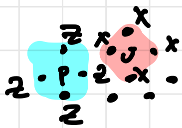
From Stabilizer codes to Fracton Phases

Based on work of Haah, Ruba-Yang ...
and conversations with Hermele, Shirley, Spiegel, Wickenden, Yang

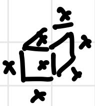
Part I: Stabilizer Codes and Charges.

What this is about? Lattice models

Ising : $H = -\sum_{\lambda \in \mathbb{Z}} Z_\lambda Z_{\lambda+1}$ . . . Z-Z . . Not Lagrangian

Toric Code: $H = -\sum_{p} \begin{smallmatrix} Z - Z \\ | \quad | \\ Z - Z \end{smallmatrix} - \sum_{v} \begin{smallmatrix} & X & \\ X & + & X \\ & X & \end{smallmatrix}$ 

Trivial/Ancilla: $H = -\sum_{\lambda \in \mathbb{Z}^d} X_\lambda$

X-Cube: $H = -\sum_{c}$  $- \sum_{v} \left(Z \overset{Z}{\underset{Z}{+}} Z + Z \diagup Z + Z \diagup Z \right)$

All exhibit different behaviour ground state and excitation but can be modeled using common framework.

Key:
- products of Paulis
- commuting
- translation invariant
- Locally generated
- finite range.

Goal:
- Algebraic framework
- Mobility
- Fractons
- Phases.

Part 1: Pauli Group

Let $$\mathcal{A}_{loc} = \bigotimes_{\lambda \in \mathbb{Z}^d} M_n(\mathbb{C})^{\otimes m}$$

\# qudits

on site dim.

spatial dim

Generalized Pauli's

Let $Xe_i = e_{i+1}$ shift

$Ze_i = \zeta e_i$ clock

$$ZX = \underbrace{e^{2\pi i/n}}_{= \zeta} XZ$$

$\mathcal{A}_{loc}$ is spaned by Pauli Group $\mathcal{P} \subset U(\mathcal{A}_{loc})$ and phases

$$zX^a Z^b := z \prod_{\lambda \in \mathbb{Z}^d} \underbrace{X_{\lambda,1}^{a_{\lambda,1}} Z_{\lambda,1}^{b_{\lambda,1}}}_{\text{on 1st qudit}} \otimes \cdots \otimes \underbrace{X_{\lambda,m}^{a_{\lambda,m}} Z_{\lambda,m}^{b_{\lambda,m}}}_{\text{on } m\text{'th qudit}}$$

where $z \in U(1)$ and $a_{\lambda,j}, b_{\lambda,j} = 0$ for all but fin. many $\lambda \in \mathbb{Z}^d$.

Encode this algebraically:

Let $R = \mathbb{Z}_n[\mathbb{Z}^d] \cong \mathbb{Z}_n[x_1^{\pm}, \dots, x_d^{\pm}]$

$$\lambda = (\lambda_1, \dots, \lambda_d) \mapsto x^\lambda = x_1^{\lambda_1} \cdots x_d^{\lambda_d}$$

Involution:

$$x_i \longmapsto x_i^{-1} = \bar{x}_i$$

$$(a;b) = (a_{\lambda,1}, \dots, a_{\lambda,m}; b_{\lambda,1}, \dots, b_{\lambda,m})_\lambda$$

$$\mapsto (a_1(x), \dots, a_m(x); b_1(x), \dots, b_m(x)) \in R^m \oplus R^m =: P$$

Pauli module.

$$a_i(x) := \sum_{\lambda \in \mathbb{Z}^d} a_{\lambda,i} x^\lambda$$

Central extension:

forgets phase

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] & \mathcal{P} \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
z \prod_{\lambda,j} X_{\lambda,j}^{a_{\lambda,j}} Z_{\lambda,j}^{b_{\lambda,j}} \arrow[r, mapsto] \arrow[d, "\mu"'] & (a_1, \dots, a_m; b_1, \dots, b_m) \arrow[d, mapsto] \\
\mu\text{-translate} \arrow[r, mapsto] & \bar{x}^\mu (a_1, \dots, a_m; b_1, \dots, b_m)
\end{tikzcd}$$

Action of R records translation on the lattice

Encode phase in symplectic form:

$$\zeta^{ad-bc} X^a Z^b X^c Z^d = X^c Z^d X^a Z^b \quad ; \quad [a,b]\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}$$

For $p=(a_1,\dots,a_m;b_1,\dots,b_m)^T$ $\quad \omega(p,q)=\sum_{\lambda,j} a_{\lambda,j}d_{\lambda,j}-b_{\lambda,j}c_{\lambda,j}$

$q=(c_1,\dots,c_m;d_1,\dots,d_m)^T$ $\quad h_p, h_q$ commute iff $\omega(p,q)=0$.

Note $\omega(x^\lambda p, x^\lambda q)=\omega(p,q)$ (translation inv.)

Let $\Omega: P\otimes_R P\to R$ be $\Omega(p,q)=\sum_\lambda \omega(x^\lambda p,q)x^\lambda$

Then $\Omega(p,q)=p^\dagger\begin{bmatrix}0&I\\-I&0\end{bmatrix}q$ $\quad p^\dagger=(\bar a_1,\dots,\bar a_m,\bar b_1,\dots,\bar b_m)$

and $\Omega(p,q)_0=\omega(p,q)$ ← constant term.

Rmk: (P,Ω) is a symplectic space.

$\Omega(p,q)=0$ iff h_q commutes with all translates of h_p.

Let $\iota: L\hookrightarrow P$ be a submodule.

$$L^\perp=\{p\in P \mid \Omega(p,\ell)=0 \text{ for all } \ell\in L\}.$$

Def^n: L is isotropic if $L\subset L^\perp$: stabilizer code (SC)

Prop: If L is isotropic, there exists a gp homo

$H: L\to \mathcal{P}$, $\ell\mapsto H_\ell$ s.t.

$$\begin{tikzcd} & L \arrow[dl, "H"'] \arrow[d, hook] \\ \mathcal{P} \arrow[r] \arrow[ur, phantom, "\circ"] & P \end{tikzcd}$$

$H_\ell=\phi(\ell)h_\ell$ ← phase

Note L is finitely generated. Choose $\ell_1,\dots,\ell_n$ gen.

$$\mathcal{H} = -\sum H_{e_1} + \cdots + H_{e_k} + h.c.$$

is commuting, locally generated Pauli Hamiltonian of finite range.

A frustration free ground state is a state $|0\rangle$ s.t.

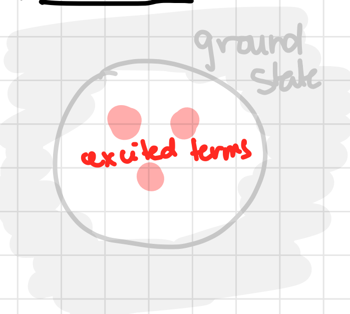
$$H_\ell |0\rangle = |0\rangle \quad \text{for all } \ell \in L.$$

$\uparrow$ stabilizers

Thm H has a unique frustration free ground state iff L is Lagrangian i.e. $L = L^\perp$ "topological"

Def$^{\underline{n}}$: A Lagrangian stabilizer Code is $L \subset P$ s.t. $L = L^\perp$.

Local Excitations : Specify which terms are excited



Let $\mathrm{Hom}^{loc}(L, \mathbb{Z}_n)$ be the homomorphisms

$$s_0 : L \longrightarrow \mathbb{Z}_n \quad \text{s.t. } s_0(x^\lambda \ell) = 0 \quad |\lambda| \gg 0$$

$(r\varphi)(\ell) = \bar{r}\varphi(\ell)$

$$\mathrm{Hom}^{loc}(L, \mathbb{Z}_n) \cong \mathrm{Hom}_R(L, R) = L^*$$

$$s_0 \longmapsto \ell \mapsto \sum s_0(x^{-\lambda}\ell)x^\lambda$$

We get pure state $|s\rangle$ s.t. $H_\ell |s\rangle = \zeta^{s_0(\ell)} |s\rangle$

Thm $|s\rangle$ is in the superselection sector of $|0\rangle$

iff $|s\rangle = h_p |0\rangle$ for some $p \in P$ $\quad H_\ell|s\rangle = H_\ell h_p|0\rangle$

$= \zeta^{\omega(p,\ell)} h_p |0\rangle$

iff $\exists\, p \in P$ s.t. $s(\ell) = \Omega(p, \ell)$ for all $\ell \in L$.

Def$^{\underline{n}}$ $s(\ell) = \Omega(p, \ell)$ is locally creatable.

Def^n The charge module $Q_L = \dfrac{L^*}{P/L} = \dfrac{\text{local excitations}}{\text{locally creatable}}$

Phase invariant. $\Omega(\ell, \ell') = 0$

Charge Complex: Let $L \subset P$ be isotropic.

free res of P/L — exact — charges — charge complex

$$\begin{tikzcd}
 & 0 \arrow[r] & L \arrow[r, hook, "\iota"] & P \arrow[r] & P/L \arrow[r] & 0 & & \\
F_3 \arrow[r] & F_2 \arrow[r, "d_2"'] & F_1 \arrow[r, "d_1"'] \arrow[u] & F_0 \arrow[r, "d_0"] \arrow[u, equal] \arrow[d, "\cong", "\Omega"'] & F_{-1} \arrow[r, "d_{-1}"] \arrow[d, equal] & F_{-2} \arrow[r, "d_{-2}"] \arrow[d, equal] & F_{-3} \arrow[r] \arrow[d, equal] & \cdots \\
 & & & F_0^* \arrow[r, "d_1^*"] \arrow[d, equal] & F_1^* \arrow[r, "d_2^*"] & F_2^* \arrow[r, "d_3^*"] & F_3^* & \\
 & & & P^* \arrow[r, "\iota^*"] & L^* \arrow[u] & & &
\end{tikzcd}$$

code Lagr. iff $=0$.

$$\Omega : P \xrightarrow{\cong} P^*$$
$$p \longmapsto \Omega(p,-)$$

Prop: a) $H_0(F) = \dfrac{\ker(d_0)}{\operatorname{im}(d_1)} = \dfrac{\ker(\iota^*\Omega)}{\operatorname{im}(\iota)} = \dfrac{L^{\perp}}{L}.$

b) $H_{-1}(F) = \dfrac{\ker(d_2^*)}{\operatorname{im}(d_1^*)} = \dfrac{L^*}{P} = Q_L \cong \operatorname{Ext}^1_R(P/L, R)$

Ruba-Yang

Def$^{\underline{n}}$ The ith dimensional charge module is

$$Q_L^i = \operatorname{Ext}_R^{i+1}(P/L, R)$$

$d \leq 3$

1) Q_L^0 pointlike excitations

2) Q_L^1 loop excitations

3) Q_L^2 membrane excitations

Def$^{\underline{n}}$ A Lagrangian code is pointlike if $Q_L^i = 0$ if $i > 0$.

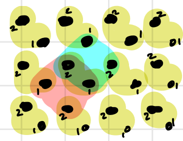
Rmk: If all $Q_L^i = 0$, related to invertible phases and QCA's

See, e.g.: Haah-Fidkowski-Hastings, Chen-Hsin, Shirley et.al.
Ruba-Yang, Geiko-Shukin. Ellison et.al.

Part II: Mobility and Phase.

Example (2D Toric Code) $\quad H = -\sum_v A_v - \sum_p B_p$

$d=2,\ n=2,\ m=2$



> Excercise: Write a code for Ising. Prove not Lagrangian

$$A_v = X_{(-1,0),1}\, X_{(0,0),1}\, X_{(0,-1),2}\, X_{(0,0),2}$$

$$B_p = Z_{(0,0),1}\, Z_{(0,1),1}\, Z_{(0,0),2}\, Z_{(1,0),2}$$

$$\ell_v := (1+\bar x,\ 1+\bar y\ ;\ 0\ ,\ 0\ )$$

$$\ell_p := (\ 0\ ,\ 0\ ;\ 1+y,\ 1+x\ )$$

L_{TC} is the R-sub. of $P = R^2$ gen. by these.

$$0 \to R^2 \xrightarrow[\begin{bmatrix} \bar t_x & 0 \\ \bar t_y & 0 \\ 0 & t_y \\ 0 & t_x \end{bmatrix}]{d_1} \underset{}{\overset{P}{\overset{\|}{R^4}}} \xrightarrow[\begin{bmatrix} 0 & 0 & t_x & t_y \\ \bar t_y & \bar t_x & 0 & 0 \end{bmatrix}]{d_0} R^2 \to 0$$

$t_x = 1-x$

$t_y = 1-y$

exact so Lagrangian

$$Q_{TC} = \operatorname{coker}(d_0) \cong \mathbb{Z}_2\{\underset{[e_1]}{\overset{\|}{e}},\ \underset{[e_2]}{\overset{\|}{m}}\} = \frac{L_{TC}^*}{P/L_{TC}}.$$

Finite

Example (X-Cube) $\quad H = -\sum_c A_c - \sum_v B_v^x + B_v^y + B_v^z$ 

$d=3,\ n=2,\ m=3$

$$R^3 \xrightarrow[\begin{bmatrix} \bar t_x \bar t_y & 0 & 0 \\ \bar t_y \bar t_z & 0 & 0 \\ \bar t_x t_z & 0 & 0 \\ 0 & t_z & 0 \\ 0 & t_x & t_x \\ 0 & 0 & t_y \end{bmatrix}]{} \overset{}{\underset{P}{\underset{\|}{R^6}}} \xrightarrow[\begin{bmatrix} 0 & 0 & 0 & t_x t_y & t_y t_z & t_x t_z \\ \bar t_z & \bar t_x & 0 & 0 & 0 & 0 \\ 0 & \bar t_x & \bar t_y & 0 & 0 & 0 \end{bmatrix}]{} R^3$$

$$Q_{XC} = R[c]/(t_x t_y c,\ t_y t_z c,\ t_x t_z c) \oplus R[v^y, v^z]/(t_z v^z,\ t_x(v^y+v^z),\ t_y v^y)$$

Not finite

<u>Mobility</u>: In the toric code $x^i y^j e = e$ $\forall i,j$. So we can move the e-excitation to anywhere in space by application of local operators. $Z\ Z\ Z\ Z\ Z\ Z$ Same for m.

Q_{TC} is fully mobile. Local operators implement mobility

In X-Cube, excitations have restricted mobility

Fractons : c can't move
Planon : $t_x c$ can move in yz-plane
Lineon : o^y can move along y direction.

all Fractons

<u>Defn</u> Let M be an R-module. The annihilator ideal is

$$\mathrm{Ann}(M) = \{ r \in R \mid rm = 0 \ \forall m \in M \}.$$

The (Krull) dimension of M, $\dim M$, is the maximal k s.t.

$$\mathrm{Ann}(M) \subset p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_k$$

for p_i prime ideals.

<u>Note</u> $\dim(M) \le \dim(R) = d$: $(p) \subsetneq (p, 1-x_1) \subsetneq \cdots \subsetneq (p, 1-x_1, \ldots, 1-x_d)$

<u>Ex</u>: $\dim(Q_{TC}) = 0$ since

$$\mathrm{Ann}(Q_{TC}) = (1-x, 1-y) = p_0 \subseteq \mathbb{F}_2[x^{\pm}, y^{\pm}]$$

maximal.

<u>Ex</u>: $\dim(Q_{XC}) = 1$ since

$$\mathrm{Ann}(Q_{XC}) = (t_x t_y, t_x t_z, t_y t_z) \subset \underbrace{(t_x, t_y)}_{p_0 =} \subsetneq \underbrace{(t_x, t_y, t_z)}_{p_1}$$

Haah, Ruba-Yang

Mobility Lemma:

$$\dim(M) = 0 \quad \text{iff} \quad \exists k \text{ s.t. } (x_1^k - 1, \dots, x_d^k - 1) \subset \mathrm{Ann}(M)$$

Interpretation: m can be moved (nearby) anywhere by local operators.

(topological/liquid topological)

Def^n L is **fully mobile** if $\dim Q_L^i = 0$ for all $i \geq 0$.

L is **fractonic** if $\dim Q_L^i > 0$ for some i.

(non-liquid topological)

Thm (Haah, Ruba-Yang) Let L be a Lagrangian Stab. code

$$\dim Q_L^i \leq d - 2 - i$$

In particular: $d=1$: $Q_L = 0$ <u>Note</u> $n=p$ invertible phase

toric code, Double semion ($\mathbb{Z}_4$)

$d=2$: $\dim Q_L = 0$ <u>Note</u> No fractons in 2D.

<u>Pure fractonic</u>: All submodules of Q_L have $\dim = 1$

3D Toric code

$d=3$: $\dim Q_L = 0$ fully mobile or

Hybrid X-Cube

$\dim Q_L = 1$ fractonic.

<u>Hybrid fractonic</u> Some fully mobile excitations

X-Cube, Haah's Code, Chamon, Shirley-Slagle-Chen (SSC)

Pf For any R-module (Gorenstein, equicodim. Cohen Macaulay)

$$\dim \mathrm{Ext}_R^i(M, R) \leq d - i \quad \text{for all } i$$

<u>Note</u> $P/L \hookrightarrow L^* \hookrightarrow F_1^*$

$$\dim(Q_L^i) = \dim(\mathrm{Ext}^{i+1}(P/L, R)) = \dim(\mathrm{Ext}^{i+2}(F_1^*/P/L, R)) \leq d - 2 - i.$$

∎

Stacking: Fix d and $\underset{\uparrow \text{aesthetic}}{n}$

$$\underset{m}{(L \subset P)} \oplus \underset{m'}{(L' \subset P')} = \underset{m+m'}{L \oplus L' \subset P \oplus P'}$$

Corresponds to usual stacking operation $H \otimes 1 \oplus 1 \otimes H'$

So codes form a monoid. $\boxed{Q_L \oplus Q_{L'} \simeq Q_{L \oplus L'}}$.

Ancillas: $m=1$ $\quad L_X = R_X \subset R_X \oplus R_Z = P \rightsquigarrow H = \sum_{\lambda \in \mathbb{Z}^d} X_\lambda + X_\lambda^\dagger$

trivial invertible phase

Charge complex: $R \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} R^2 \xrightarrow{[0\ 1]} R$

So $Q^i_{L_{tr}} = 0$ for all i.

Isometry: Def^n An iso. of R-modules $\alpha : P \xrightarrow{\cong} P$ s.t.

$$\Omega(\alpha(p), \alpha(q)) = \Omega(p,q)$$

is called a symplectic isomorphism.

We say that L and L' are isometric if $\exists\, \alpha : P \xrightarrow{\cong} P$ s.t. $\alpha(L) = L'$.

Example $L_X \subset R^2$ is isom. to $L_Z \subset R^2$ via the Hadamard iso $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$.

Example: Permuting qudits

Decoupled layers: let $d' < d$ and $\varphi : \underset{R_{d'}}{\mathbb{Z}^{d'}} \hookrightarrow \underset{R_d}{\mathbb{Z}^d}$ an inj. homo.

$$\mathrm{ind}_\varphi : \mathrm{Mod}(\mathbb{Z}_n[\mathbb{Z}^{d'}]) \longrightarrow \mathrm{Mod}(\mathbb{Z}_n[\mathbb{Z}^d])$$

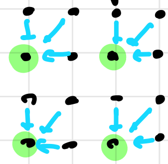
$$M \longmapsto R \otimes_{\mathbb{Z}_n[\mathbb{Z}^{d'}]} M.$$

$$P_{d'} = R^m_{d'} \oplus R^m_{d'} \longmapsto P_d = R^m_d \oplus R^m_d$$

$$L' \longmapsto \mathrm{ind}_\varphi(L)$$

Def^n L is decoupled layers if it is isometric to $\mathrm{ind}_\varphi(L')$ for some φ.

<u>Restructuring</u> A lattice transformation is a homomorphism $\phi: \mathbb{Z}^d \to \mathbb{Z}^d$ s.t. $\det(\phi) \geq 1$.



$$res_\phi : Mod(R) \longrightarrow Mod(R)$$
$$M \longmapsto res_\phi(M) \qquad r \cdot_\phi m = \phi(r) m.$$

Example: $\phi = \begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_d \end{bmatrix}$ $N_i \geq 1$: coarsening of translation symmetry.

Example: σ even permutation matrix : rotating the lattice

<u>Note</u> : $res_\phi(R^n \oplus R^n) \cong R^{\det\phi\, n} \oplus R^{\det\phi\, n}$.

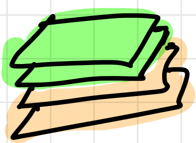
more qudits.

$res_\phi(L)$ is Lagrangian.

$res_\phi(Q_L) \cong_{ab} Q_{res_\phi(L)}$

res_ϕ is an exact functor commutes with duals

<u>Rmk</u> : If L is decoupled layers $\exists\, \phi$ s.t. $res_\phi(L) \cong L^{\oplus \det\phi}$



<u>Def</u>^{n} Fix n and d. Lagrangian stabilizer codes are in the same <u>phase</u> if they are equivalent under equivalence rel^{n} generated by:

1) isometry $L \sim \alpha(L)$
2) restructuring $res_\phi(L) \sim L$
3) Stabilization $L \sim L \oplus L_X$

Phases form a monoid under stacking.

<u>Thm</u> (Haah) Let $n = p$ be prime
- $d=1$, phases are trivial
- $d=2$, phases are $\mathbb{Z}_{\geq 0}$: gen. by 2D $\mathbb{Z}_p$-Toric code.

Stacks of TC

For $d=1$, composite n. Various Witt classification results:
Ruba Ruba-Yang, Gecko-Shuklin

<u>Thm</u> (Ellison-Chen-Dua-Shirley-Tantivasadakarn, Williamson)
$d=2$, n = composite
Double semion <u>not</u> $\cong$ toric code
Any Witt trivial (twisted quantum double) MTC is a code.

<u>Thm</u> (Ruba-Yang) $d=2$, n = composite
Any code gives rise to Witt trivial MTC.

$$0 \to R^2 \xrightarrow[\begin{bmatrix} \bar t_x & 0 \\ \bar t_y & 0 \\ 0 & t_y \\ 0 & t_x \end{bmatrix}]{d_1} \underset{}{\overset{P}{\overset{\parallel}{R^4}}} \xrightarrow[\begin{bmatrix} 0 & 0 & t_x & t_y \\ \bar t_y & \bar t_x & 0 & 0 \end{bmatrix}]{d_0} R^2 \to 0$$

exact so Lagrangian

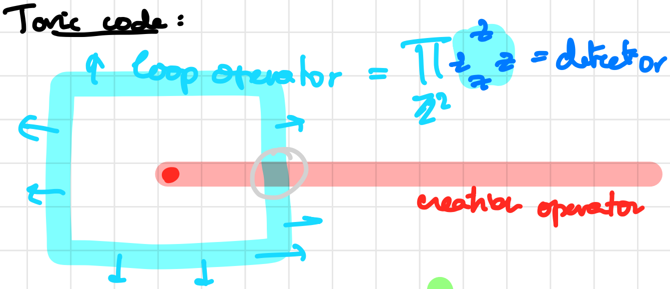
<u>Errata</u>: · Split codes are invertible phases but not known if trivial. evolved from vacuum by Clifford QCA but maybe not finite depth Quantum circuits.

· might consider phases with not all α's allowed not to kill non-trivial invertible phases. .. diff. classification

· Ising is gapped just not topological (no unique g.s.)

Part II Mutual Statistics.

Toric code:



$b(e,m) = 1 \neq 0$ $= \omega(\text{loop}, \text{creation})$

$b(e,e) = 0$

$b(m,m) = 0$

Perfect sym bil. form: $b_{TC}: Q_{TC} \times Q_{TC} \longrightarrow \mathbb{Z}_2$ mutual statistics

Any 2D system has it $(Q_L, b_L) \sim (Q_L, \theta_L)$ ← quadratic refinement.

2D theories of abelian anyons are classified by $(Q_L, \theta_L) \leftrightarrow$ MTC.

How to see braiding in homological context? The more general pairings are "excitation - detector" pairings.
? "products of stabilizer with extended supp"

Def^n

$\mapsto$ injective cogenerator $\text{Hom}(M, \tilde{R}) \neq 0$

Let $\tilde{R} = \text{Hom}(R, \mathbb{Z}_n) \cong \mathbb{Z}_n[[x_1^{\pm}, \dots, x_d^{\pm}]]$ no restriction $\sum_{\lambda \in \mathbb{Z}^d} a_\lambda x^\lambda$

For any M, $\text{Hom}(M, \mathbb{Z}_n) \cong \text{Hom}_R(M, \tilde{R})$

$L^\vee$ = extended excitations = $\text{Hom}_R(L, \tilde{R})$

Let $P \otimes_R \tilde{R}$ ← operators with ∞-support

$L \otimes_R \tilde{R}$ ← combinations of stabilizers with ∞-supp.

Exact seq^n:

$$0 \to \text{Tor}_1(P/L, \tilde{R}) \to L \otimes_R \tilde{R} \longrightarrow P \otimes_R \tilde{R} \longrightarrow P/L \otimes_R \tilde{R} \longrightarrow 0$$

Example: belongs to Tor. $\begin{bmatrix} 0 \\ 0 \\ 1+y \\ 1+x \end{bmatrix} \otimes \sum_{i,j} x^i y^j = D \mapsto 0$ $\prod$

Def$^{\underline{n}}$ The ith module of detectors

$$D_L^i := \mathrm{Tor}^R_{i+1}(P/L, \tilde{R}) \cong \mathrm{Tor}_i^R(L, \tilde{R}) \text{ for } i>0.$$

Lemma: Let M be f.g and N be injective. There is an iso

$$\mathrm{Tor}_i^R(M,N) \xrightarrow{\cong} \mathrm{Hom}(\mathrm{Ext}^i_R(M,R),N)$$
$$\text{" } m \otimes n \longmapsto \phi \mapsto \phi(m)n. \text{ "}$$

Excitation-dectector Pairing

There is an non-degenerate bilinear map

$$D_L^i \times Q_L^i \longrightarrow \mathbb{Z}_n$$

Pf: $\mathrm{Tor}^R_{i+1}(P/L, \tilde{R}) \xrightarrow{\cong} \mathrm{Hom}(\mathrm{Ext}^{i+1}_R(P/L, R), \tilde{R})$

Take constant term of the associate map $D_L^i \otimes_R Q_L^i \to \tilde{R}$.

For $i=0$: $D_L \times Q_L \longrightarrow \mathbb{Z}_n$

Rmk: (Yang-Yu) There is a map

$$D_L^i := \mathrm{Tor}^R_{i+1}(P/L, \tilde{R}) \longrightarrow Q_L^{d-2-i} \cong \mathrm{Ext}_R^{d-i-1}(P/L, R)$$

If L is fully mobile, this is an iso.

Pf Flat res. $0 \to R \to T_{d-1} \to \cdots \to T_0 \to \tilde{R} \to 0$ of $\tilde{R}$

support exclude rays

op's with supp excludes to an orthant.

Also used. dectectors as op's that create excitations whose support is at the boundary of a higher dim. strata

$$F_\bullet \otimes \tilde{R} \xleftarrow[\text{q.i}]{\cong} F_\bullet \otimes T_\bullet \xrightarrow{*} F_\bullet[d]$$

on i'th homology gives the map. If code fully mobile $*$ is iso

This gives rise to mutual statistics for fully mobile theories

Thm (Ruba-Yang) If L is fully mobile, there's a perfect pairing

$$Q_L^i \times Q_L^{d-2-i} \longrightarrow \mathbb{Z}_n$$

If $d=2$, $i=0$ this is the braiding.

<u>Mutual statistics</u> for <u>pointlike theories</u>

Recall L is pointlike if $Q_L^i = 0$ for $i \neq 0$. Only have

$$\mathrm{Tor}_1(P/L, \tilde{R}) \times Q_L \longrightarrow \mathbb{Z}_n$$

<u>Lemma</u>: If L is pointlike, $\mathrm{Tor}_1(P/L, \tilde{R}) \cong \mathrm{Tor}_2(Q_L, \tilde{R})$.

So braiding:

$$b: \mathrm{Tor}_2(Q_L, \tilde{R}) \times Q_L \longrightarrow \mathbb{Z}_n$$

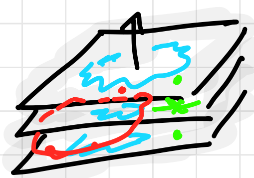
Completely described in terms of Q_L.

<u>MetaThm</u> (Shirley-Wickenden-Yang-B.-Hermele)

For pointlike theories with n odd (Q_L, b_L) is a complete invariant.

$\rightsquigarrow$ Shift perspective and study (Q_L, b_L) and its properties.

<u>Planon-only</u> <u>fracton orders</u>



study p.o codes by focusing on those

<u>Def</u>$^{\underline{n}}$ A planon-only code is a pointlike stabilizer code in $d=3$ which is in the restructured phase of a code with

1) $(1-x, 1-y) \subset \mathrm{Ann}(Q_L)$
2) Q_L is pure fractonic (no fully mobile excitations)

Fix $n=p^k$ Let $\mathbb{K} = R/(1-x,1-y) = \mathbb{Z}_{p^k}[z^{\pm}]$, $\tilde{\mathbb{K}} = \mathbb{Z}_{p^k}[[z^{\pm}]]$

Note Q_L is a $\mathbb{K}$-module. Let S_L be the charge module viewed as $\mathbb{K}$-module.

Study planar only codes by looking at codes satisfying 1) & 2)

Lemma For p.o. codes, S_L is $\mathbb{K}$-torsion free. i.e. if $rs=0$ and $p \nmid r$, then $s=0$.

Lemma: For a p.o. thy $Q_L \otimes_{\mathbb{K}} \tilde{\mathbb{K}} \cong D_L$ as $\mathbb{K}$-modules.

Pf use flat res.

Upshot: $\tilde{S}_L := Q_L \otimes_{\mathbb{K}} \tilde{\mathbb{K}}$.

Thm There is a non-degenerate bilinear pairing

$$\begin{array}{ccc} \tilde{b}: \tilde{S}_L \times S_L & \longrightarrow & \mathbb{Z}_n \\ \uparrow & \nearrow_{b} & \\ S_L \times S_L & & \end{array}$$

← nondegenerate

b satisfies:

1) (trans inv.) $b(z^k s, z^k s') = b(s,s')$ $\forall s,s'$
2) (locality) $b(s, t^k s') = 0$ if $|k| \gg 0$

Rmk (WSBH)

This is equivalent to a perfect sym. pairing:

$$B: S_L \otimes_{\mathbb{K}} S_L \to \mathbb{K}.$$

- Wickenden-Qi, Dua-Hermele
- Song-Prem-Huang, Martin-Delgado
- Pai-Hermele
- Song-Tantivasadakarn, Shirley-Hermele

Example (Shirley-Slagle-Chen SSC thy)

Describe it as pair (S,B) $\exists$ code

Let $S = \dfrac{\mathbb{Z}_4[z^{\pm}]\{m_1, m_2, e_1, e_2\}}{\substack{2e_1 = 2e_2 = 0 \\ 2m_1 = (\bar{z}-1)e_2 \quad 2m_2 = (z-1)e_1}}$

$$B = \begin{array}{c} \begin{matrix} m_1 & m_2 & e_1 & e_2 \end{matrix} \\ \begin{bmatrix} 0 & z-1 & 2 & 0 \\ \bar{z}-1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

Examples. Decoupled layers of 2D codes: induced from (A, b)

↑ with trivial

no interlayer action.

$(\mathbb{Z}_{p^k}[z^{\pm}] \otimes A,\ \mathbb{Z}_{p}^{k}[z^{\pm}] \otimes b)$

Thm (Wickenden-Shirley-B.-Hermele + Yang)

- All prime fusion order p.o. codes are decoupled layers.
- p^2, phases are generated by $\mathbb{Z}_p$-SSC. and decoupled layers

General theories: Not all phases come from codes. Need a more general framework.

Wilbur-Shirley

- Generalize the charge complex

More general forms. → Braided fusion complexes (to appear soon)

- Taking homology is destructive if not pointlike
- Quasi-iso type that matters .. derived

Plaquon-only (S, B) — S is a torsion free fin. gen. $\mathbb{K}$-mod. B is a perfect symm. bil. form. } $\mathcal{P}$

This is an exact category with duality

Witt, Grothendieck Witt, K-groups.

$$W(\mathcal{P}) = W(\mathbb{F}_p),\quad GW(\mathcal{P}) = GW(\mathbb{F}_p),\quad K(\mathcal{P}) \cong K(\mathbb{F}_p)$$

For p.o. theories, the fun is in the non-split nature so localizing invariants too destructive. Group completion + restructuring give algebraic invariants of rings that I don't recognize.... exciting and interesting

Example: Haah's Cubic Code $d=3, n=2, m=2$

$$R^2 \longrightarrow R^4 \longrightarrow R^2$$

$$\begin{bmatrix} 1+xy+xz+yz & 0 \\ 1+x+y+z & 0 \\ 0 & 1+\bar{x}+\bar{y}+\bar{z} \\ 0 & 1+\bar{x}\bar{y}+\bar{x}\bar{z}+\bar{y}\bar{z} \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1+\bar{x}\bar{y}+\bar{x}\bar{z}+\bar{y}\bar{z} & 1+\bar{x}+\bar{y}+\bar{z} \\ 1+x+y+z & 1+xy+xz+yz & 0 & 0 \end{bmatrix}$$

$$Q_{HC} = R\{a\}/(1+\bar{x}\bar{y}+\bar{x}\bar{z}+\bar{y}\bar{z},\ 1+\bar{x}+\bar{y}+\bar{z}) \oplus R\{b\}/(1+x+y+z,\ 1+xy+xz+yz)$$

(domains $x^\lambda - 1 \neq 0$)

infinite.

Type II: No planons
No lineons

Example (Chamon) $d=3, n=2, m=1$

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} x+\bar{x}+y+\bar{y} \\ y+\bar{y}+z+\bar{z} \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} y+\bar{y}+z+\bar{z} & x+\bar{x}+y+\bar{y} \end{bmatrix}} R \longrightarrow 0$$

$$Q_{Ch} = R\{e\}/(y+\bar{y}+z+\bar{z},\ x+\bar{x}+y+\bar{y})e$$

Not CSS (Calderbank-Shor-Steane)

Infinite.

Example (3D Toric Code) Kitaev/Hamma-Zanardi-Wen

fully mobile

loop + point excitations

$$d_1 = \begin{bmatrix} 1-\bar{x} & & & \\ 1-\bar{y} & & & \\ 1-\bar{z} & & & \\ & 0 & z-1 & y-1 \\ & z-1 & 0 & 1-x \\ & 1-y & 1-x & 0 \end{bmatrix}$$

Example (Shirley-Slagle-Chen, SSC or twisted 1-foliated) 3D, $\mathbb{Z}_4$-code

$$\sigma = \begin{pmatrix} 2 & d_z & d_x & d_y & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & -d_x & 0 & -d_y & 0 & \bar{d}_z & 2 \\ 0 & 0 & 2 & 0 & d_z & d_y & 0 & 0 & \bar{d}_x & 0 \\ 0 & 0 & 0 & 2 & 0 & -d_x & d_z & 0 & \bar{d}_y & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & -d_y & 0 & \bar{d}_x \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -d_x & 0 & \bar{d}_y \end{pmatrix}$$

Example (Hybrid XC – Tantivasadakarn-Ji-Vijay)
$\mathbb{Z}_4$ – 3d Point & loop excitations

$$\sigma = \begin{pmatrix} yz-y-z+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ xz-x-z+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ xy-x-y+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ -y+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ -z+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & \bar{x}-1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{y}-1 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\bar{z}+1 & -\bar{z}+1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\bar{z}+2 & 0 & 2\bar{y}+2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\bar{z}+2 & 0 & 0 & 0 & 0 & 2\bar{x}+2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\bar{y}+2 & 0 & 2\bar{x}+2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example (Double Semion)

$$\sigma = \begin{pmatrix} \bar{x}-1 & 0 & 0 & 2 \\ \bar{y}-1 & 0 & 2x & 0 \\ 1-y & 2+2y & 2 & 0 \\ x-1 & 2+2x & 0 & 2\bar{y} \end{pmatrix}$$

Ellison-Chen-Dua-Shirley-Tantivasadakarn-Williamson

Fully mobile $\mathbb{Z}_4$-code 2D code

$Q_{DS} = \mathbb{Z}_2^2$