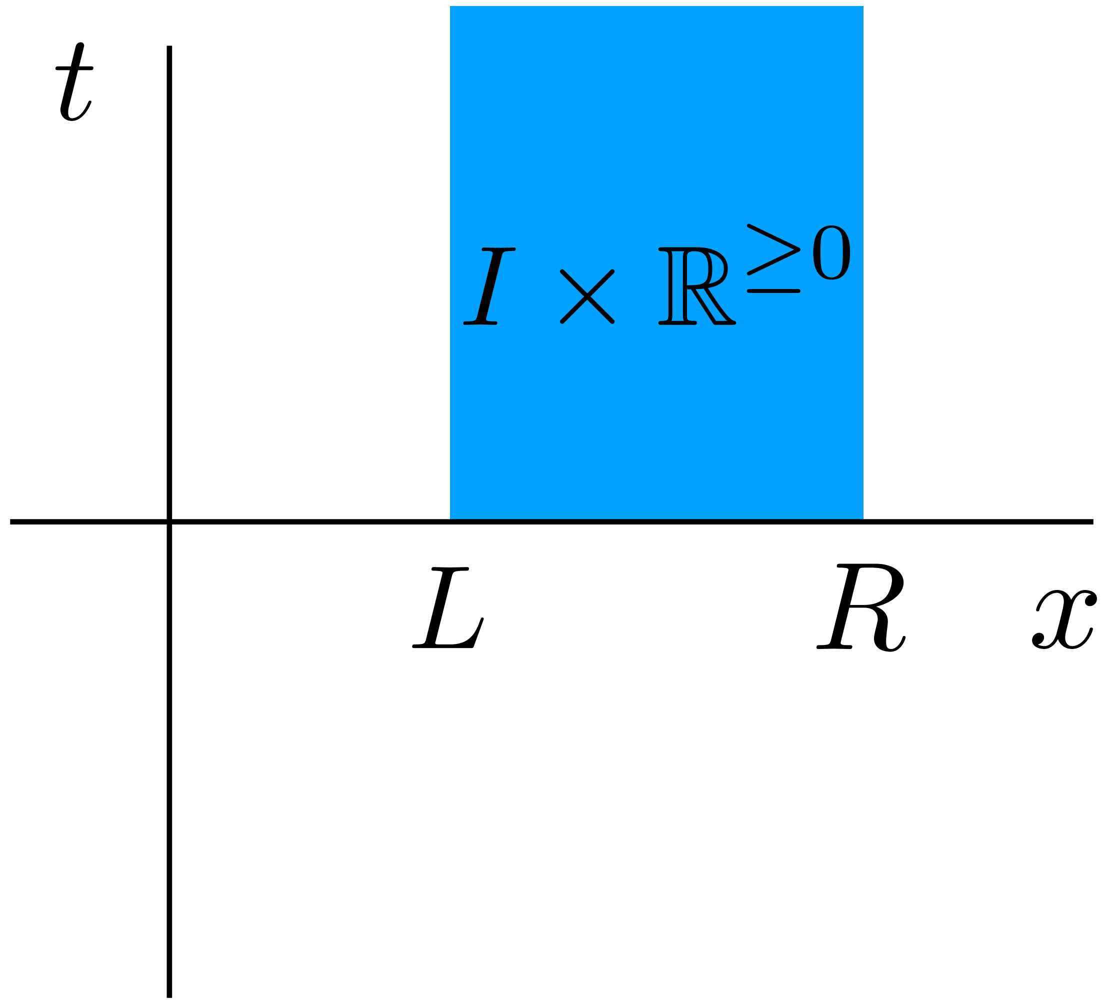
pumps $U(1)_V$ charge from one endpoint to the other

- can be justified by building 2+1d SPT with this edge anomaly and explicitly gauging $U(1)_V \times U(1)_A$, c.f. Else-Nayak and Levin-Gu

1+1d: lattice diagnostic of anomaly

- continuum intuition: imagine breaking $U(1)_A$ and gauging $U(1)_V$:

$$S \sim \int dt\, dx\, \theta\, F_V \qquad (F_V = dA_V)$$



then acting with $\exp(2\pi i Q_A^I)$ amounts to shifting

$$\theta \to \theta + 2\pi$$

so that

$$S \to S + 2\pi \int_{I \times \mathbb{R}^{\geq 0}} F_V = S + 2\pi \int_{\partial(I \times \mathbb{R}^{\geq 0})} A_V$$

charge pumped from L to R

3+1d (bosonic): Not-on-site symmetry action

$$Q_V = \frac{i}{2\pi} \int_{M^3} \frac{d}{d\phi}$$

$$Q_A = \int_{M^3} (d\phi - \lfloor d\phi \rceil) \cup d\lfloor d\phi \rceil$$

(need simplicial manifold with branching structure)

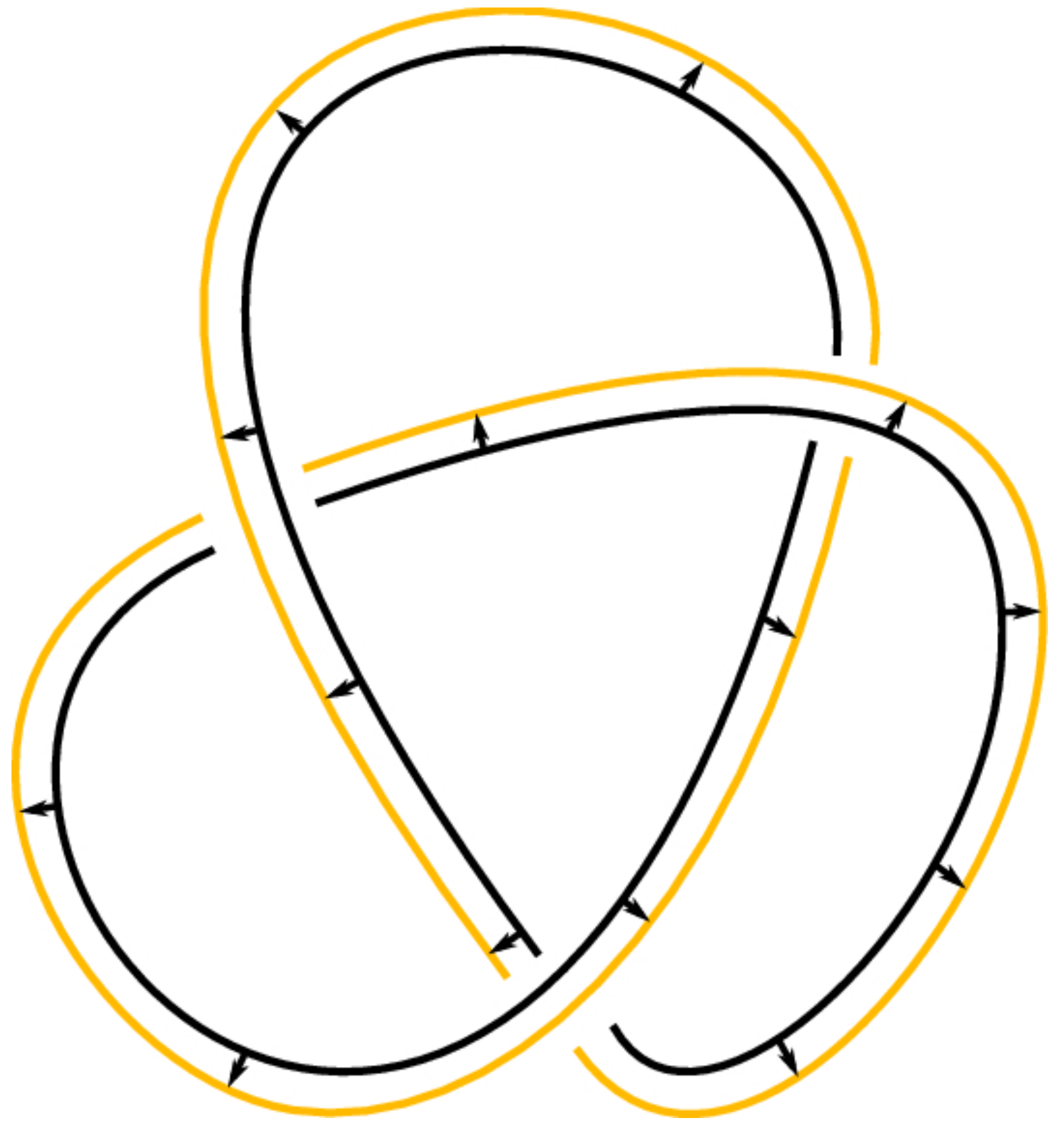
(Fidkowski, Xu, Zhang arXiv:2510.17969)

- on a manifold M^3 with no boundary:

$$\exp(i\theta Q_A) = \exp\left(-i\theta \int_{M^3} \lfloor d\phi \rceil \cup d\lfloor d\phi \rceil\right)$$

self-linking number of vortex configuration $d\lfloor d\phi \rceil$

Xu Senthil 2013



3+1d (bosonic): Not-on-site symmetry action

- on a manifold M^3 with boundary:

pumps 2+1d bosonic SPT of $U(1)_V$ to the boundary of M^3

$$U_A(2\pi) = \exp\left(2\pi i \int_{\partial M^3} \phi \cup d\lfloor d\phi \rceil\right)$$

DeMarco, Wen 2020, 2021

- continuum intuition: break $U(1)_A$ and gauge $U(1)_V$:

$$S \supset \int dt\, dx\, \theta\, F_V \wedge F_V$$

then acting with $\exp(2\pi i Q_A^{M^4})$, where $M^4 \subset \mathbb{R}^4$, amounts to shifting $\theta \to \theta + 2\pi$

$$S \to S + 2\pi \int_{M^4} F_V \wedge F_V = S + 2\pi \int_{\partial M^4} A_V \wedge dA_V \qquad (F_V = dA_V)$$

bosonic U(1) SPT

3+1d (bosonic): Not-on-site symmetry action

action of 4+1d anomaly in-flow SPT phase:

$$S \sim A_A \wedge dA_V \wedge dA_V \sim -\frac{1}{4\pi^2} dA_A \wedge A_V \wedge dA_V$$

inserting 2 pi
axial flux...

pulls in a 2+1d bosonic
SPT state with U(1)_V Hall
conductivity

compare 1+1 d case:

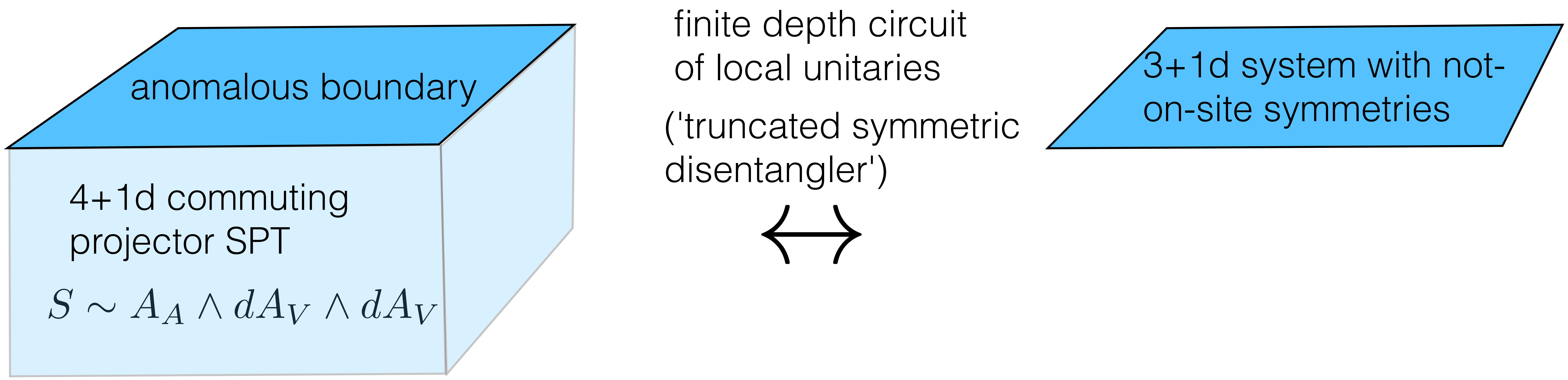
$$S \sim dA_A \wedge A_V$$

inserting 2 pi
axial flux...

pulls in a 0+1d bosonic
charge of U(1)_V

3+1 d bosonic lattice model with two U(1) symmetries

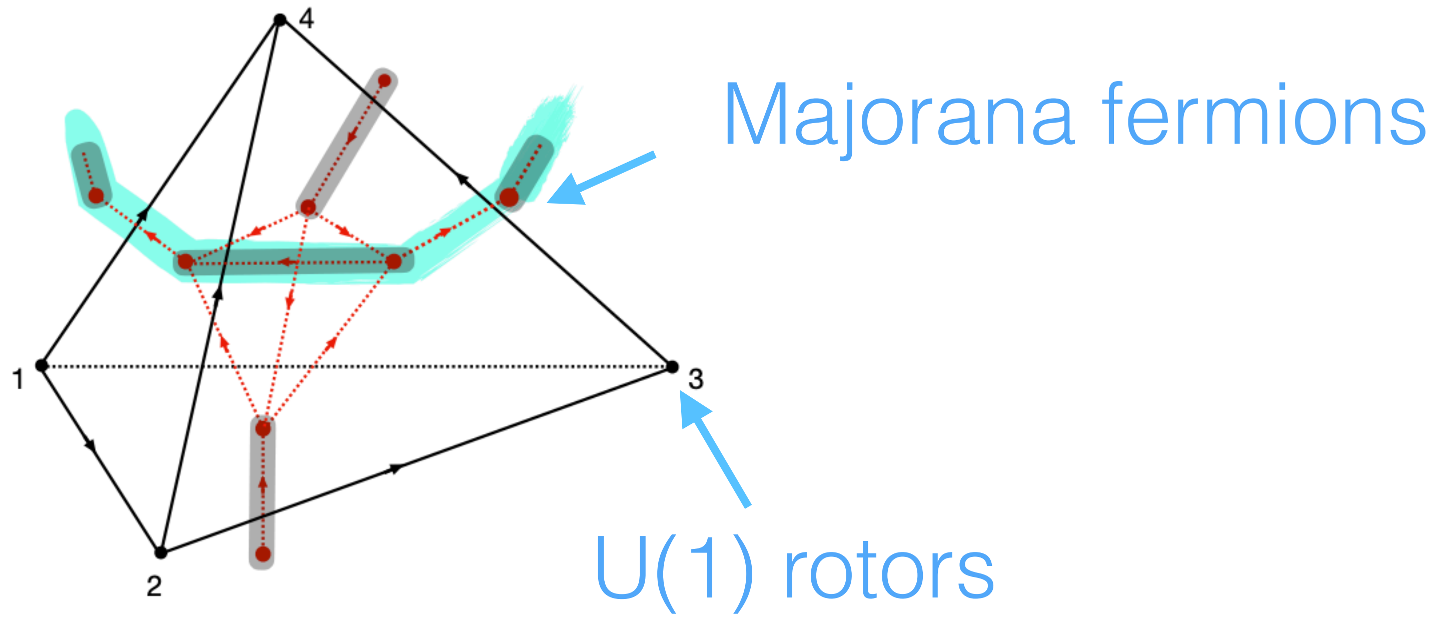
- can construct 4+1d **commuting projector** SPT phase whose 3+1d boundary can be modeled with our not-on-site symmetries Else, Nayak 2014

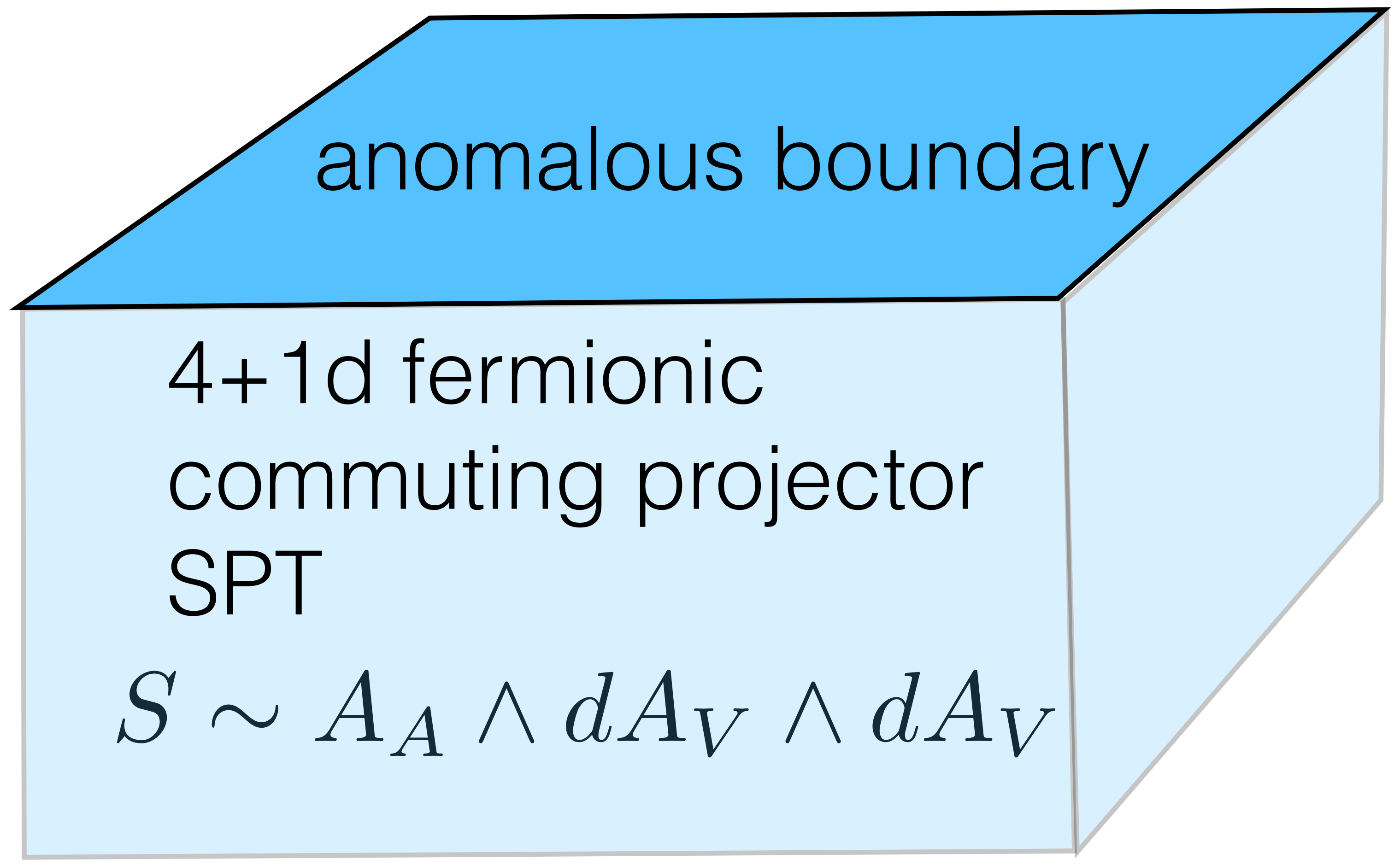


3+1 d **fermionic** lattice model with two U(1) symmetries

- decorate our system with **fermionic** degrees of freedom and modify the two U(1) generators so:

$$U_A(\pi) = (-1)^F$$

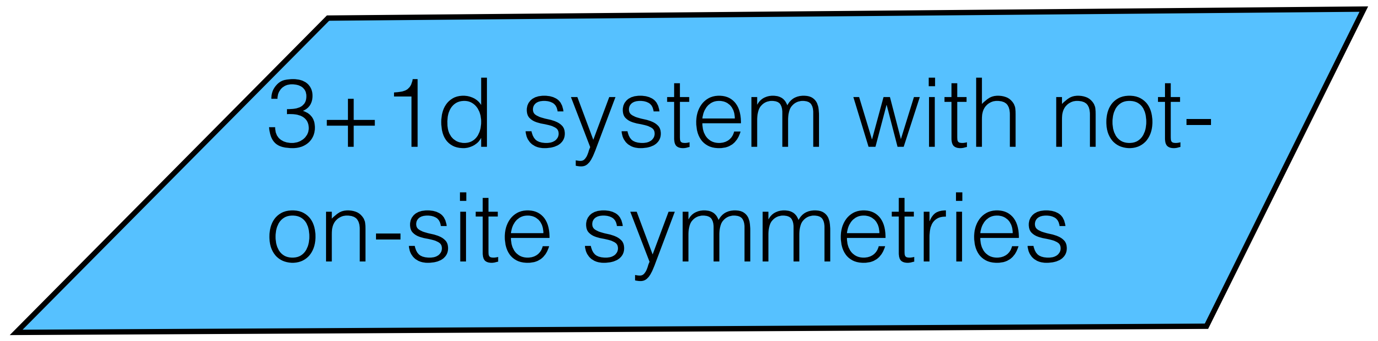




finite depth circuit
of local unitaries

('truncated symmetric
disentangler')

$\leftrightarrow$

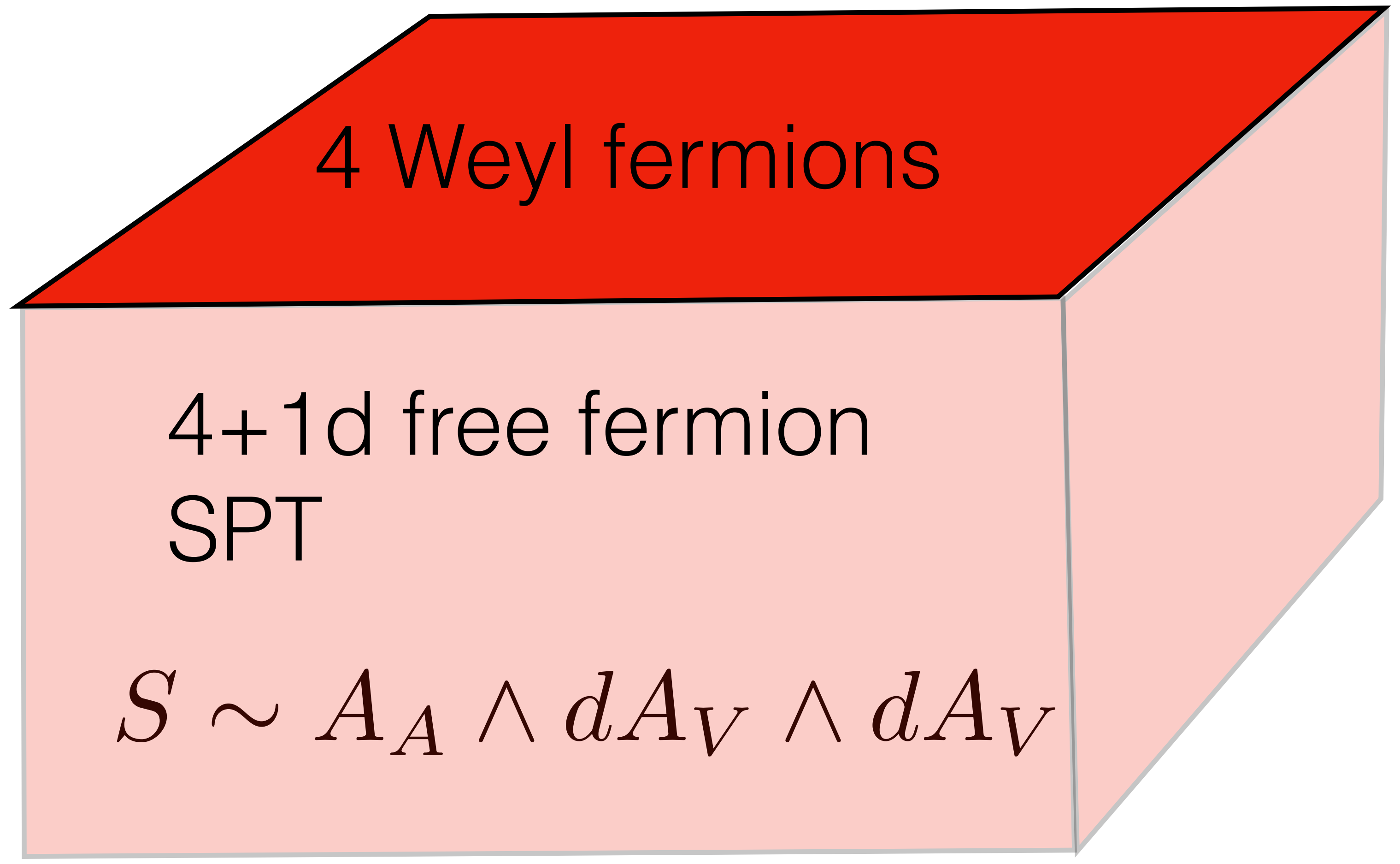


3+1 d fermionic field theory saturating the anomaly

- naive guess - Dirac fermion - has mixed U(1) gravitational anomaly for $U(1)_A$

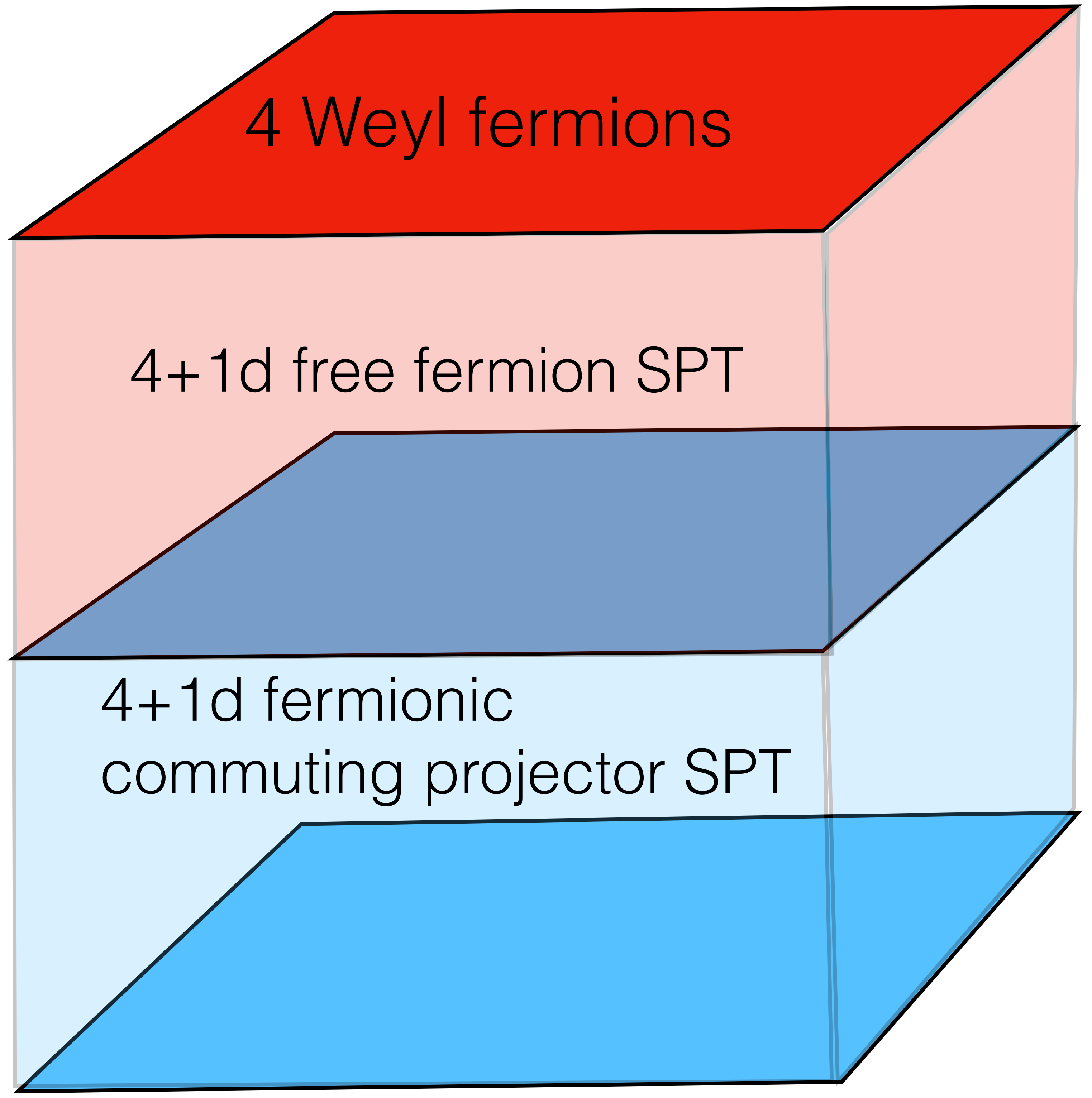
- instead: 4 left-handed Weyl fermions with charges:

	ψ_1	ψ_2	ψ_3	ψ_4
$U(1)_V$:	1	-1	0	0
$U(1)_A$:	1	1	-1	-1



Kaplan 1992

Put together the two SPTs



putative trivially gapped interface

can be modeled as 3+1d system with not-on-site actions of $U(1)_V$ and $U(1)_A$

- now make stacks of such systems, with

$$Q = \sum_\alpha (q_V^\alpha Q_V^\alpha + q_A^\alpha Q_A^\alpha)$$

being anomaly-free:

$$\sum_\alpha q_A^\alpha q_V^\alpha q_V^\alpha = 0$$

- can disentangle not-on-site action of Q, and hence gap out bottom boundary in exactly solvable zero correlation length setting

(an exactly solvable version of symmetric mass generation, c.f. Eichten Preskill 1986)

U(1) hypercharge in Standard Model

	ψ_1	ψ_2	ψ_3	ψ_4
$U(1)_V$:	1	-1	0	0
$U(1)_A$:	1	1	-1	-1

- Standard model (single generation, assuming sterile neutrino):

$$SU(3) \times SU(2) \times U(1)_Y : (3,2)^1 + (\bar{3},1)^{-4} + (\bar{3},1)^2 + (1,2)^{-3} + (1,1)^6$$

quark hypercharges (x 3):

lepton hypercharges:

$$(2,-4,1,1) = 3*(1,-1,0,0) + (-1)*(1,1,-1,-1)$$

$$(6,0,-3,-3) = 3*(1,-1,0,0) + 3*(1,1,-1,-1)$$

U(1) hypercharge in Standard Model

- Assuming the free fermion / commuting projector interface can be trivially gapped, this gives a well controlled way to realize the hypercharge sector of one generation of the Standard Model, with the U(1) hypercharge acting onsite

    - a heuristic for trivially gapping the interface: can break U(1)_V, which results in a state whose U(1)_A charge is the self-linking number of the U(1)_V vortices. This cancels between the commuting projector and free fermion SPTs, allowing U(1)_V to be restored through quantum disordering the U(1)_V order parameter

- Since the hypercharge U(1) is onsite, it can straightforwardly be gauged using the usual methods (minimal coupling on the spatial lattice)

Future directions

- incorporating chiral SU(2); Witten anomaly and mixed U(1) - SU(2) anomaly

- implications of rotors vs. finite dimensional Hilbert spaces

- numerical simulation?