Well-posedness for the intermediate nonlinear Schrödinger equations

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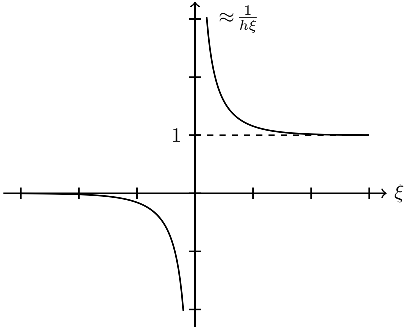
The INLS equations

- Intermediate nonlinear Schrödinger (INLS) equations:

$$i\partial_t u = -u'' \pm 2iu\Pi_{+,h}\big(|u|^2\big)',$$

$$\widehat{\Pi_{+,h}f}(\xi) = \tfrac{1}{2}(1 + \coth(h\xi))\widehat{f}(\xi)$$

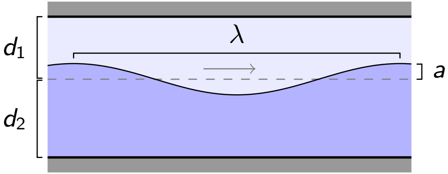


Physical origins

- Intermediate long wave (ILW) equation:

$$\partial_t v = -\frac{1}{h}\partial_x v + T_h \partial_x^2 v - 2v\partial_x v$$

$$\widehat{T_h f}(\xi) = -i\coth(h\xi)\widehat{f}(\xi)$$



- Long wavelength, small amplitude limit: $\lambda \sim \frac{d_1^2}{a} \gg 1$
- Unidirectional waves

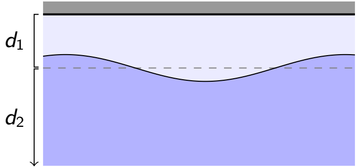
- Modulation theory for ILW [Pelinovsky '95]

$$v(t,x) = \epsilon^{-1} u(t,x) e^{i\theta(t,x;\epsilon)} + O(1) \quad \text{solves ILW}$$

$$\overset{\epsilon\to 0}{\Longrightarrow} \qquad u(t,x) \quad \text{solves defocusing INLS:}$$

$$i\partial_t u = -u'' - 2iu\Pi_{+,h}\big(|u|^2\big)', \qquad \Pi_{+,h} = \tfrac{1}{2}(1 + iT_h)$$

Infinite-depth limit



- Infinite depth limit: $d_2 \gg d_1 \implies h \to \infty$

$$\begin{aligned} \widehat{T_h f}(\xi) &= -i \coth(h\xi)\widehat{f}(\xi) \\ &\to -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi) \qquad \text{as } h \to \infty \\ &= \widehat{Hf}(\xi) \end{aligned}$$

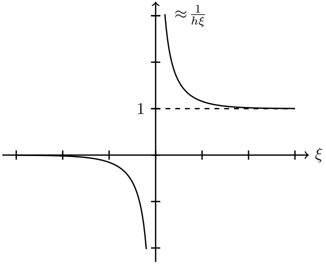
- ILW $\to$ the Benjamin–Ono equation:

$$\partial_t v = H\partial_x^2 v - 2v\partial_x v$$

$$\Pi_{+,h} = \tfrac{1}{2}(1 + iT_h) \to \tfrac{1}{2}(1 + iH) = \Pi_+ \quad \text{as } h \to \infty$$

- Cauchy–Szegő projection:

$$\widehat{\Pi_+ f}(\xi) := 1_{[0,\infty)}(\xi)\widehat{f}(\xi)$$



$$\widehat{\Pi_{+,h} f}(\xi) = \begin{cases} \tfrac{1}{2}(1 + \coth(h\xi))\widehat{f}(\xi) & \text{if } 0 < h < \infty \\ 1_{[0,\infty)}(\xi) & \text{if } h = \infty \end{cases}$$

The CCM equations

- As $h \to \infty$, INLS $\to$ continuum Calogero–Moser (CCM) equations

$$i\partial_t u = -u'' \pm 2iu\Pi_+\left(|u|^2\right)'$$

- Cauchy–Szegő projection:

$$\widehat{\Pi_+ f}(\xi) := 1_{[0,\infty)}(\xi)\widehat{f}(\xi)$$

- The Hardy space

$$L^2_+ := \{f \in L^2 : \widehat{f}(\xi) = 0 \text{ for } \xi < 0\}$$

is invariant under the flow

Physical origins

- Focusing CCM: continuum limit of Calogero–Moser particle system [Abanov–Bettelheim–Wiegmann '09]

$$x_1(t), \dots, x_n(t) \quad \text{solve CM:} \qquad \frac{d^2 x_j}{dt^2} = \sum_{k \neq j} \frac{1}{(x_j - x_k)^3}$$

$$\rightsquigarrow \quad \rho(t,x) = \sum \delta(x - x_j(t)), \;\; v(t,x) \quad \text{solve} \quad \partial_t \rho + (\rho v)' = 0$$

$$\rightsquigarrow \quad u(t,x) = \sqrt{\rho} e^{i \int^x (v + \pi\rho)\, dx} \quad \text{solves focusing CCM}$$

- "Chiral" solutions $\Longleftrightarrow u \in L^2_+$

CCM is also known as...

- Intermediate nonlinear Schrödinger equation [Pelinovsky '95, ...]
- Nonlocal derivative nonlinear Schrödinger equation [Matsuno '00, ...]
- Bidirectional Benjamin–Ono equation [Abanov–Bettelheim–Wiegmann '09]
- Calogero–Moser derivative nonlinear Schrödinger equation [Gérard–Lenzmann '24, ...]
- Calogero–Sutherland derivative nonlinear Schrödinger equation [Badreddine '24, ...]
- Continuum Calogero–Moser equation [Killip–L.–Vişan '25, ...]

Prior work

- Special families of solutions
  - defocusing INLS: dark multisolitons [Pelinovsky '95]
  - defocusing CCM: dark multisolitons [Matsuno '00, ...]
  - focusing INLS: multisolitons [Tutiya '09]
  - focusing CCM: multisolitons [Matsuno '23, ...]
- Scattering transform
  - defocusing INLS: [Pelinovsky–Grimshaw '95]
  - defocusing CCM on L^2_+: [Matsuno '04]
  - focusing CCM on L^2_+: [Frank–Read '25]

Well-posedness for defocusing CCM

Authors	Year	$H^s_+(\mathbb{R})$	$H^s_+(\mathbb{T})$
Gérard–Lenzmann	2024	$s > \frac{3}{2}$	$s > \frac{3}{2}$
de Moura	2007	$s \geq 1$	
de Moura–Pilod	2010	$s > \frac{1}{2}$	
Badreddine	2024		$s \geq 0$
Killip–L.–Vişan	2025	$s \geq 0$	

- Scaling symmetry $\implies$ critical H^s regularity is $s = 0$

Well-posedness for focusing CCM

- Om $\mathbb{R}$ and $\mathbb{T}$, focusing CCM is well-posed for

$$\|u\|_{L^2}^2 < 2\pi$$

- Solitons:

$$u(t,x) = \sqrt{\lambda} R(\lambda x + x_0), \qquad R(x) = \frac{\sqrt{2}}{x+i}$$

- Mass is

$$M(u) = \int |u|^2 \, dx = \int |R|^2 \, dx = 2\pi$$

for all $\lambda > 0$

- Pseudo-conformal solitons with mass 2π blow up in finite time, but not in $L^2_+(\mathbb{R})$ (or $H^{\frac{1}{2}}(\mathbb{R})$) [Gérard–Lenzmann '24]
- No solutions in $H^1(\mathbb{R})$ with mass 2π blow up in finite or infinite time [Gérard–Lenzmann '24]
- There exist smooth solutions in $L^2_+(\mathbb{R})$ with mass $2\pi + \epsilon$ that blow up in finite or infinite time [Hogan–Kowalski '24]
- There exist smooth solutions in $L^2_+(\mathbb{R})$ with mass $2\pi + \epsilon$ that blow up in finite time [Kim–Kim–Kwon '24]
- There exist smooth solutions in $L^2_+(\mathbb{T})$ with mass $2\pi + \epsilon$ that blow up in finite time [Chen–Lenzmann '26]

Well-posedness for INLS (and CCM)

Authors	Year	Result	$H^s(\mathbb{R})$	$H^s(\mathbb{T})$
classical energy methods		LWP	$s > \frac{3}{2}$	$s > \frac{3}{2}$
de Moura	2007	GWP	$s \geq 1$	
de Moura–Pilod	2010	LWP	$s > \frac{1}{2}$	
Barros–de Moura–Santos	2019	LWP	$B^{1/2}_{2,1}$	
Chapouto–Forlano–L.	2025	GWP	$s > \frac{1}{4}$	
Chapouto–Forlano–L.	2026	GWP		$s \geq \frac{1}{2}$

- Zhidkov spaces: $|u(x)| \to$ constant as $x \to \pm\infty$
  - defocusing CCM: [Chen '25]
  - defocusing INLS: [Akahori–Badreddine–Ibrahim–Kishimoto '25]

Main results

Theorem 1 (Chapouto–Forlano–L. '25). Fix $s > \frac{1}{4}$. For any $0 < h \leq \infty$, the INLS equations

$$i\partial_t u = -u'' \pm 2iu\Pi_{+,h}\big(|u|^2\big)'$$

are locally well-posed in $H^s(\mathbb{R})$.

- This result is new even for CCM, as $u \notin L^2_+$
- Also applies to the model

$$i\partial_t u = -u'' \pm 2iu\Pi_{+,h}\big(|u|^2\big)' + \gamma|u|^2 u, \quad \gamma \in \mathbb{R}$$

  derived in [Pelinovsky–Grimshaw '96]

- Lipschitz continuous data-to-solution map

- Gauge transform from [de Moura–Pilod '10]:

$$v = \mathbf{P}_{+,\mathrm{hi}}\left[e^{\mp 2i\int_{-\infty}^{x}|u|^2}u\right], \qquad w = \mathbf{P}_{-,\mathrm{hi}}u$$

$$i\partial_t v = -v'' \mp 2i\mathbf{P}_{+,\mathrm{hi}}\left[v\mathbf{P}_{-}(|u|^2)'\right] + \dots$$

- Bootstrap argument for $u \in X^{s-1,1} + X^{s-\frac{1}{4}-,\frac{1}{2}+}$ and $v, w \in X^{s,\frac{1}{2}+}$

- Bilinear Strichartz estimate

$$\left\| |\partial|^{\frac{1}{2}}\left(\overline{e^{it\partial^2}\phi} \cdot e^{it\partial^2}\psi\right)\right\|_{L^2_{t,x}} \lesssim \|\phi\|_{L^2_x}\|\psi\|_{L^2_x}$$

  for $\mathbf{P}_{-}(|u|^2)'$, inspired by the work [Molinet–Pilod '12] on the Benjamin–Ono equation

Theorem 2 (Chapouto–Forlano–L. '26). Fix $s \geq \frac{1}{2}$. For any $0 < h \leq \infty$, the INLS equations are locally well-posed in $H^s(\mathbb{T})$.

- Previous record was $s > \frac{3}{2}$
- In the gauge variables (v, w), the v equation has an extra term:
$$\partial_t v = \pm\tfrac{1}{\pi} M\big(|u|^2\big)\partial_x v + \dots$$
but the w equation does not
- The boost $\widetilde{v}(t,x) := v(t, x \mp \frac{1}{\pi}M(|u|^2)t)$ is not Lipschitz continuous in $u \in H^s$
- Continuous data-to-solution map (not Lipschitz)
- Bootstrap argument for $u, \widetilde{u}, \widetilde{v}, w$

Theorem 3 (Chapouto–Forlano–L. '25). Fix $\frac{1}{4} < s \leq 1$. For any $0 < h \leq \infty$, the INLS equations

$$i\partial_t u = -u'' \pm 2iu\Pi_{+,h}\big(|u|^2\big)'$$

are globally well-posed for data in $H^s(\mathbb{R})$ that is small in $L^2(\mathbb{R})$.

- Small-data assumption is necessary: focusing CCM has explicit pseudo-conformal blow-up solutions [Gérard–Lenzmann '24]

Theorem 4 (Chapouto–Forlano–L. '25). Fix $\frac{1}{4} < s \leq 1$. Given $u(0) \in H^s(\mathbb{R})$ sufficiently small, the solutions $u_h(t)$ of INLS converge as $h \to \infty$ to the solution $u_\infty(t)$ of CCM in $C(\mathbb{R}; H^s)$.

Theorem (Chapouto–Forlano–L. '26).
Theorems 3–4 also hold on $\mathbb{T}$ for $s \geq \frac{1}{2}$.

Lax pair for CCM

- $u(t)$ in L^2_+ solves CCM $\iff$ $\frac{d}{dt}L = [P, L]$ for the operators

$$L := -i\partial \mp u\Pi_+\overline{u}, \qquad P := i\partial^2 \pm 2u\Pi_+\partial\overline{u} \qquad \text{on } L^2_+$$

Proposition. For $u \in L^2_+$, the operator

$$L_u f = -i\partial f \mp u\Pi_+(\overline{u}f)$$

with domain H^1_+ is selfadjoint.

- Estimate from [Gérard–Lenzmann '24]:

$$\|\Pi_+(\overline{u}f)\|_{L^2} \lesssim \|u\|_{L^2}\|f\|_{H^{\frac{1}{2}}} \quad \text{for all } u, f \in L^2_+$$

- So:

$$\langle f, u\Pi_+(\overline{u}f)\rangle = \langle \Pi_+(\overline{u}f), \Pi_+(\overline{u}f)\rangle \lesssim \|u\|^2_{L^2}\|f\|^2_{H^{\frac{1}{2}}}$$

Lax pair for INLS

- 2×2 matrix Lax operator: [Pelinovsky–Grimshaw '95]

  Proposition (Chapouto–Forlano–L. '25). For any $0 < h \leq \infty$, $u(t)$ solves INLS on $\mathbb{R}$ if and only if the operators

  $$L := -i\partial \mp u\Pi_{+,h}\overline{u}, \qquad P := i\partial^2 \pm 2u\Pi_{+,h}\partial\overline{u} \qquad \text{on } L^2$$

  satisfy $\frac{d}{dt}L = [P, L]$.

- New Lax pair for INLS, resembles that of CCM
- This is "new" for CCM, as $u \notin L^2_+$
- For CCM, eigenvalue equation for L is formally equivalent to that for the 2×2 matrix Lax operator [J. Chen–Pelinovsky '25]
- Lax pair for CCM on L^2 in gauged variables: [Kim–Kim–Kwon '24]

- If $u(t)$ solves INLS, then

$$\frac{d}{dt}L = [P, L], \qquad \frac{d}{dt}u = Pu$$

- L is selfadjoint, P is anti-selfadjoint

Lemma. If $F : \mathbb{R} \to \mathbb{R}$ is a smooth bounded function, then

$$t \mapsto \left\langle u(t), F(L_{u(t)})u(t) \right\rangle$$

is conserved.

Proof:

$$\frac{d}{dt}\langle u, F(L)u \rangle = \left\langle Pu, F(L)u \right\rangle + \left\langle u, [P, F(L)]u \right\rangle + \left\langle u, F(L)Pu \right\rangle$$

$$= 0 \qquad \square$$

- $F(L) = 1$: Mass

$$M(u) = \langle u, u \rangle = \int |u|^2 \, dx$$

- $F(L) = L$: Momentum

$$P(u) = \langle u, Lu \rangle = \int \overline{u}\big(-iu' \mp u\Pi_{+,h}\big(|u|^2\big)\big) \, dx$$

- $F(L) = L^2$: Hamiltonian

$$H(u) = \langle u, L^2 u \rangle = \tfrac{1}{2} \int \big|-iu' \mp u\Pi_{+,h}\big(|u|^2\big)\big|^2 \, dx$$

$$\vdots$$

- $F(L) = (L^2 + \kappa^2)^s$: a-priori estimates in H^s for $\frac{1}{4} \leq s \leq 1$

$$\langle u, (L_u^2 + \kappa^2)^s u \rangle \approx \langle u, (L_0^2 + \kappa^2)^s u \rangle$$

- [Killip–L.–Vișan '25]: For CCM on L^2_+, can use

$$\langle u, (L_u + \kappa)^{2s} u \rangle \approx \langle u, (L_0 + \kappa)^{2s} u \rangle$$

since $L_0 := -i\partial \geq 0$ on L^2_+

- If $0 \leq s \leq 1$, then

$$0 \leq A \leq B \quad \Longrightarrow \quad A^s \leq B^s$$

Proposition. For $u \in H^{\frac{1}{4}}$, the operator

$$L_u f = -i\partial f \mp u\Pi_{+,h}(\overline{u}f)$$

with domain H^1 is selfadjoint, and

$$\frac{1}{2}(L_0^2 + \kappa^2) \leq L_u^2 + \kappa^2 \leq \frac{3}{2}(L_0^2 + \kappa^2)$$

as quadratic forms for all $\kappa \gg 1$.

- For high frequencies, we have $\Pi_{+,h} \approx \Pi_+$:

$$\|u\Pi_+(\overline{u}f)\|_{L^2} \lesssim \|u\|_{L^4}^2 \|f\|_{L^\infty} \lesssim \|u\|_{H^{\frac{1}{4}}}^2 \|f\|_{H^1}$$

- For low frequencies, we have $\Pi_{+,h} \approx i\partial^{-1}$:

$$\|u\partial^{-1}(\overline{u}f)\|_{L^2} \leq \|u\|_{L^2}\|\partial^{-1}(\overline{u}f)\|_{L^\infty} \leq \|u\|_{L^2}\|\overline{u}f\|_{L^1} \leq \|u\|_{L^2}^2\|f\|_{L^2}$$

Circle case

Proposition (Chapouto–Forlano–L. '26). For any $0 < h \leq \infty$ and $c \in \mathbb{R}$, $u(t)$ solves INLS on $\mathbb{T}$ if and only if the operators

$$L := -i\partial \mp u(\Pi_{+,h} + c\mathbf{P}_0)\overline{u}, \qquad P := i\partial^2 \pm 2u\Pi_{+,h}\partial\overline{u} \qquad \text{on } L^2$$

satisfy

$$\frac{d}{dt}Lf = [P, L]f + \tfrac{1}{2}u\big\langle(1 + T_h^2)\big(|u|^2\big)', \overline{u}f\big\rangle + \tfrac{1}{2}u(1 + T_h^2)\big(|u|^2\big)'\langle u, f\rangle.$$

- For $h = \infty$, this is a Lax pair. It is "new", as $u \notin L^2_+$.
- A-priori estimates in H^s for $\frac{1}{2} \leq s < 1$: Use $h = \infty$ Lax pair and a Gronwall argument

Thank you!