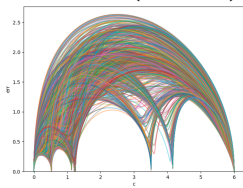


Entanglement Bootstrap and Symmetric Conformal Field Theories

John McGreevy (UCSD)

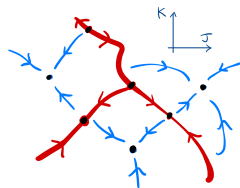
based on 2303.05444 with **Ting-Chun (David) Lin**
2509.04596 with **Xiang Li** and **Ting-Chun (David) Lin**
2606.12540 with **Rolando Ramirez Camasca, Xiang Li,**
Ting-Chun (David) Lin



Big goal: Find all fixed points of the renormalization group.

Phase of matter $\equiv$ basin of attraction of an RG fixed point.

QFT $\equiv$ RG flow between UV and IR fixed points.



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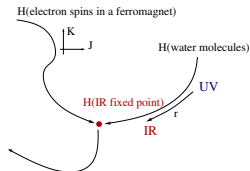
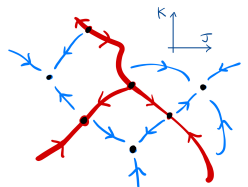
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Fixed points are rare and precious. This explains the experimental phenomenon of universality.



[MIT TSG]



e.g. ferromagnet near the Curie transition
(let $t \equiv \frac{T_c - T}{T_c}$)

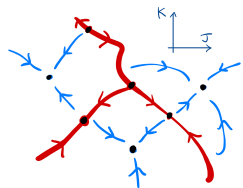
specific heat: $c_v \sim t^{-\alpha}$

magnetic susceptibility: $\sim t^{-\gamma}$

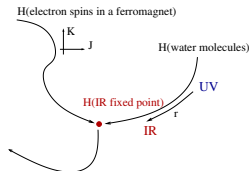
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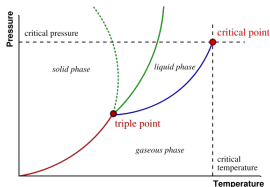
specific heat: $c_v \sim t^{-\alpha}$

compressibility: $\sim t^{-\gamma}$

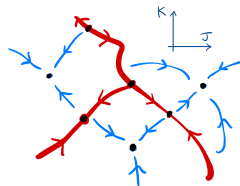
with the
same α, γ !



[MIT TSG]

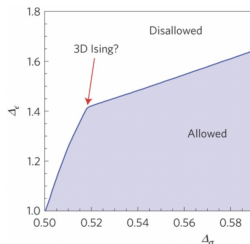


Big goal: Find all fixed points of the renormalization group.



Strategies so far have not been systematic:

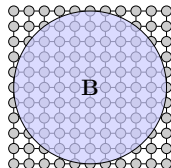
- do experiments and simulations.
- construct special examples from supersymmetry and string theory.
- find free fixed points, do perturbation theory.
- find fixed points that are close to violating simple unitarity bounds (conformal bootstrap).



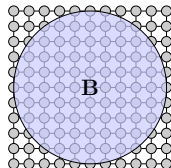
[fig: Poland Simmons-Duffin 2016]

Central Dogma of Entanglement Bootstrap: All the universal properties of a state of matter are encoded in a single representative density matrix on a ball.

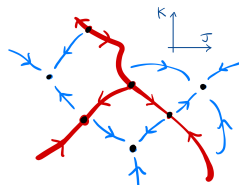
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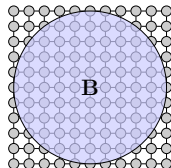
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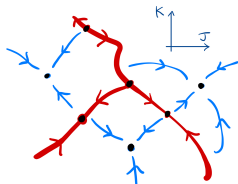
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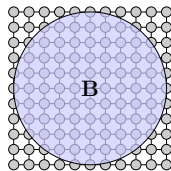


Why we care about locality:

While the full many-body wavefunction in the thermodynamic limit is an object of terror, the reduced density matrix of a small-enough ball is finitary.

Reasons to believe the Central Dogma

Central Dogma: All the universal properties of a state of matter are encoded in a single representative density matrix on a ball ρ_B .

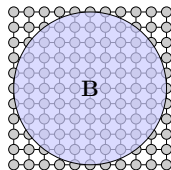


Some evidence:

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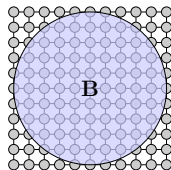
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- In a groundstate of a local Hamiltonian, the entanglement entropy satisfies an *area law*, $S_B \equiv -\text{tr}_B \rho_B \log \rho_B = |\partial B|/\epsilon^{d-1} + \text{smaller}$ (modulo some important and well-understood exceptions)

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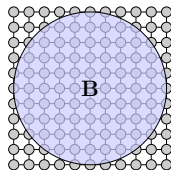
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- In a groundstate with liquid topological order, for $B \sim \text{disk}$, $S_B = \frac{|\partial B|}{\epsilon} - \gamma$, $\gamma = \text{topo. entanglement entropy (TEE)}$ extracts anyon data [Levin-Wen, Kitaev-Preskill 06, Grover-Turner-Vishwanath]

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[Levin-Wen, Kitaev-Preskill 06, Grover-Turner-Vishwanath]

- There is more info in ρ_B than entropies: $\rho_X = e^{-K_X}$
 $K_X \equiv -\log \rho_X$ *entanglement Hamiltonian*

$$J_\psi(A, B, C) \equiv \mathbf{i} \langle \psi | [K_{AB}, K_{BC}] | \psi \rangle = \frac{\pi c_-}{3}$$

chiral central charge, the net number of edge modes of a FQH state

[Kim-Shi-Kato-Albert 21, Zou-Shi-Sorce-Lim-Kim 22] also: [Kitaev-Preskill, Li-Haldane]

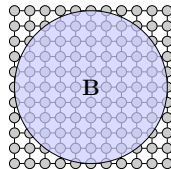


Local conditions

A consequence of the above Central Dogma is that there should be *local* conditions on a wavefunction that tell us which category of state it represents:

Is it topologically ordered? Is it a CFT groundstate?

Is it a fracton state?

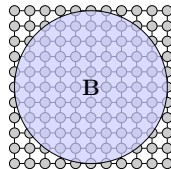


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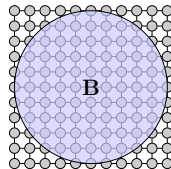
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1. impose the conditions as axioms and prove structural properties of the universal data.
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So the goal of Entanglement Bootstrap is to understand the structure of this universal info and learn to extract it from the local density matrix.

Because this universal info is usually encoded in a QFT, this can teach us about QFT.

So far

- ▶ Entanglement Bootstrap for liquid topological order

[2112.08398, 2301.07119 with **Bowen Shi** (UCSD, UQM postdoc $\rightarrow$ BIMSA faculty) and **Jin-Long Huang** (UCSD $\rightarrow$ Hong Kong City U), based on foundational work by Bowen Shi, Isaac Kim, Kohtaro Kato 1906.09376, 2008.11793]

- ▶ Entanglement Bootstrap for gapped domain walls

[Bowen Shi, Isaac Kim 2008.11793]

- ▶ Entanglement Bootstrap for 1+1d CFT

[2303.05444 with **Ting-Chun (David) Lin** (UCSD $\rightarrow$ Stanford), 2509.04596 with XL and TCL.]

- ▶ Entanglement Bootstrap for Mixed States

[Tai-Hsuan Yang, Bowen Shi, Jong-Yeon Lee, 2506.04221]

- ▶ Entanglement Bootstrap for 2+1d chiral topological states

2403.18410 2404.03725 2408.10306 2510.23720 with **Xiang Li** (UCSD $\rightarrow$ Oxford), BS, TCL, Isaac Kim

- ▶ Entanglement Bootstrap for CFT in any dimension

[2606.12540 with **Rolando Ramirez Camasca**, XL, TCL]

Entanglement Bootstrap for liquid topological order

[Topological order: a gapped phase of matter with local **excitations** not creatable by any local operator (anyons).]

Liquid: the set of anyons does not depend on geometry (no fractons).]

$\implies$ we expect a low-energy *topological quantum field theory* (TQFT).

Axioms of EB for liquid topological order

Quantum marginal problem (Hard):

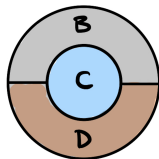
given ρ_{AB} and ρ_{BC} with $\rho_B = \text{tr}_A \rho_{AB} = \text{tr}_C \rho_{BC}$, can we merge them into ρ_{ABC} with marginals ρ_{AB}, ρ_{BC} ?

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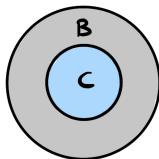
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- **A1:** $0 = \Delta(B, C, D)_\rho \equiv (S_{BC} + S_{CD} - S_B - S_D)_\rho$
for all regions of the form:



Merging is possible.

- **A0:** $0 = \Delta(B, C)_\rho \equiv (S_{BC} + S_C - S_B)_\rho$
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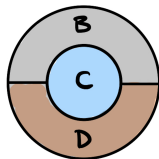
Merging is unique.

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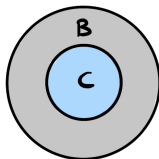
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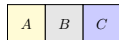
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A1 $\implies I(A : C|B)_\sigma \equiv S_{AB} + S_{BC} - S_B - S_{ABC} = 0$
(quantum Markov chain).



Entanglement Bootstrap for liquid topological states

[Shi-Kato-Kim 1906.09376, Huang-JM-Bowen Shi 2112.08398, 2301.07119]

- From **A0**, **A1**, much of the structure of 2+1d TQFT can be derived.

Anyon types, quantum dimensions, fusion rules, a non-degenerate S -matrix, generalized symmetry operators...

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Anyon types, quantum dimensions, fusion rules, a non-degenerate S -matrix, generalized symmetry operators...

- The same axioms work in any dimension.

In $D > 2 + 1$, the correct determining structure is not entirely clear yet.

- A0** and **A1** hold at *fixed points* of the RG.

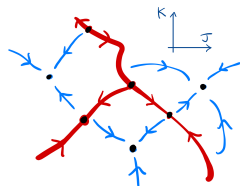
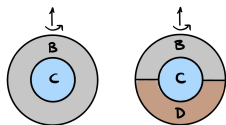
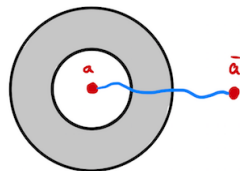


Frontier: robust EB.

Strategy 1: Show that (small-enough) violation of **A0** and **A1** decays under coarse-graining.

Strategy 2: replace the axioms with a version that is true everywhere in the phase [Kim-Kitaev-Moses-Ranard,

Kim-Shi, in progress]



Instructions for doing Entanglement Bootstrap

0. Pick a category of states of matter (Liquid TO, CFT, fractons....).
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Ideally, the violation of these conditions is an RG monotone.
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b) Find states that satisfy the conditions and use them to enumerate universality classes.



[adapted from Allie Brosh]

Entanglement Bootstrap for 1+1d CFT

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The behavior of the EE of an interval A is $S_A = \frac{c}{3} \log |A|/\epsilon$

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Not a sufficient condition for CFT.

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We know the entanglement Hamiltonian of a single interval for 1+1d CFT with groundstate $|\psi\rangle$ [Bisognano-Wichmann 75, Casini-Huerta-Myers 11, Cardy-Tonni 16]

$$K_A \equiv -\log \rho_A = \int_A \beta_A(x) \mathbf{h}(x) + S_A \mathbb{1}$$

where $\mathbf{h}(x)$ is the Hamiltonian density.

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$$\eta = \frac{(x_2-x_1)(x_4-x_3)}{(x_3-x_1)(x_4-x_2)}$$

$$h(\eta) = -\eta \log \eta - (1-\eta) \log(1-\eta)$$

Fact: $\beta_{AB}(x) + \beta_{BC}(x) - \eta(\beta_A(x) + \beta_C(x)) - (1-\eta)(\beta_B(x) + \beta_{ABC}(x)) = 0, \forall x.$

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$$K_\Delta \equiv K_{AB} + K_{BC} - \eta(K_A + K_C) - (1-\eta)(K_B + K_{ABC}) = \frac{c}{3} h(\eta) \mathbb{1}.$$

$$K_X \equiv K_X \otimes \mathbb{1}_{\bar{X}}$$

Entanglement Bootstrap for 1+1d CFT

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So far: continuum formulae. To compare with quantum critical lattice models, divide up the 1d space into equal-size intervals; regard each as a ‘site’ (with infinite-dimensional $\mathcal{H}$).

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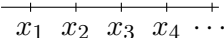
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A horizontal line with five tick marks. Below the first four tick marks are the labels x_1 , x_2 , x_3 , and x_4 . Below the fifth tick mark is an ellipsis $\cdots$.

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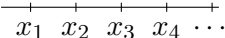
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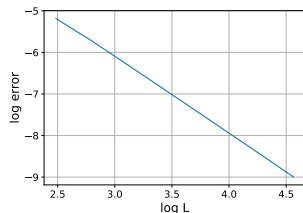
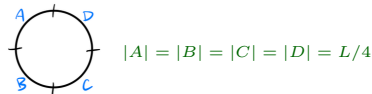
$$\text{error} \equiv \sqrt{\langle \psi | K_\Delta^2 | \psi \rangle - \langle \psi | K_\Delta | \psi \rangle^2}$$

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$$K_\Delta |\psi\rangle = \frac{c}{3} h(\eta) |\psi\rangle \quad (\text{VFPE})$$

is a more robust statement, but still a very strong constraint (N equations on N components).

This is a plot of the variance of K_Δ as a function of system size for the critical Ising model. $\rightarrow$



Entanglement Bootstrap for 1+1d CFT

Claim: all universal data encoded in local density matrix of a ground state.

But what is the full set of universal data?

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Reconstruct the **groundstate** (merging): solve the FPE for K_{ABC} :

$$\tilde{K}_{ABC} = (\tilde{K}_{AB} - \tilde{K}_B + \tilde{K}_{BC}) - \frac{\eta}{1 - \eta} \left(\tilde{K}_{AB} + \tilde{K}_{BC} - \tilde{K}_A - \tilde{K}_C \right), \quad \tilde{K} \equiv K - \langle K \rangle.$$

$\implies$ We can reconstruct ρ_{ABC} given ρ_{AB} and ρ_{BC} .

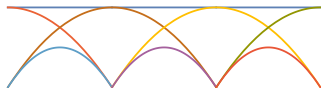
Reduces to Markov/Petz reconstruction when $\eta = 0$.

What about H ?

Entanglement Bootstrap for 1+1d CFT [Ting-Chun David Lin, JM 2303.05444]

The density matrix on ABC can be used to reconstruct a *local* lattice **Hamiltonian** for the CFT.

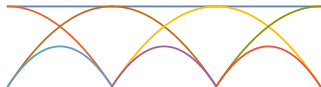
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$H_{\text{rec}} = \sum_i \hat{\Delta}(i-1, i, i+1, i+2) \geq 0$ by
operator weak monotonicity [Ting-Chun Lin,
Kim, Hsieh 22]

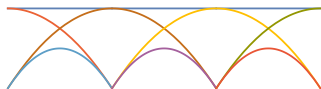
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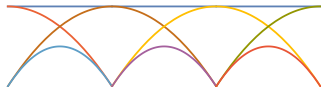
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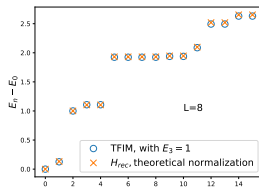
It works even when the local Hilbert space is small:

$$H_{\text{rec}} = \sum_{i=-\infty}^{\infty} (K_{[i,i+2]} - K_{[i,i+1]}) \stackrel{\text{CFT}}{=} \int dx \mathbf{h}(x)$$

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[e.g., Ising chain on 8 sites]

A systematic search for CFTs in very small spaces

Wonderful surprise: VFPE is the condition to extremize the function

$$\boxed{c_{\Delta}(|\psi\rangle)h(\eta)/3 \equiv S_{AB} + S_{BC} - \eta(S_A + S_C) - (1 - \eta)(S_B + S_{ABC})} \geq 0 .$$

Key: $dS = \langle d\psi|K|\psi\rangle + \langle\psi|K|d\psi\rangle$

The value of this function at the critical point is the central charge.

A systematic search for CFTs in very small spaces

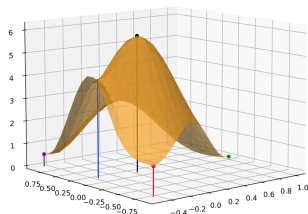
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A systematic search for CFTs in very small spaces

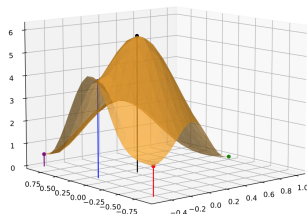
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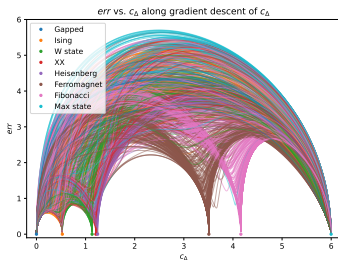
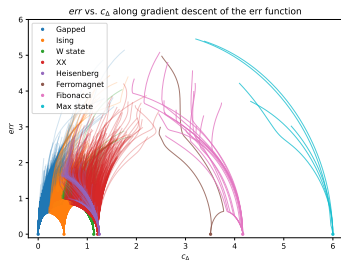


Moreover: it is numerically effective even for tiny systems!
In the table at right A, B, C (and their complement on S^1) is each a single, measly qubit:

c_{Δ}	Description	Explicit form up to on-site unitary
0	cat states	$a 0000\rangle + b 1111\rangle$
0.526	Ising CFT	
1.132	W state	$\frac{1}{2}(0001\rangle + 0010\rangle + 0100\rangle + 1000\rangle) - \frac{1}{2}(0101\rangle + 1010\rangle) +$
1.211	XX model	$\frac{1}{2\sqrt{2}}(0011\rangle + 0110\rangle + 1100\rangle + 1001\rangle) - \frac{1}{\sqrt{3}}(0101\rangle + 1010\rangle) +$
1.245	Heisenberg	$\frac{1}{2\sqrt{3}}(0011\rangle + 0110\rangle + 1100\rangle + 1001\rangle) + \frac{1}{\sqrt{3}} 0000\rangle +$
3.510	ferromagnet	$\frac{1}{\sqrt{6}}(0111\rangle + 1011\rangle + 1101\rangle + 1110\rangle) + \frac{1}{\sqrt{5}}(0000\rangle + \sqrt{\frac{\sqrt{5}-1}{2}}(0101\rangle + 1010\rangle))$
4.165	Fibonacci chain	
6.000	maximum point	$\frac{1}{2}(0000\rangle + 1111\rangle + 0101\rangle + 1010\rangle)$

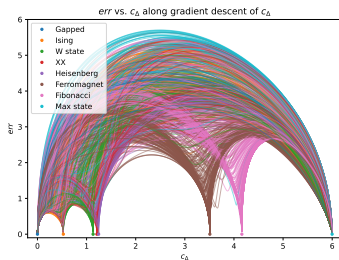
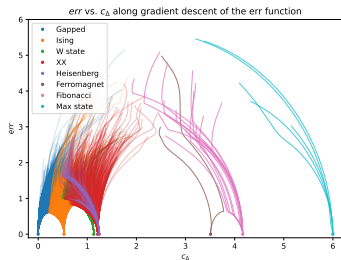
A systematic search for CFTs in very small spaces

4 qubits:

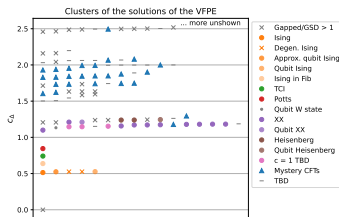
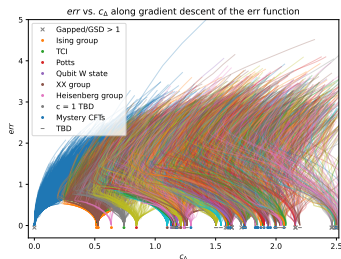


A systematic search for CFTs in very small spaces

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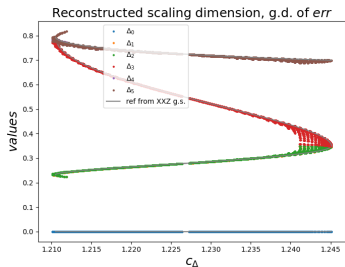
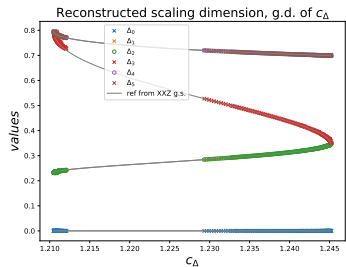
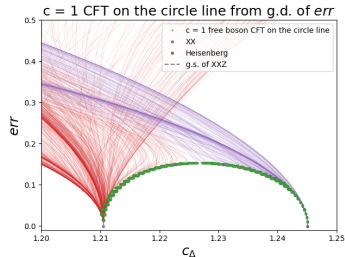
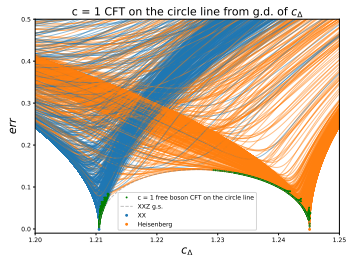


4 qutrits:



Consistent with c - d conjecture [Latorre-Sierra, 2403.17242]: in a nearest-neighbor lattice model with local Hilbert space $\mathbb{C}^d$, we'll only find $c \geq d - 1$.

Approximate solutions of VFPE



Entanglement Bootstrap for 1+1d CFT with symmetry

Imposing symmetries, we can find CFTs with larger c .

EB for CFT with symmetry

[in progress]

Setup: Pick G , take $\mathcal{H} = R^{\otimes 4} \equiv A \otimes B \otimes C \otimes D$

EB for CFT with symmetry

[in progress]

Setup: Pick G , take $\mathcal{H} = R^{\otimes 4} \equiv A \otimes B \otimes C \otimes D$

If R is multiplicity-free, G -invariant states are labelled by a in

$$R \otimes R = \oplus_a R_a$$

$$= \sum_b F_{ab}^{RRRR}$$

Translation-invariant states are eigenvectors of F_{ab}^{RRRR} with eigenvalue 1.

Let's search for solutions of the VFPE among translation-invariant G -singlets.

EB for CFT with symmetry

[in progress]

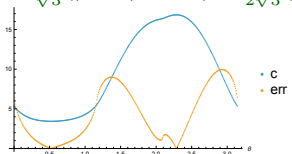
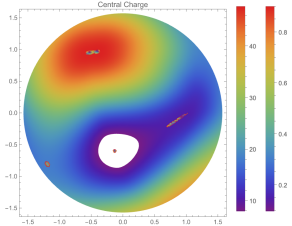
Search for solutions of VFPE among translation-invariant G -singlets:

Symmetry G	Known CFTs	Result
$SU(2), R = \mathbf{2}$	$c((SU(2))_1) = 1$	$-\frac{1}{\sqrt{3}}(0101\rangle + \dots) + \frac{1}{2\sqrt{3}}(\dots)$

EB for CFT with symmetry

[in progress]

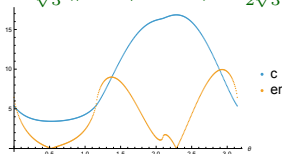
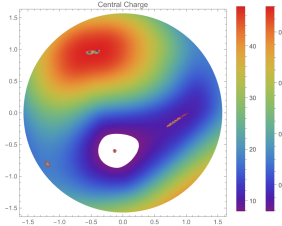
Search for solutions of VFPE among translation-invariant G-singlets:

Symmetry G	Known CFTs	Result
$SU(2), R = 2$ $G_2, R = 7$	$c((SU(2))_1) = 1$ $c((G_2)_1) = 2.8$	$-\frac{1}{\sqrt{3}}(0101\rangle + \dots) + \frac{1}{2\sqrt{3}}(\dots)$  The plot shows two curves: a blue solid line for 'c' and an orange solid line for 'err'. The x-axis is labeled 'theta' and ranges from 0 to 3.5. The y-axis ranges from 0 to 15. The blue curve starts at ~4.5, dips to ~3.5 at theta=0.5, peaks at ~16 at theta=2.5, and ends at ~5.5. The orange curve starts at ~4.5, dips to ~0 at theta=0.5, peaks at ~9 at theta=1.5, dips to ~0 at theta=2.5, and ends at ~10. A legend on the right shows a blue dot for 'c' and an orange dot for 'err'.
$E_8, R = 248$	$c((E_8)_1) = 8$	 The heatmap shows the Central Charge in a circular domain. The x and y axes range from -1.5 to 1.5. A color bar on the right indicates values from 10 to 40. The plot shows a complex distribution of values, with a prominent red region (high values) in the upper left and a white region (low values) in the lower right.

EB for CFT with symmetry

[in progress]

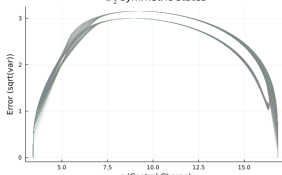
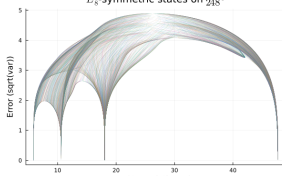
Search for solutions of VFPE among translation-invariant G-singlets:

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$E_8, R = 248$	$c((E_8)_1) = 8$	 The heatmap shows the Central Charge in the complex plane for the E_8 singlet. The color scale ranges from 10 (dark blue) to 40 (red). A white region indicates a zero of the central charge.
Fib, $R = \tau$	$c(\text{tri-Ising}) = 0.7$	$c_\Delta = 0.8512$ (unique)
Haagerup, $R = \rho$	$c \stackrel{?}{=} 2$	$c_\Delta = 2.175$ (unique)
Monster, $R = 196833$	$c(\text{Monster}) = 24$	$c_\Delta = 29.897$

EB for CFT with symmetry

[in progress]

Search for solutions of VFPE among translation-invariant G-singlets:

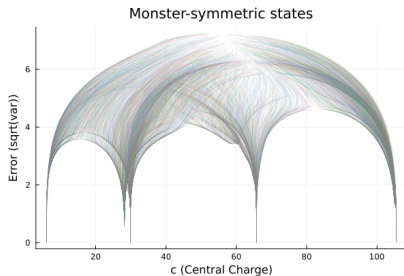
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$E_8, R = 248$	$c((E_8)_1) = 8$	E_8-symmetric states on 248^4 
Fib, $R = \tau$	$c(\text{tri-Ising}) = 0.7$	$c_\Delta = 0.8512$ (unique)
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EB for CFT with symmetry

[in progress]

- $G = \text{Monster}$, $R = \mathbf{196883}$

Compare: $c(\text{Monster CFT}) = 24$

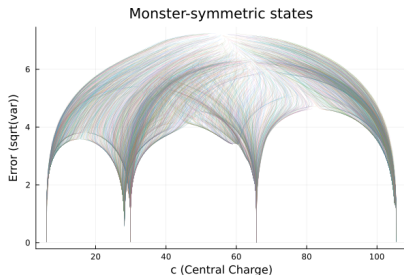


EB for CFT with symmetry

[in progress]

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Compare: $c(\text{Monster CFT}) = 24$



Analytic approximate solution: Suppose $R = R^*$. If $R \otimes R = R \oplus \dots$, a unique invariant 3- R tensor:

$$\begin{array}{c}
 \begin{array}{c}
 A \quad B \\
 \diagdown \quad \diagup \\
 \text{---} \text{---} \text{---} \\
 \diagup \quad \diagdown \\
 D \quad C
 \end{array}
 \end{array}
 = \sum_a \left(F_{Ra}^{RRRR} \right)^2 d_a^{-1/2}
 \begin{array}{c}
 \begin{array}{c}
 A \quad B \\
 \diagdown \quad \diagup \\
 \text{---} \text{---} \text{---} \\
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 \end{array}$$

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Current bottleneck:

$6j$ symbols for *e.g.* the Monster do not appear in the literature!

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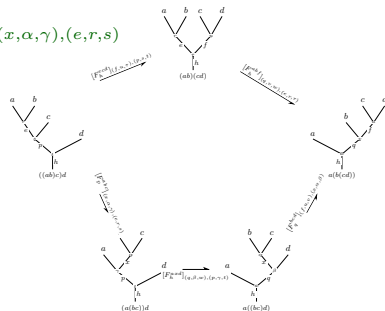
Following [Searle, 1988 (for Lie groups)], we can bootstrap their values:

- F is unitary: $\sum_{f,t,u} \overline{[F_d^{abc}]_{(f,t,u),(e,r,s)}} [F_d^{abc}]_{(f,t,u),(e',r',s')} = \delta_{ee'} \delta_{rr'} \delta_{ss'}$.

- F satisfies the pentagon equation

$$\sum_{x,\alpha,\beta,\gamma} [F_q^{bcd}]_{(f,u,v),(x,\alpha,\beta)} [F_h^{axd}]_{(q,\beta,w),(p,\gamma,t)} [F_p^{abc}]_{(x,\alpha,\gamma),(e,r,s)}$$

$$= \sum_{\tau} [F_h^{abf}]_{(q,v,w),(e,r,\tau)} [F_h^{ecd}]_{(f,u,\tau),(p,s,t)}$$

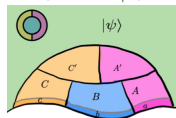


Theorem: the vector fixed point equation

$$\left(\eta \hat{\Delta}(AA', B, CC') + (1 - \eta) \hat{I}(A : C|B) \right) |\psi\rangle \propto |\psi\rangle$$

$\Leftrightarrow \delta c(\psi) = 0$ for all norm-preserving state variations.

$$1 = e^{-6\Delta/c} + e^{-6I/c}$$



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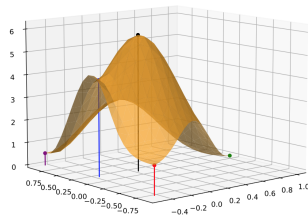
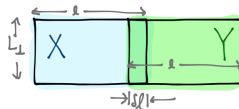
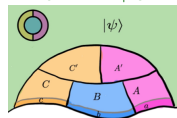
In the case of a 1+1d relativistic QFT groundstate, for a certain choice of intervals, this $c(\psi)$ is closely related to the entropic c -function of Casini and Huerta [hep-th/0405111]:

$$c(\psi) \approx 6 \frac{\Delta}{\ln 1/\eta} = 6 \frac{2S(\ell) - 2S(\ell - \delta\ell)}{2\delta\ell} = 6\ell \partial_\ell S(\ell) \equiv c_{\text{CH}}.$$

The above theorem shows that $c(\psi)$ is stationary, not just under variation of couplings, but under arbitrary variations of the state.

Q: can we relate c'' to the spectrum of operator dimensions, and c''' to the OPE coefficients?

$$1 = e^{-6\Delta/c} + e^{-6I/c}$$

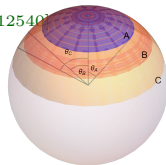


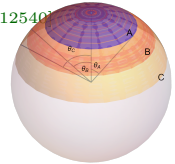
EB for CFT in any dimension

Let $A_i(\Omega), i = 1..k$ be concentric disks centered at Ω ,

$$D_{\text{rec}} \equiv \int_{S^d} d^d\Omega \sum_i \lambda_i K_{A_i}(\Omega) \ .$$

Choose λ_i to cancel the UV stuff.





EB for CFT in any dimension

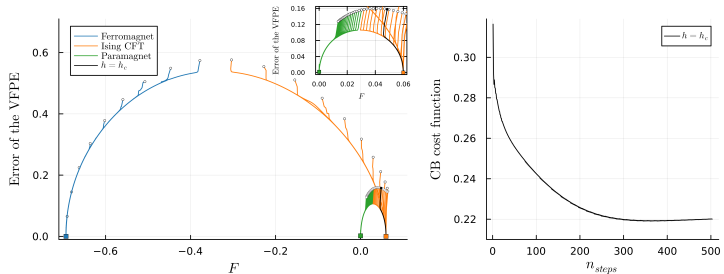
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Choose λ_i to cancel the UV stuff.

- Starting from a given, human-chosen model with a quantum critical point, we can do gradient descent on $\sigma(D_{\text{rec}})$ to move closer to the ideal CFT at fixed system size.

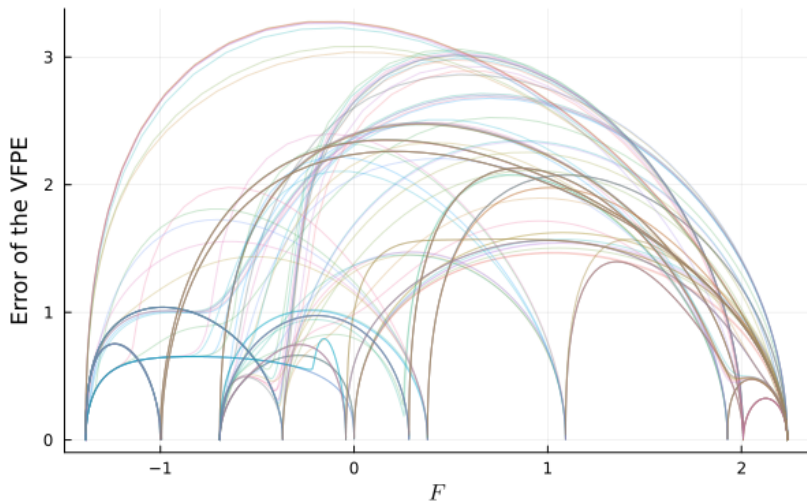
Gradient descent of the error function in the Ising model



- Search is doable, maybe we can find new CFTs.

Preliminary search results

CFT search: F-gradient flow trajectories



4 orbitals, with spin $\frac{1}{2}$, at half-filling.

Final words.

Central Dogma of Entanglement Bootstrap: all the universal data about a state of matter is encoded in a local region of a single representative wavefunction.

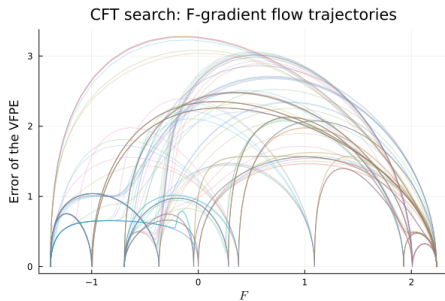
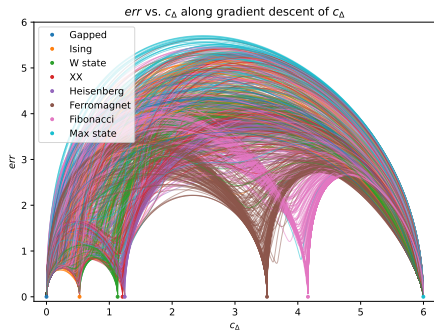
So far: liquid bulk topological order, gapped interfaces between topological orders, 1+1d CFT, 2+1d chiral states, 2+1d CFT.

Important open problem: show that the violations of the local conditions (**A0** & **A1**, VFPE) are RG monotones.

Future: Non-relativistic fixed points? Non-unitary CFT? Fractons?
Thermal states? Non-equilibrium steady states? String theory?

The end.

Thanks for listening.



Entanglement Bootstrap for 2+1d states with gapless boundary

The following statements also apply to ordinary 1+1d CFTs.

Entanglement Bootstrap for 2+1d chiral states

[Isaac Kim, Xiang Li, Ting-Chun David Lin, JM, Bowen Shi, 2403.18410]

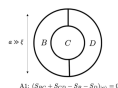
Next challenge: gapped states in 2+1d with gapless boundaries. *e.g.* $c_- \neq 0$.

Consider such a chiral state on a cylinder. By dimensional reduction, this is a (non-chiral) 1+1d CFT:

$$\left(\eta \hat{\Delta}(A, B, C) + (1 - \eta) \hat{I}(A : C|B) \right) |\psi\rangle = \frac{c}{3} h(\eta) |\psi\rangle$$

$$\hat{\Delta}(A, B, C) \equiv K_{AB} + K_{BC} - K_A - K_C$$

$$\hat{I}(A : C|B) \equiv K_{AB} + K_{BC} - K_B - K_{ABC}.$$

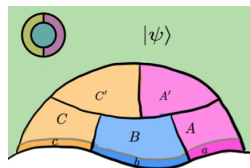
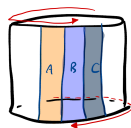


Bulk A1 plus some weak assumptions

$\Rightarrow$ fixed point equation on a single

boundary:

$$\left(\eta \hat{\Delta}(AA', B, CC') + (1 - \eta) \hat{I}(A : C|B) \right) |\psi\rangle \propto |\psi\rangle.$$



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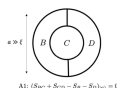
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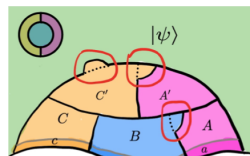
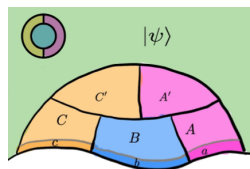
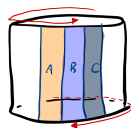
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- This is a combination from which the UV cutoff cancels.
- Bulk A1 $\implies$ independent of deformations in bulk.



Boundary conformal geometry from the state

A useful perspective:

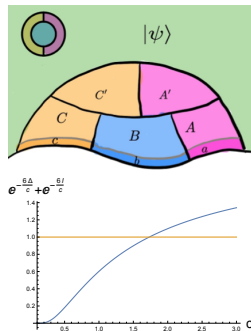
- given any three contiguous bdry intervals, compute $\Delta(AA', B, CC') \equiv S_{AA'B} + S_{BCC'} - S_{AA'} - S_{CC'}$ and $I(A : C|B) \equiv S_{AB} + S_{BC} - S_B - S_{ABC}$.
- Determine $c = c(\psi)$ as the solution to

$$1 = e^{-6\Delta/c} + e^{-6I/c}.$$

- Determine $\eta = \eta(\psi)$ by

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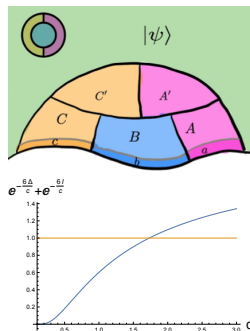
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Contexts: (1) No bulk, 1+1d CFT. Then this follows from $S(\ell) = \frac{c}{3} \log \frac{\ell}{a}$. Then η is the ordinary geometric cross-ratio, $\Delta = -6c \log \eta$, $I = -6c \log(1 - \eta)$.

(2) Gapped boundary: $c = 0$. In this case, the fixed-point equation reduces to boundary A1 of [Kim-Shi, 2008.11793]



(3) CFT on the boundary of a 2+1d chiral state.

(4) Rough edge of chiral state.

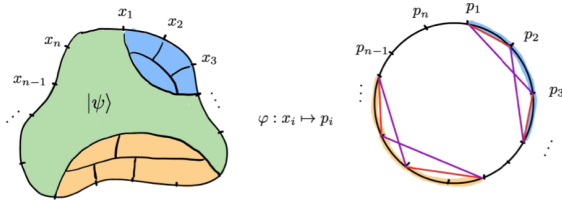
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$$(\Delta, I) \rightarrow (c, \eta) : 1 = e^{-6\Delta/c} + e^{-6I/c}, \quad \frac{\Delta}{I} = \frac{\ln \eta}{\ln(1 - \eta)}.$$

Theorem: The η s computed in this way from a fixed point state (with a weak non-degeneracy condition) are the cross-ratios of intervals on a circle, and c is the same for any choice of regions.

i.e. $\exists$ a map $\varphi : \partial\Sigma \rightarrow S^1$ with $\eta(x_i, x_j, x_k, x_l) = \frac{\overline{p_i p_j p_k p_l}}{\overline{p_i p_k p_j p_l}}$.



Idea of proof: Associativity of decomposition of K_{abcd} .

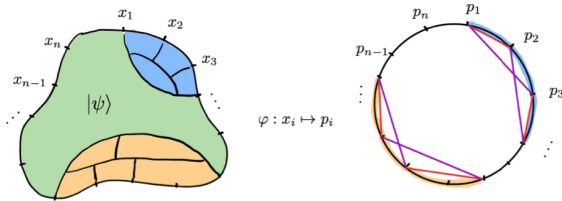
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Idea of proof: Associativity of decomposition of K_{abcd} .

Furthermore, certain modular flows produce a new state that still satisfies the VFPE. This gives a deformation of φ : an element of $\text{Diff}(S^1)$. In this way we can build a representation of the Virasoro algebra.

Extract Virasoro generators

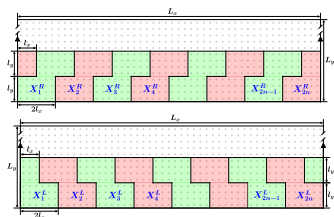
[Isaac Kim, Xiang Li, Ting-Chun David Lin, JM, Bowen Shi, 2404.03725]

Based on the expectation $\mathcal{K}_{[\theta_1, \theta_2]} = \int_0^{2\pi} d\theta \beta^{[\theta_1, \theta_2]}(\theta) \mathcal{T}(\theta) + \alpha_{[\theta_1, \theta_2]} \mathbb{I}$,
we can extract the chiral Virasoro generators:

$$\tilde{L}_n = \mathbb{L}_n^{(e)} + i\mathbb{L}_n^{(o)}, \quad \tilde{L}_{-n} = \mathbb{L}_n^{(e)} - i\mathbb{L}_n^{(o)}$$

$$\text{with } \mathbb{L}_n^{(o)} = \frac{A_n}{2} \sum_{j=1}^{2n} (-1)^{j-1} \left(K_{X_j^R} + K_{X_j^L} \right)$$

$$\text{and } \mathbb{L}_n^{(e)} = \frac{A_n}{2} \sum_{j=1}^{2n} (-1)^{j-1} \left(K_{Y_j^R} + K_{Y_j^L} \right),$$



It works: (data for $p + ip$ groundstate)

n	$\langle \Psi [\tilde{L}_n, \tilde{L}_{-n}] \Psi \rangle$	CFT predictions
1	$O(10^{-8})$	0
2	0.24399045	0.24399193
3	0.97200578	0.97201215
4	2.42663026	2.42665060
5	4.85014000	4.85020037
6	8.48474652	8.48491574

CFT emerges even without
translation symmetry.

Entanglement Bootstrap for CFT in any dimension

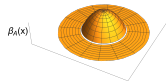
Entanglement Bootstrap for 2+1d CFT

[Xiang Li, Ting-Chun David Lin, JM, Rolando

Ramirez Camasca, 2606.12540]

In any dim $D = d + 1$, for a *round* ball A , K_A for a CFT groundstate takes the form

$$K_A^{\text{CFT}} = \int_A d^d x \beta_A(x) T_{00}(x) + \mathbb{1} f_A .$$



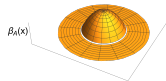
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Again, we can look for linear combinations $K_F \equiv \sum_A \lambda_A^F K_A$ such that the operator part cancels: $\sum_A \lambda_A^F \beta_A(x) = 0$, and $K_F |\text{gs}\rangle_{\text{CFT}} = F |\text{gs}\rangle_{\text{CFT}}$.

In $D > 1 + 1$, this requires an *integral* over regions.

We can also look for combinations $H_{\text{rec}} \equiv \sum_A \lambda_A K_A$ such that $\sum_A \lambda_A \beta_A(x) = 1$.

Entanglement Bootstrap for 2+1d CFT

[Xiang Li, Ting-Chun
David Lin, JM,
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Simplest and most numerically-robust answer so far, on S^d : let $A_i(\Omega)$, $i = 1..k$ be concentric disks centered at Ω ,

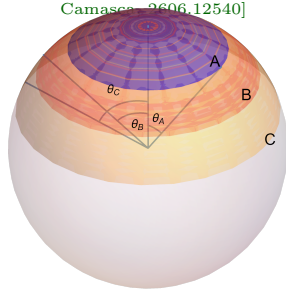
$$H_{\text{rec}} \equiv \int_{S^d} d^d\Omega \sum_i \lambda_i K_{A_i(\Omega)} \equiv \sum_i \lambda_i \overline{K_{A_i}} .$$

Basic idea:

$$\int d\Omega K_{A(\Omega)}^{\text{CFT}} \equiv \overline{K_A^{\text{CFT}}} = \int_{S^d} d^d\Omega T_{00}(\Omega) = H^{\text{CFT}} .$$

Choose λ_i to cancel the UV stuff in

$$K = K^{\text{CFT}} + K^{\text{UV}} .$$



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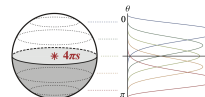
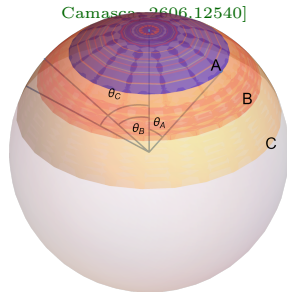
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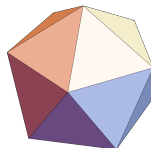
How to check? For 2+1d CFT

- ... on the icosahedron [Brower et al 12, Lao-Rychkov 23]
- ... on fuzzy sphere with orbital cuts
- ... on fuzzy sphere with real-space cuts [Hu-Zhu-He

2401.17362]



Ising model on Icosahedron



Begin with $k = 2$: $H_{\text{rec}} = \lambda_A \overline{K_A} + \lambda_B \overline{K_B}$.

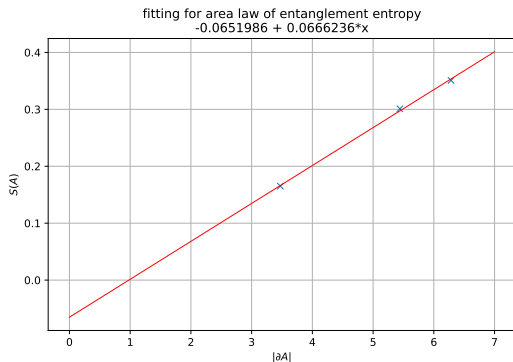
Choose $\lambda_{A,B}$ to cancel the area law contributions in

$$S_A = \langle K_A \rangle = |\partial A|/\epsilon - F.$$

$|\partial A| = 2\pi \sin(\theta_A)$ with θ_A determined by $|A| = \int_A d\Omega = 2\pi(1 - \cos(\theta_A))$

$$\frac{|A|}{4\pi} = \frac{\text{num of sites in } A}{\text{total num of sites}}$$

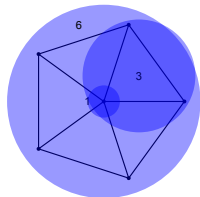
Pretty good area law
(and F !):



Ising model on Icosahedron

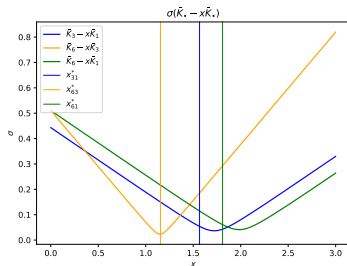


Possible balls:



Error of VFPE:

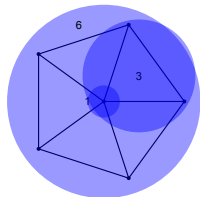
$$\sigma^2 \equiv \langle H_{\text{rec}}^2 \rangle - \langle H_{\text{rec}} \rangle^2$$



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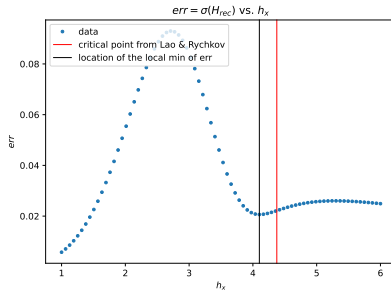
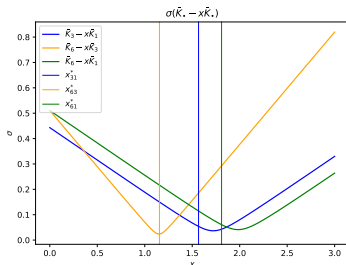


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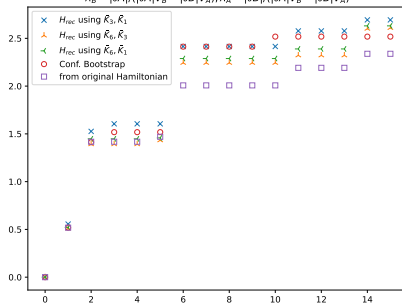


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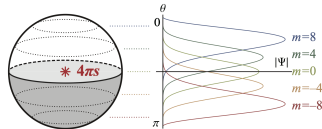
$$\sigma^2 \equiv \langle H_{\text{rec}}^2 \rangle - \langle H_{\text{rec}} \rangle^2$$



spectrum of $H_{\text{rec}} = \lambda_B \tilde{K}_B - \lambda_A \tilde{K}_A$, $A \subset B$, after a shift s.t. $E_0 = 0$
 $\lambda_B = |\partial A|/(|\partial A|V_B - |\partial B|V_A)$, $\lambda_A = |\partial B|/(|\partial A|V_B - |\partial B|V_A)$

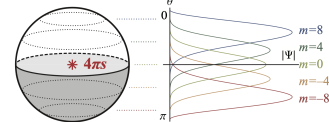
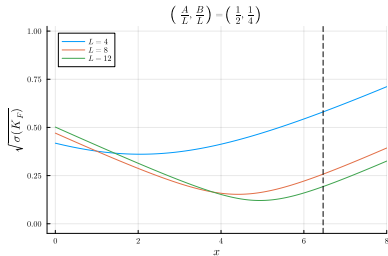


Ising model on FS, orbital cuts

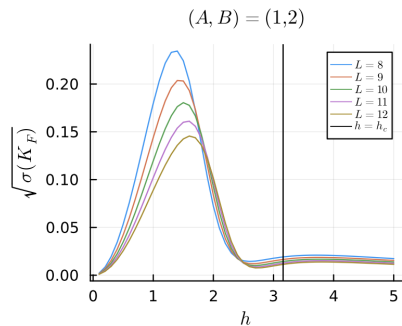


[fig from 2407.15914 Zhou-Zou]

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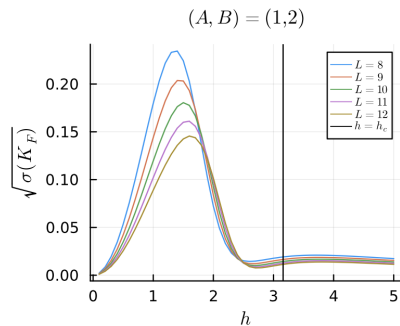
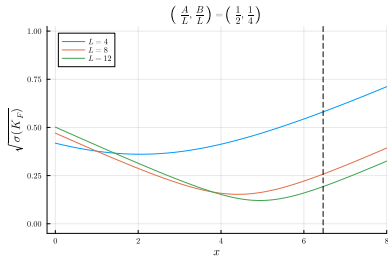


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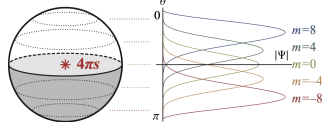


($\exists$ similar results for other CFTs.)

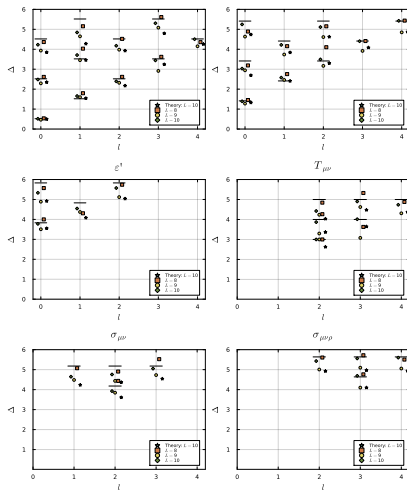
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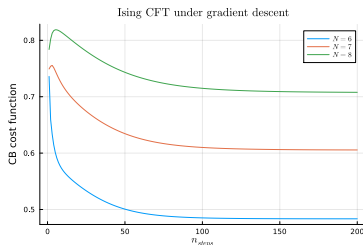


(Blue stars = 0-parameter fit.)

Getting closer to the fixed point

Starting from the groundstate of a human-made Hamiltonian that approximates the fixed point, we can improve upon it by doing gradient descent on the error of the vector fixed-point equation.

Here the cost function is a measure (cooked up by Ruihua Fan et al) of the extent to which the low-lying spectrum fits into multiplets of conformal symmetry and agrees with conformal bootstrap results.

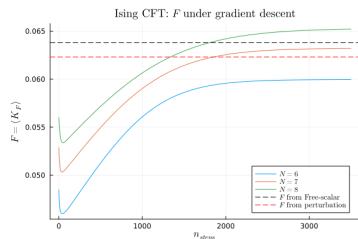
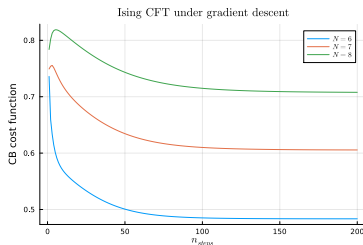


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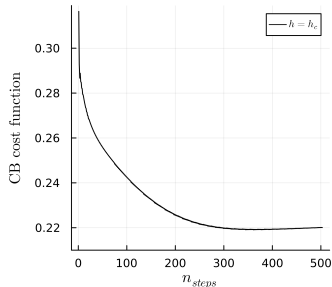
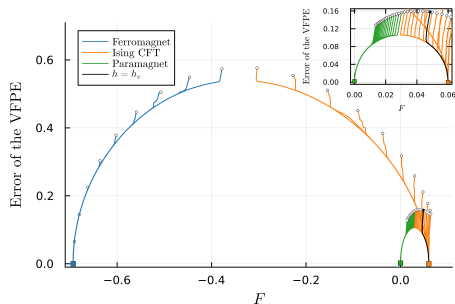
This also improves upon the estimate of $F = \langle K_F \rangle$.



Why are we doing this?

- Starting from a given, human-chosen model with a quantum critical point, we can do gradient descent on $\sigma(H_{\text{rec}})$ to move closer to the ideal CFT at fixed system size.
- Yin-Chen He [2506.14904] recently made an interesting analytic ansatz for the groundstate of the fuzzy sphere transverse-field Ising model. Our method gives a good way to extract the spectrum of scaling dimensions.
- Search is doable, maybe we can find new CFTs.

Gradient descent of the error function in the Ising model



Sheaves of quantum states

[Ting-Chun (David) Lin]

A *presheaf* $\mathcal{F}$ on a topological space X is:

- for each open set $U \subset X$, a set $\mathcal{F}(U) \ni$ sections of $\mathcal{F}$ over U .
- for each inclusion $U \hookrightarrow V$, a restriction map, $\text{res}_{V,U} : \begin{array}{ccc} \mathcal{F}(V) & \rightarrow & \mathcal{F}(U) \\ f & \mapsto & f|_U \end{array}$

with $\text{res}_{U,U} = \text{id}$, and $U \hookrightarrow V \hookrightarrow W$ implies the corresponding maps commute.

e.g.: functions on X (perhaps with some adjectives).

a good reference is [R. Vakil, *The Rising Sea*]

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Appearances of presheaves in quantum physics:

- In algebraic QFT, the *net of operator algebras* is

$\mathcal{F}_{\text{AQFT}}(U) \equiv \{\text{bounded ops on } U\} \sim \text{observables on } U$.

- Instead, let $\mathcal{F}(U) \equiv \{\text{density matrices } \rho \text{ on } U\}$ ($\rho \geq 0, \text{tr} \rho = 1$)

with $\text{res}_{V,U} : \begin{array}{ccc} \mathcal{F}(V) & \rightarrow & \mathcal{F}(U) \\ \rho & \mapsto & \text{tr}_{V \setminus U} \rho \end{array}$

Sheaves of quantum states

A *sheaf* is a presheaf such that for any open cover of U , $\{U_i\}_{i \in I}$:

- **Gluing axiom** (existence of gluing): given $f_i \in \mathcal{F}(U_i)$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, $\forall i, j$ then $\exists f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$.
- **Identity axiom** (uniqueness of gluing): given $f_{1,2} \in \mathcal{F}(U)$ with $f_1|_{U_i} = f_2|_{U_i} \forall i$, then $f_1 = f_2$.
(works for functions on X .)

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The presheaf of *all* quantum states is not a sheaf, for good reasons:

- **Gluing does not always exist.** Because of monogamy of entanglement:

e.g. let a, b, c = qubits, $|\text{EPR}\rangle \equiv (|00\rangle + |11\rangle) / \sqrt{2}$,

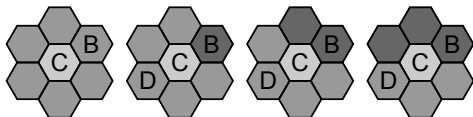
$\sigma_{ab} = |\text{EPR}\rangle\langle\text{EPR}| = \sigma'_{bc}$. Then $\sigma_{ab}|_b = \sigma'_{bc}|_b = \mathbb{1}_b/2$. But $\nexists \rho_{abc}$ with

$\rho_{abc}|_{ab} = \sigma_{ab}, \rho_{abc}|_{bc} = \sigma'_{bc}$, since:
$$\underbrace{S_{ab} + S_{bc}}_{=0} \stackrel{\text{SSA}}{\geq} \underbrace{S_b}_{=\log 2} + \underbrace{S_{abc}}_{\geq 0}$$

- **Gluing is not unique.** Topological order means multiple groundstates that are locally indistinguishable.

Sheaves of quantum states

Consider instead $\mathcal{F}(U) \equiv \{\text{density matrices on } U \text{ satisfying } \mathbf{A0}, \mathbf{A1}\}$.



$$\mathbf{A0}: 0 = \Delta(B, C)_\rho \equiv (S_{BC} + S_C - S_B)_\rho,$$

$$\mathbf{A1}: 0 = \Delta(B, C, D)_\rho \equiv (S_{BC} + S_{CD} - S_B - S_D)_\rho$$

- $\mathbf{A1} \implies \exists$ gluing (Merging lemma 3.8 of 1906.09376): given $\rho_i \in \mathcal{F}(U_i)$ with $\rho_i|_{U_i \cap U_j} = \rho_j|_{U_i \cap U_j}$, $\forall i, j$ then $\exists \rho \in \mathcal{F}(U)$ with $\rho|_{U_i} = \rho_i$.

idea: $\mathbf{A1} \implies I(A : C|B) = 0$, which means

$$\rho_{AB} \mapsto \rho_{ABC} = \rho_{BC}^{1/2} \rho_B^{-1/2} \rho_{AB} \rho_B^{-1/2} \rho_{BC}^{1/2}.$$

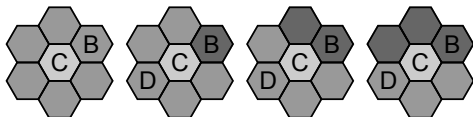
- $\mathbf{A0} \implies !$ gluing (Prop 3.5 of 1906.09376): for all *contractable* U , given $\rho_{1,2} \in \mathcal{F}(U)$ with $\rho_1|_{U_i} = \rho_2|_{U_i} \forall i$ then $\rho_1|_{U_-} = \rho_2|_{U_-}$.

idea: $\mathbf{A0} \implies$ local purity $\rho_{BC} = W_B (\rho_B \otimes |\psi_{B_2C}\rangle\langle\psi_{B_2C}|) W_B^\dagger$,

$W_B : \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \rightarrow \mathcal{H}_B$ is an isometry.

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Given a reference state σ , we can further restrict to sections that locally agree with σ , to pick out a particular topological order.

This gives a construction of TQFT.

The reference state is the groundstate of the reconstructed Hamiltonian

[Xiang Li, Ting-Chun

Lin, JM, to appear]

Prop: Assume that $|\Psi\rangle$ satisfies the operator fixed-point equation for every choice of three contiguous intervals. Then $|\Psi\rangle$ is the groundstate of $H_{\text{rec}} \equiv \frac{\cot \frac{\pi}{L}}{L} \sum_{i=1}^L \left(\tilde{K}_{[i,i+2]} - \tilde{K}_{[i,i+1]} \right)$.
 $\tilde{K} \equiv K - \langle \Psi | K | \Psi \rangle$.

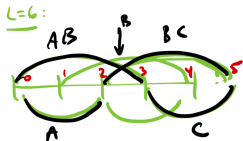
Idea of proof:

$$\begin{aligned} H_L &\equiv \sum_{i=1}^L (K_{[i,i+L/2]} - K_{[i,i+L/2-1]}) \\ &= \frac{1}{2} \sum_i \hat{\Delta} \left(i, i + \frac{L}{2} - 1, i + \frac{L}{2}, i + L - 1 \right) \end{aligned}$$

Steps:

1. $H_L \geq 0$ by operator weak monotonicity ($\hat{\Delta} \geq 0$). [Kim, TCL, Hsieh 22]
2. $H_L = a_L H_{\text{rec}} + b_L$.
3. $\lim_{L \rightarrow \infty} a_L = aL$, $a \geq 0$, $a < \infty$,
 $\lim_{L \rightarrow \infty} b_L = 0$.

(L even)



Some practical applications

[Xiang Li, Ting-Chun

Lin, JM 2509.04596]

- ▶ The multiplicative normalization of $H_{\text{rec}} = \sum_i \lambda_i K_i$ is completely fixed by $\sum_i \lambda_i \beta(x) \stackrel{!}{=} 1$. There is no need to search for $|T\rangle$ at $\Delta = 2$.

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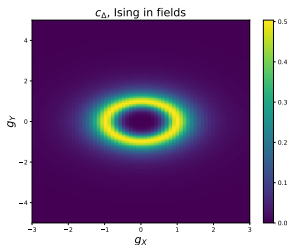
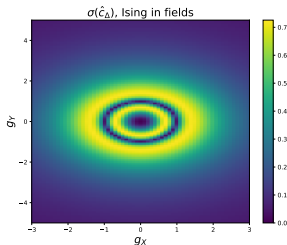
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- ▶ Minimal error of VFPE detects fixed points of RG:
Ising chain in various fields:

$$H = - \sum_{j=1}^L Z_j Z_{j+1} - g_X \sum_j X_j - g_Y \sum_j Y_j$$



PBC, $L = 8$ ($A = [1, 2]$, $B = [3, 4]$, $C = [5, 6]$)

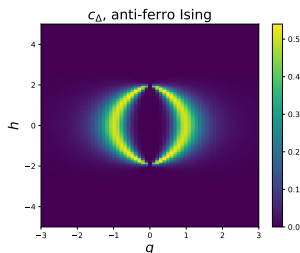
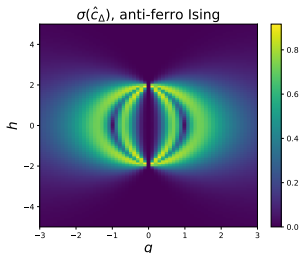
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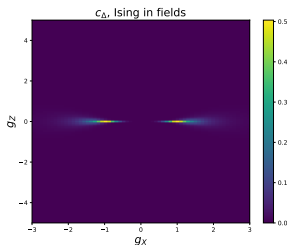
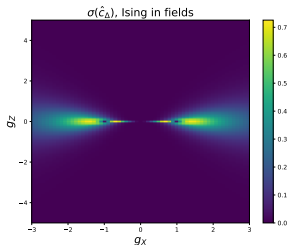
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Entanglement Bootstrap for 1+1d CFT (Recap)

Claim: the local condition for 1+1d CFT is the *vector fixed-point equation*

$$\boxed{K_{\Delta}|\psi\rangle = \frac{c}{3}h(\eta)|\psi\rangle} \text{ (VFPE)}$$

$$\begin{array}{cccc} & A & B & C \\ \hline & | & | & | \\ x_1 & x_2 & x_3 & x_4 \end{array}$$

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$$h(\eta) = -\eta \log \eta - (1 - \eta) \log(1 - \eta)$$

$$K_{\Delta} \equiv K_{AB} + K_{BC} - \eta(K_A + K_C) - (1 - \eta)(K_B + K_{ABC}) \text{ from the state } |\psi\rangle$$

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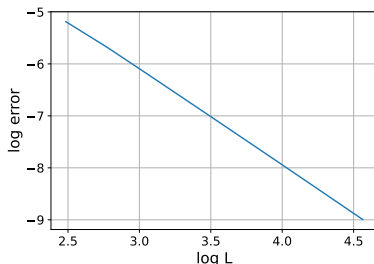
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- ▶ Improves with system size in quantum critical lattice models.
- ▶ On 4 sites, $K_{\Delta}|\Psi\rangle = H_{\text{rec}}|\Psi\rangle$ by purity.

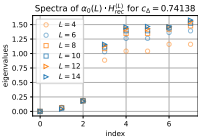
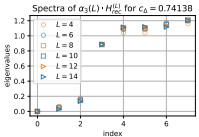
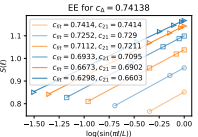
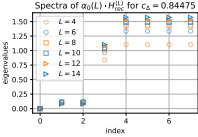
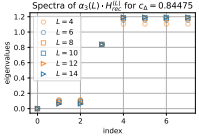
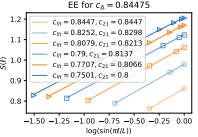
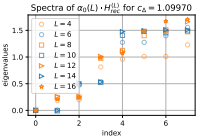
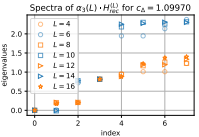
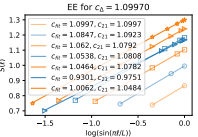
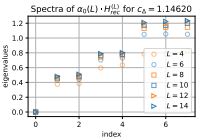
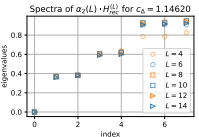
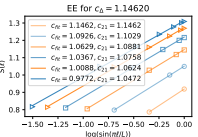


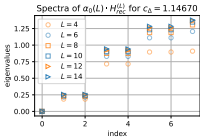
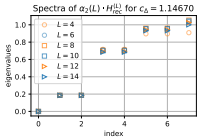
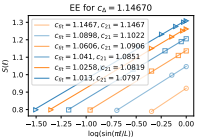
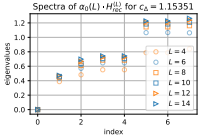
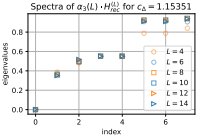
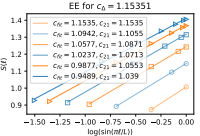
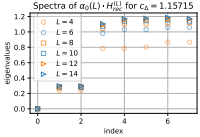
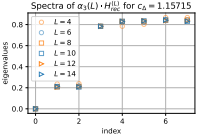
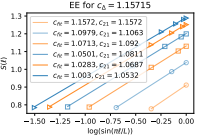
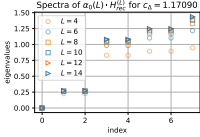
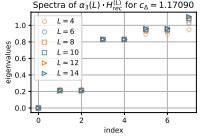
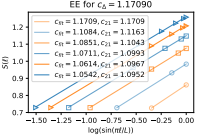
$$\text{error} \equiv \sqrt{\langle \psi | K_{\Delta}^2 | \psi \rangle - \langle \psi | K_{\Delta} | \psi \rangle^2}$$

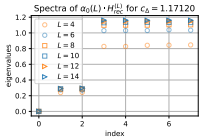
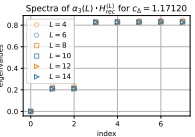
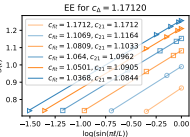
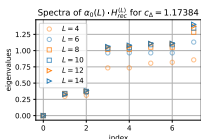
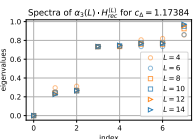
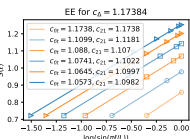
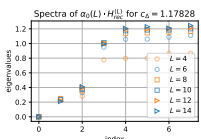
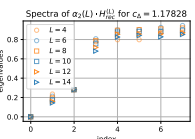
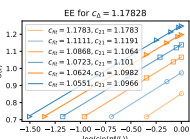
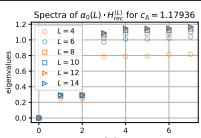
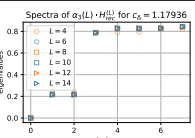
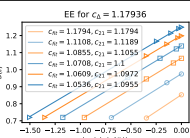


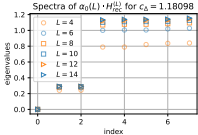
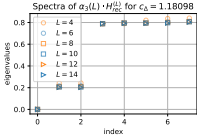
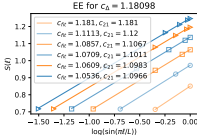
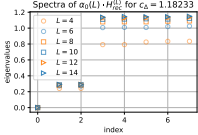
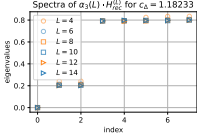
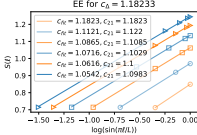
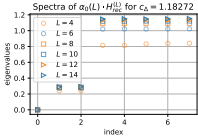
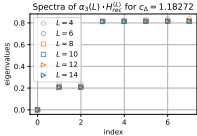
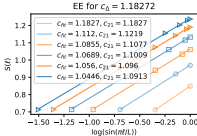
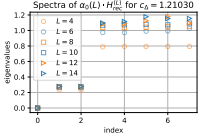
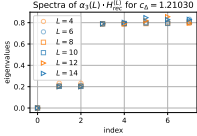
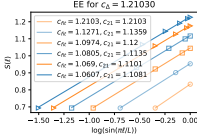
IS THIS A CFT?

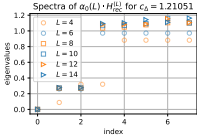
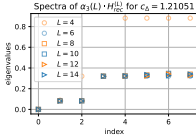
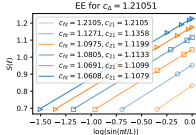
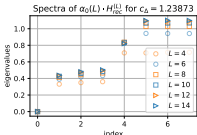
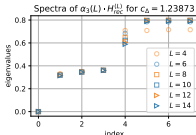
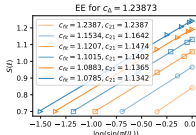
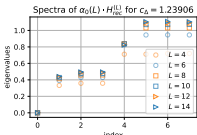
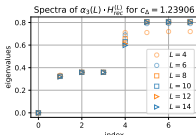
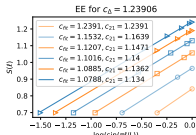
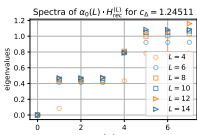
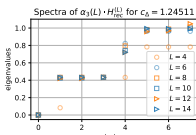
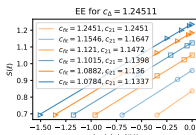
Chigh	name	spectra		
0.51360	Ising			
0.52445	$\approx$ qubit Ising			
0.52639	qubit Ising			
0.63776	Ising in Fib			

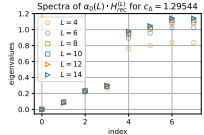
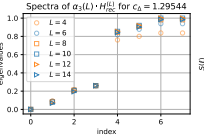
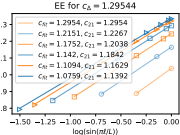
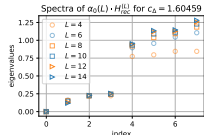
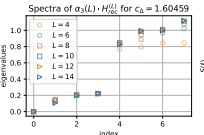
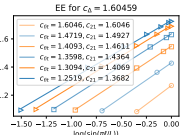
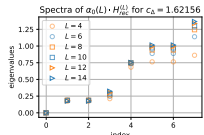
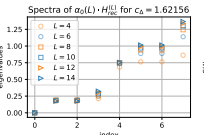
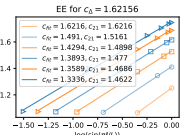
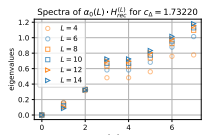
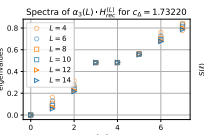
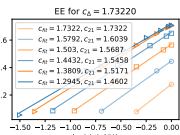
Chigh	name	spectra
0.74138	TCI	  
0.84475	Potts	  
1.09970	$c = 1$, XX	  
1.14620	$c = 1$	  

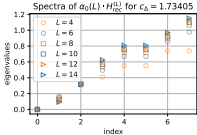
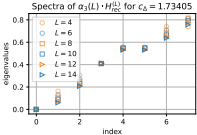
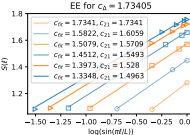
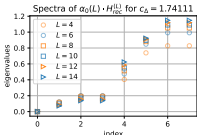
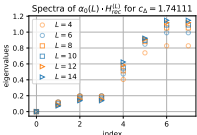
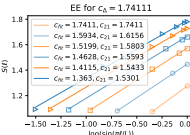
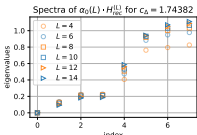
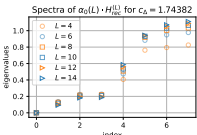
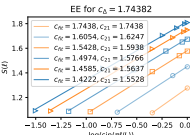
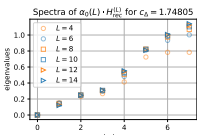
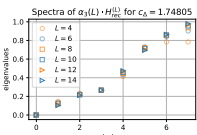
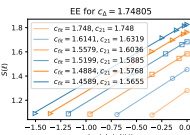
Chigh	name	spectra
1.14670	$c = 1$	  
1.15351	$c = 1$	  
1.15715	$c = 1, \text{XX}$	  
1.17090	$c = 1, \text{XX}$	  

Chigh	name	spectra
1.17120	$c = 1,$ XX	  
1.17384	$c = 1,$ XX	  
1.17828	Mystery CFT 0	  
1.17936	$c = 1,$ XX	  

Chigh	name	spectra
1.18098	$c = 1$, XX	  
1.18233	$c = 1$, XX	  
1.18272	$c = 1$, XX	  
1.21030	$c = 1$, XX	  

c_{high}	name	spectra
1.21051	$c = 1$, qubit XX	  
1.23873	$c = 1$, Heisenberg	  
1.23906	$c = 1$, Heisenberg	  
1.24511	$c = 1$, qubit Heisenberg	  

Chigh	name	spectra
1.29544	Mystery CFT 1	  
1.60459	Mystery CFT 2	  
1.62156	Mystery CFT 3	  
1.73220	Mystery CFT 4	  

Chigh	name	spectra
1.73405	Mystery CFT 4'	  
1.74111	Mystery CFT 5	  
1.74382	Mystery CFT 6	  
1.74805	Mystery CFT 6'	  

Chigh	name	spectra		
1.82871	Mystery CFT 7			
1.83075	Mystery CFT 8			
1.84206	Mystery CFT 9			
1.84526	Mystery CFT 10			

C_{high}	spectra	comments
1.13233	Spectra of $a_0(L) \cdot H_0^L / H_0^L$ for $c_h = 1.13233$ EE for $c_h = 1.13233$	qubit W state
1.50237	Spectra of $a_0(L) \cdot H_0^L / H_0^L$ for $c_h = 1.50237$ EE for $c_h = 1.50237$	Spectrum is drifting. The number of low-lying states (below a gap) is ℓ_L .
1.58353	Spectra of $a_0(L) \cdot H_0^L / H_0^L$ for $c_h = 1.58353$ EE for $c_h = 1.58353$	
1.59520	Spectra of $a_0(L) \cdot H_0^L / H_0^L$ for $c_h = 1.59520$ EE for $c_h = 1.59520$	Perturbation of a CFT into a gapped phase with GSD

Strict area law versus chirality

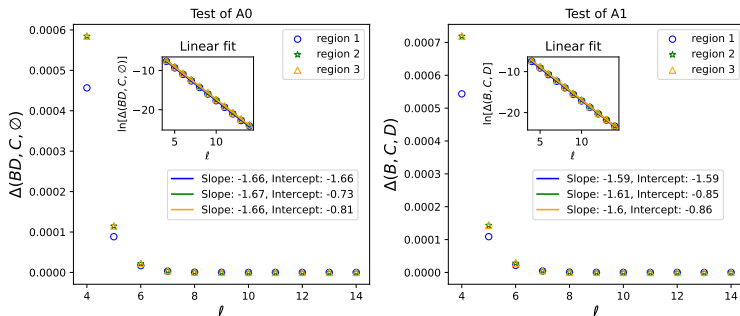
[Xiang Li, Ting-Chun David Lin, JM, Bowen Shi, 2408.10306]

In 2+1d gapped phases, $c_- \neq 0$ is an obstruction to gapped boundaries.

$$c_- = \frac{3i}{\pi} \langle \psi | [K_{AB}, K_{BC}] | \psi \rangle \quad [\text{Kim-Shi-Kato-Albert 2022}]$$



Above we studied examples with $\xi > 0$ and approximate A0, A1:



Q: $\exists$ a $\xi = 0$ fixed point state in a tensor product Hilbert space with finite local dimension?

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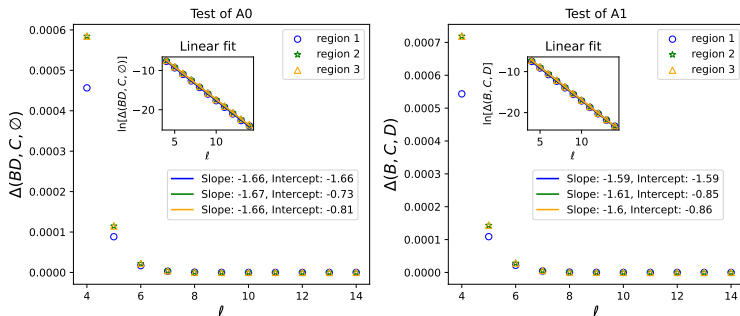
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A: No.

Strict area law versus chirality

[Xiang Li, Ting-Chun David Lin, JM, Bowen Shi, 2408.10306]

The following properties cannot coexist:

1. strict area law (A1)
2. finite local Hilbert space dimension
3. non-zero modular commutator.

Property 2) can be replaced by finite entropy.

Property 3) can be replaced by a version related to Hall conductivity [Ruihua Fan et al 2023].

This is a strong motivation to develop a robust version of Entanglement Bootstrap for liquid TO.

[Work in progress by Isaac Kim, Alexei Kitaev, Daniel Ranard, Bowen Shi, Milo Moses]

Instantaneous Modular Flow

[Isaac Kim, Xiang Li, Ting-Chun David Lin, JM, Bowen Shi, 2404.03725]

The proof uses a new QI primitive: $\mathcal{I}_A(t) : |\psi\rangle \rightarrow \rho_A^{\text{it}}|\psi\rangle$, $\rho_A \equiv \text{tr}_{\bar{A}}|\psi\rangle\langle\psi|$.

‘instantaneous’ means $\mathcal{I}$ always uses the ρ of the state it acts on:

$\mathcal{I}_A(t)\mathcal{I}_B(s)|\psi\rangle = (\rho'_A)^{\text{it}}\rho_B^{\text{is}}|\psi\rangle$ where ρ'_A comes from $\rho_B^{\text{is}}|\psi\rangle$.

Nice things: $\mathcal{I}_A(t) = \mathcal{I}_{\bar{A}}(t)$.

$\mathcal{I}_A(s)\mathcal{I}_B(t) = \mathcal{I}_B(t)\mathcal{I}_A(s)$ if $A \subseteq B$ or $B \subseteq A$.

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We can find a family of states ($|\Psi(t)\rangle = \mathcal{I}_{AB}(t)|\Psi\rangle$) with

$$\frac{d}{dt}S_{BC}(t) = \mathbf{i}\langle\Psi(t)|[K_{AB}(t), K_{BC}(t)]|\Psi(t)\rangle \stackrel{A1}{=} \frac{\pi c_-}{3}$$



so that $S_{BC}(t)$ is unbounded or negative.

Bulk reconstructed Hamiltonian

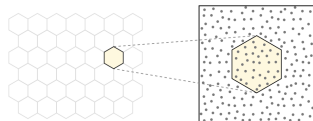
Exact **A0**, **A1** gives a commuting parent H [Kim-Lin-Ranard-Shi 2404.05867]

Assume **A0**, **A1** on regions bigger than ξ .

Let D be a big region $\gg \xi$ with the topology of a disk.

A1 $\implies$

$$K_D = \sum_{f \in D} K_f - \sum_{e \in D \setminus \partial D} K_e + \sum_{v \in D \setminus \partial D} K_v$$



Compare: $\chi = F - E + V$

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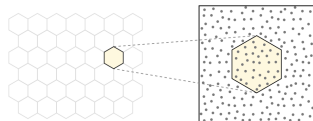
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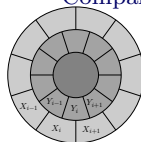
A1 $\Rightarrow$

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$$= K_{D_{int}} + \underbrace{K_{D_{\partial}}}_{=\sum_i \hat{\Delta}(X_{i-1}Y_{i-1}, X_i, X_{i+1}Y_i) = H_{rec}^{\partial} \geq 0}$$



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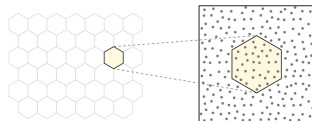
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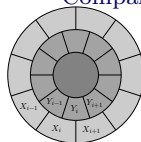
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$$\langle H_{rec}^{bulk} \rangle \text{ on } M \stackrel{\text{closed}}{=} \gamma \chi(M) \geq$$

$\gamma_{universal} \chi(M)$ [Kim-Levin-Lin-Ranard-Shi 2302.00689]^{*}

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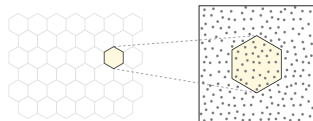
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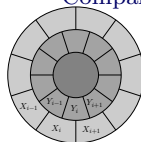
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$$H_{rec}^{bulk} \stackrel{\mathbf{A1}}{=} \sum_j \hat{\Delta}(B_j, C_j, D_j) - \gamma \text{ where each } B_j C_j D_j \text{ is an } \mathbf{A1} \text{ region}$$

The iterative process $|\psi\rangle \mapsto H_{rec}^{bulk}(\psi) \mapsto |\psi\rangle \mapsto H_{rec}^{bulk}(\psi) \mapsto \dots$ decreases error of **A0**, **A1**, $J - \frac{\pi c_-}{3}$, VFPE.



Compare: $\chi = F - E + V$



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Corner entanglement for chiral gapped phases

$$\mathbf{A1} \implies K_D |\psi\rangle \stackrel{|D| \gg \xi}{\simeq} \int_{\partial D} dx \mathcal{O}(x) |\psi\rangle \quad [\text{Xiang Li et al, 2510.23720}]$$

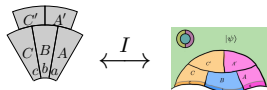
- for x in a smooth component of ∂D , $\mathcal{O}(x) \simeq T^{\text{CFT}}(x)$ as if $\rho = e^{-\beta H^{\text{CFT}}}$ with $\beta = \xi/R_{\text{bdy}}$ small – high temperature.
- for x near a *corner*, the CFT is at low temperature.

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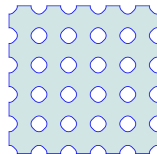
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Two perspectives: 1) symmetries of a topological theory $\ni$ conformal group $\ni$ inversion map $I : x^\mu \mapsto -\frac{x^\mu}{x^2}$.



2) FQH states can be filled with holes.

[Brehm-Runkel 21, Sopenko 23, Cheng-Chen-Gu-Hung 23, Kane-Mukhopadhyay-Lubensky 01]

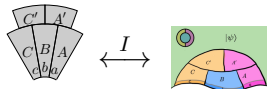


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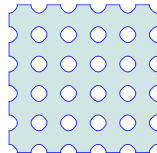
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- for x near a *corner*, the CFT is at low temperature.

Two perspectives: 1) symmetries of a topological theory $\ni$ conformal group $\ni$ inversion map $I : x^\mu \mapsto -\frac{x^\mu}{x^2}$.



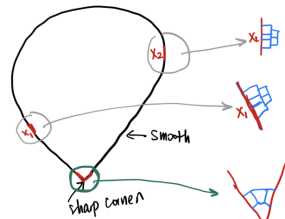
2) FQH states can be filled with holes.

[Brehm-Runkel 21, Sopenko 23, Cheng-Chen-Gu-Hung 23, Kane-Mukhopadhyay-Lubensky 01]



Strategy: replace the tip of the corner with a hole $\rightarrow$ actual edge CFT in its groundstate
 $\rightarrow K_X \simeq \int_{\theta_0}^{\theta_1} d\theta \beta_X(\theta) T^{\text{CFT}}(\theta)$.
 (really: $\hat{\Delta} = \hat{\Delta}^{\text{CFT}}$)

generically c_{tot} will take the minimum possible value.



Corner entanglement for chiral gapped phases

[Xiang Li et al, 2510.23720.]

Consequences: • $S_A = \frac{|\partial A|}{\epsilon} - \gamma + \frac{c_{\text{tot}}}{6} \log \sin \theta/2 + S_{\text{non-universal}}$

[Rodriguez-Sierra 10, Witzak-Krempa et al 21, Liu et al 24]

Δ combinations are cutoff-independent and depend on cross-ratios.

$c_{\text{tot}} \neq 0 \Leftrightarrow$ no gapped boundary. [Siva-Zou-Soejima-Mong-Zaletel 21]

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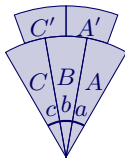
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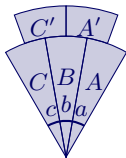
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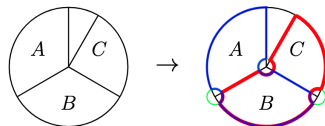
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• A systematic calculation of the modular commutator J from the edge TT OPE.

New result: $J(A, B, C) = \frac{\pi c_-}{3} \eta(a, b, c)$



Aside: Why does the fuzzy sphere work so well?

[This was explained to me by Ting-Chun David Lin.]

Besides manifest $\text{SO}(3)$ symmetry, it also preserves well the locality of the operator algebra in several senses.

Consider the following *local* Hamiltonian on “ $\mathcal{H}_{\text{full}}$ ” for continuum spinful fermions on S^2 in a magnetic field:

$$H_{\text{full}} = \Gamma \int d\Omega \psi^\dagger (D^2 - \mu)^2 \psi + H_{\text{int}}$$

with (*e.g.*) $H_{\text{int}} = - \int d\Omega h n_X(\Omega) - \int d\Omega \int d\Omega' g(\Omega - \Omega') n_Z(\Omega) n_Z(\Omega') .$
 $n_\alpha(\Omega) \equiv \psi^\dagger(\Omega) \sigma^\alpha \psi(\Omega), g(\Omega) = g_0 \delta(\Omega) - g_1 \vec{\nabla}^2 \delta(\Omega) + \dots$

Choosing μ appropriately and taking Γ large, the low-energy theory is governed by the fuzzy sphere Hamiltonian.

[Pedantic comment: $P_{\text{fuzzy}} H_{\text{full}}(\psi) P_{\text{fuzzy}} \neq H_{\text{full}}(\hat{\psi} \equiv P_{\text{fuzzy}} \psi P_{\text{fuzzy}})$ in general, but for the TFIM of [2210.13482] these are equal.]

Aside: Non-magnetic fuzzy spheres?

This property ($\exists$ a **local** H_{full} whose low-energy subspace is the image of P_{fuzzy}) is shared by various ‘non-magnetic fuzzy spheres’:

$$\hat{\psi} = P_{\text{fuzzy}} \psi P_{\text{fuzzy}} = \sum_{\ell \in \mathcal{L}} Y_{\ell m} c_{\ell m}.$$

one layer: $\mathcal{L} = \mathcal{L}_1 \equiv \{\ell_0\}$

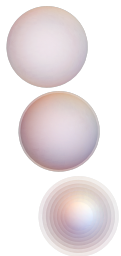
two layers: $\mathcal{L} = \mathcal{L}_2 \equiv \{\ell_0, \ell_0 + 1\}$

many layers: $\mathcal{L} = \mathcal{L}_{\text{many}} \equiv \{0, 1 \cdots \ell_{\text{max}}\}.$

[see also Gao-Wang-Lee, 2504.15338 for Gross-Neveu]

Practical drawback of $\mathcal{L}_{\text{many}}$: $N = (\ell_{\text{max}} + 1)^2$ jumps quickly with ℓ_{max} .

It would be great if we could use just one layer, because that would work in 3+1d too.



Aside: Non-magnetic fuzzy spheres?

A second notion of preserving locality of the operator algebra is:

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→ a fuzzy $\mathbb{RP}^2$!

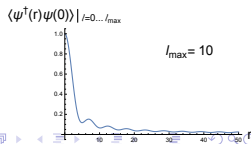
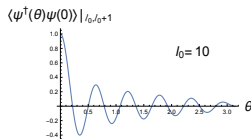
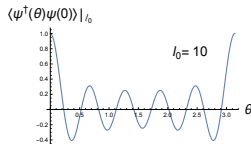
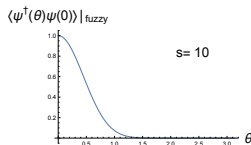
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What are we cancelling in $H_{\text{rec}} = \sum_{i=1}^k \lambda_i \overline{K_i}$?

Model of UV: $K_A = K_A^{\text{CFT}} + K_A^{\text{UV}}$

$$K_A^{\text{UV}} =$$

$\int_{\partial A} d^{d-1}x \left(O_0^{\hat{n}}(x) + \kappa O_1^{\hat{n}}(x) + \kappa^2 O_2^{\hat{n}}(x) + \dots \right)$ is supported on ∂A . ($\hat{n}$ is the normal to ∂A at x .)

Analogy: $S_{\text{SRE}} =$

$$\int_{\partial A} (a + b\kappa + c\kappa^2 + \dots)$$

[Grover-Turner-Vishwanath

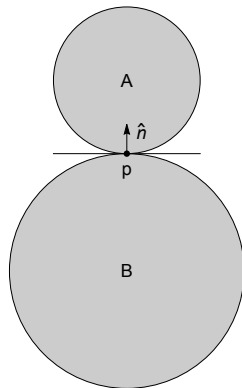
1108.4038]

For each point p and direction $\hat{n}$, we want:

- (1) $\sum_i \lambda_i |\partial A_i| = 0$. (Cancel area law.)
- (2) $\sum_i \lambda_i V_i = 1$, $V_A \equiv \int d^2\Omega \beta_A(\Omega)$. (Normalize.)
- (3) $\sum_i \lambda_i = 0$. (Positivity: If $A \subset B$, we expect $K_A - K_B \geq 0$.)

For $k = 3$ this determines the λ_i .

A check: A TQFT is a special case of a CFT. The same conditions work to determine H_{rec} for the 2+1d gapped case $H_{\text{rec}}^{\text{TQFT}} = \sum_F K_F - \sum_E K_E + \sum_V K_V$.



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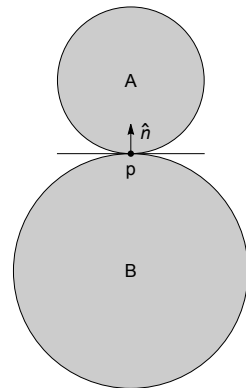
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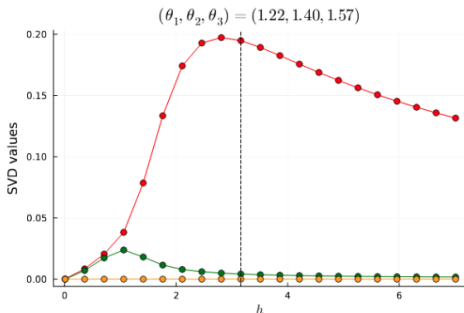
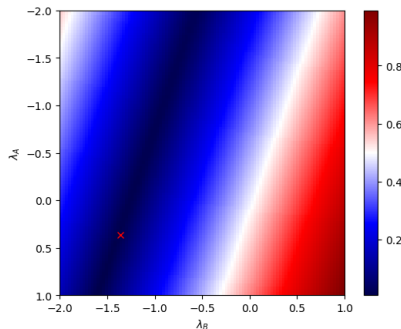
Q: why did $k = 2$ work earlier?

Purity

For both real-space and orbital cuts, the gram matrix of $\{\overline{\tilde{K}_{A_i}}|\psi\rangle\}_{i=1}^k$ has *two* small singular values (for $k \geq 2$):

$$\overline{K_A} \equiv \int d\Omega_A K_{A\Omega}, \quad \tilde{O} \equiv O - \langle O \rangle.$$

This is why $k = 2$ regions works too:



why:

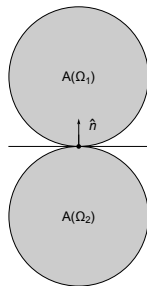
$$\int_{\partial A} (\kappa O_1^{-\hat{n}} - \kappa O_1^{\hat{n}}) |\psi\rangle = 0.$$

Analogy: In $S_{\text{SRE}} =$

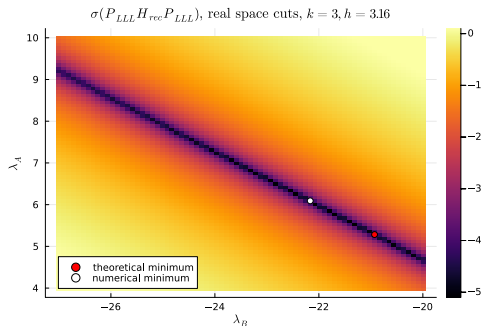
$$\int_{\partial A} (a + b\kappa + c\kappa^2 + \dots) = aR_A + bR_A^0 + cR_A^{-1} + \dots, \quad b = 0$$

by purity.

[Grover-Turner-Vishwanath 1108.4038]



Ising CFT on fuzzy sphere, real-space cuts



Current status: we think it will eventually do better than orbital cuts.

Metaphysical viewpoint

$\mathcal{H}_{\text{fuzzy}}$ is not a tensor product over space.

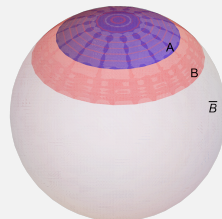
Real space cut requires enlarging $\mathcal{H}_{\text{fuzzy}}$. [Hu-Zhu-He 2401.17362]

When we enlarge $\mathcal{H}_{\text{fuzzy}} \hookrightarrow \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ we are embedding $\mathcal{H}_{\text{fuzzy}} \subset \mathcal{H}_{\text{full}}$.

Different embeddings $A \subset B$ are compatible:

$$c_m = p_A c_m^A + p_{\bar{A}} c_m^{\bar{A}} = p_B c_m^B + p_{\bar{B}} c_m^{\bar{B}} =$$

$$p_A c_m^A + p_{B \setminus A} c_m^{B \setminus A} + p_{\bar{B}} c_m^{\bar{B}}.$$

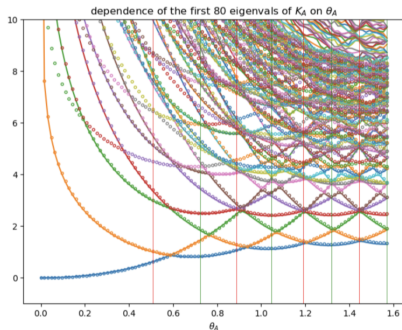


With the real-space cut, our estimate of the error is a lower bound:

$$\sigma^2 = \sigma_{LL}^2 + \sigma_{HL}^2 \geq \sigma_{LLL}^2.$$

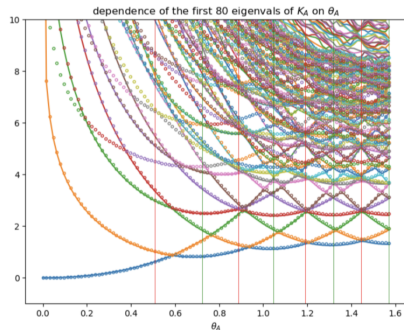
Ising model on FS, real-space cuts

circles: $h = h_*$, lines: $h \gg 1$



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circles: $h = h_*$, lines: $h \gg 1$



Observation:

$$|g_{h=\infty}\rangle = |IQH\rangle \otimes |\Rightarrow\rangle$$

$$|g_{h=0}\rangle = |IQH\rangle \otimes (|\uparrow\rangle + |\downarrow\rangle)$$

$$|g_{h=h_*}\rangle \approx |IQH\rangle \otimes |g_{\text{Ising CFT}}\rangle$$

$\uparrow K_A^{\text{IQH}}$ we understand

analytically in terms of 1+1d chiral boson, and can therefore remove.

