Semiclassical Soliton Ensembles for the Intermediate Long Wave[1] and Korteweg-de Vries[2] Equations

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Dispersive Integrable Equations
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[1] arXiv:2602.03063
[2] Joint work with K. Schmidt (UCF) and R. Buckingham (UC)
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Contents and Organization

Equations of Interest

The intermediate long wave (ILW) equation are nonlinear PDE in 1+1 dimensions which come from asymptotic models of fluid interface dynamics [5, 8, 2, 16]:

$$0 = u_t + 2uu_x + \epsilon \mathcal{T}_{\delta\epsilon}[u_{xx}]\,, \tag{1}$$

where

$$\mathcal{T}_{\delta\epsilon}[f](x) := \frac{1}{2\delta\epsilon}\,\mathrm{P.V.}\int_{\mathbb{R}} \left[\coth\left(\frac{\pi}{2\delta\epsilon}(y-x)\right) - \operatorname{sgn}(y-x)\right] f(y)\,\mathrm{d}y\,. \tag{2}$$

The parameter $\delta > 0$ is a fixed constant determined by the depth of the fluid system. In the limit $\delta \to 0$, the ILW equation recovers the Korteweg de-Vries (KdV) equation [7]

$$0 = u_t + 2uu_x + \frac{\epsilon^2}{3} u_{xxx}\,. \tag{3}$$

The parameter $\epsilon > 0$ in both equations controls the amount of linear dispersion.

Zero Dispersion Equation: Invicid Burgers'

Setting $\epsilon = 0$ in both equations recovers invicid Burgers' equation,

$$0 = u_t^{\mathrm{B}} + 2u^{\mathrm{B}} u_x^{\mathrm{B}}\,, \tag{4}$$

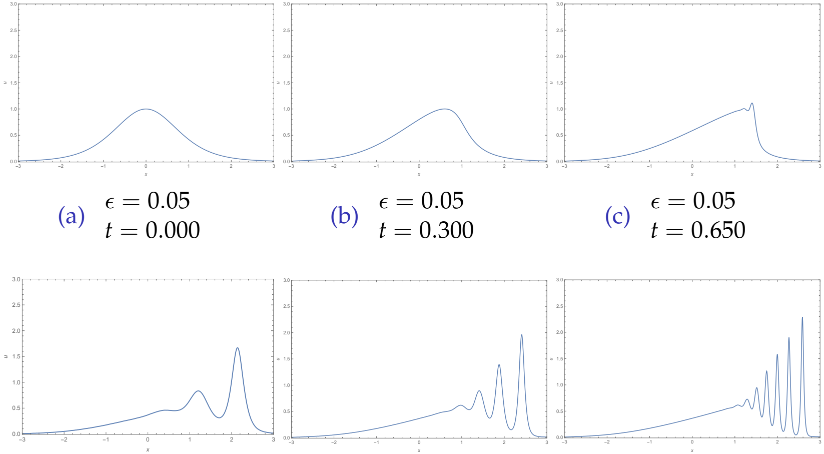
whose Cauchy problem for initial condition u_0 is solvable by the method of characteristics:

$$u^{\mathrm{B}}(x,t) = u_0(y) \quad \text{where} \quad x = y + 2u_0(y)t\,, \tag{5}$$

up to a critical time of gradient catastrophe

$$t_{\mathrm{c}} := \frac{1}{\max\limits_{x\in\mathbb{R}} -2u_0'(x)}\,. \tag{6}$$

Small-Dispersion Limit



(a) $\epsilon = 0.05$, $t = 0.000$

(b) $\epsilon = 0.05$, $t = 0.300$

(c) $\epsilon = 0.05$, $t = 0.650$

(d) $\epsilon = 0.20$, $t = 1.500$

(e) $\epsilon = 0.10$, $t = 1.500$

(f) $\epsilon = 0.05$, $t = 1.500$

Figure: Split-step simulation of the ILW eq. in the small-dispersion limit.

Inverse Scattering: Semiclassical Soliton Ensembles

Both equations have Lax pairs [3, 6] and are integrable by inverse scattering transforms

$$u(x,t) \quad \leftrightarrow \quad \left\{ \begin{array}{cl} \kappa_n & \text{(eigenvalues),} \\ c_n(t) & \text{(norming constants),} \\ r(k,t) & \text{(reflection coefficient)} \end{array} \right\} \tag{7}$$

For positive, bell-shaped initial conditions, as $\epsilon \to 0$, the number of eigenvalues (solitons) grows and the reflection coefficient is suppressed.

Definition (Semiclassical soliton ensemble (SSE))

For $N \in \mathbb{N}$, a sequence of scattering data with identically zero reflection coefficient is considered. Typically the eigenvalues and norming constants $\{\kappa_n, c_n\}_{n=1}^{N}$ are chosen from some distributions. The solution associated to this scattering data $u_N(x,t)$ is considered for a sequence of ϵ_N which goes to zero as $N \to \infty$.

Intermediate Long Wave

ILW: Formal Direct Scattering

The scattering equation [6] in the ILW Lax pair with spectral parameter $\zeta \in \mathbb{C}$ can be written as

$$\mathrm{i}\epsilon\Phi_x^+(x) + u(x)\Phi^+(x) = \zeta \mathrm{e}^{-2\delta\zeta}\Phi^-(x)\,. \tag{8}$$

where Φ is analytic on the strip $|\mathrm{Im}\,[z]\,| < \delta\epsilon$ and $\Phi^\pm$ are the lower/upper boundary values.
Eigenvalues ζ_n are when normalizable wavefunctions Φ_n exist and occur on the quadratrix [1] parameterized by $\zeta(\kappa) := \kappa\cot(2\delta\kappa) + \mathrm{i}\kappa$.

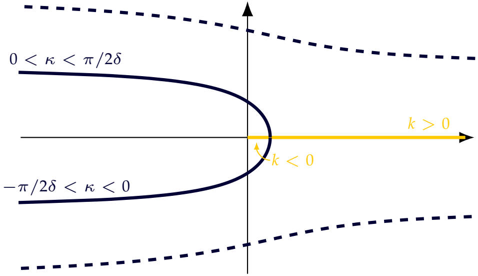


Figure: The proposed ILW scattering equation spectrum in the ζ plane.

ILW: Formal Direct Scattering (cont.)

Let $u(x) = U > 0$ and $\Phi(z) = \mathrm{e}^{-\mathrm{i}\eta z/\epsilon}$, then the scattering equation reduces to

$$\eta + U = \zeta \mathrm{e}^{-2\delta(\zeta - \eta)} \quad \implies \quad \eta_n = -U - \frac{1}{2\delta} W_n\left(-2\delta\zeta \mathrm{e}^{-2\delta(\zeta+U)}\right) \tag{9}$$

where W_n is the n^{th} branch of the Lambert W function [14, Section 4.13].

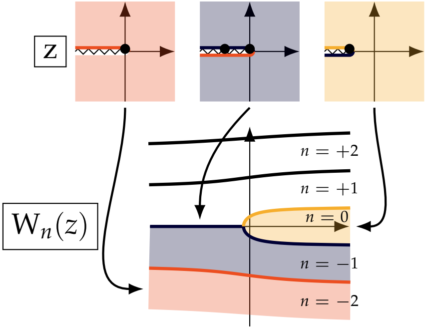


Figure: $z \mapsto W_n(z)$ demonstrated for three characteristic branches:

ILW: Asymptotic Phase

Conducting WKB, we let $\Phi(z) = A(z)\mathrm{e}^{-\mathrm{i}\theta(z)/\epsilon}(1 + \mathcal{O}(\epsilon))$. Plugging in and solving the series term-by-term gives

$$\theta_x(x) = -U - \frac{1}{2\delta} W_n\left(-\mathrm{e}^{-2\delta(u(x)-E(\zeta))-1}\right), \quad E(\zeta) := -\zeta + \frac{1}{2\delta}\left(1 + \log(2\delta\zeta)\right)$$

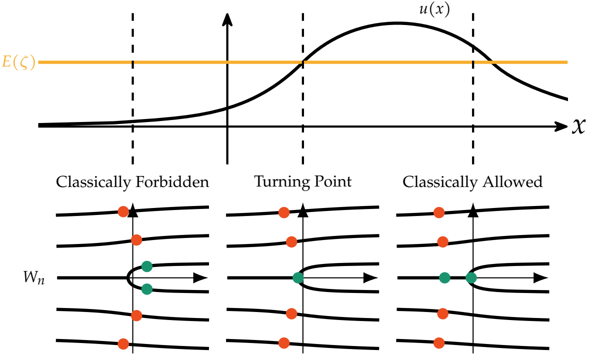


Figure: The values for the extraneous solutions ($n \neq -1, 0$) are shown in red and those for the scattering solutions ($n = -1, 0$) are shown in green.

Definition (The Modified Scattering Data)

Let $u_0 : \mathbb{R} \to \mathbb{R}^+$ be an admissible initial condition. Define the turning points $u_0(x_\pm(\zeta)) = E(\zeta)$ and functions

$$R^{\mathrm{Weyl}}(\zeta) := \frac{1}{4\delta} \int_{x_-(\zeta)}^{x_+(\zeta)} W_0 \left(-2\delta\zeta \mathrm{e}^{-2\delta(\zeta+u_0(x))}\right) - W_{-1}\left(-2\delta\zeta \mathrm{e}^{-2\delta(\zeta+u_0(x))}\right) \mathrm{d}x \,, \quad (10)$$

$$\theta_\pm(\zeta) := \mathrm{Im}\,[\zeta]\, x_\pm(\zeta) + \int_{x_\pm(\zeta)}^{\pm\infty} \mathrm{Im}\left[\zeta + \frac{1}{2\delta} W_{-1}\left(-2\delta\zeta \mathrm{e}^{-2\delta(\zeta+u_0(x))}\right)\right] \mathrm{d}x \,, \quad (11)$$

and the sequence $\epsilon_N = R^{\mathrm{Weyl}}(1/2\delta)/\pi N$. Then we define the set of distinct eigenvalues $\{\zeta_n\}_{n=1}^N$ by $R^{\mathrm{Weyl}}(\zeta_n) = \pi\epsilon_N(n-1/2)$ and associated norming constants $c_n = \mathrm{e}^{-2\theta_+(\zeta_n)/\epsilon_N}$ for all $1 \le n \le N$. Lastly, we take the reflection coefficient to be identically zero.

The formal WKB argument was first done in [12], but our formula for the norming constants is entirely new. The rigorous study of this SSE follows the analysis [9, 10] closely.

ILW: N-Soliton Formula

[6] gives an ILW N-soliton formula for zero reflection coefficient[4]

$$u_N^{\mathrm{SSE}}(x,t) = -\mathrm{i}\epsilon_N \partial_x \log\det\left(\frac{I_N + \Delta_N(x+\mathrm{i}\delta\epsilon_N, t; \epsilon_N)}{I_N + \Delta_N(x-\mathrm{i}\delta\epsilon_N, t; \epsilon_N)}\right) \tag{12}$$

where I_N is the $N \times N$ identity and Δ_N is a matrix with the structure

$$\Delta_N(z,t;\epsilon_N) = D_N(z,t;\epsilon_N)\, C_N\, D_N(z,t;\epsilon_N) \tag{13}$$

where D_N is a diagonal matrix with elements

$$[D_N(z,t;\epsilon)]_{n\,n} := c_n^{1/2} \exp\left(-\mathrm{Im}\,[\zeta_n]\,\frac{z}{\epsilon} - \mathrm{Im}\left[\left(\zeta_n - \frac{1}{2\delta}\right)^2\right]\frac{t}{\epsilon}\right) \tag{14}$$

for $1 \le n \le N$ and C_N is a Cauchy matrix which only depends on the eigenvalues

$$[C_N]_{n\ell} = \frac{\mathrm{i}}{\zeta_n - \zeta_\ell^*} \text{ for } 1 \le n, \ell \le N\,. \tag{15}$$

[4] For proof that (12) solves (1) with $\epsilon = \epsilon_N$, see [11].

ILW: SSE and Equilibrium Measure

Theorem (3.4 and 3.5 [13])

For any $t \in \mathbb{R}$, the distributional limit of (12) *is given by*

$$d - \lim_{N\to\infty} u_N^{\mathrm{SSE}}(\diamond, t) = -\frac{4\delta}{\pi}\partial_x^2 E_\infty^{\min}(\diamond, t) \tag{16}$$

where $E_\infty^{\min}(x,t) := \min_{\rho\in\mathcal{A}} \mathcal{E}[\rho^{\min}](x,t)$ *and $\mathcal{A}$ is the admissible set of densities* $(0 \leq \rho(\kappa) \leq \rho^{\mathrm{Weyl}}(\kappa) := -\frac{d}{d\kappa}R^{\mathrm{Weyl}}(\zeta(\kappa)))$ *and*

$$\mathcal{E}[\rho](x,t) := \int_0^{\kappa_{\max}} \left(V(\zeta(\kappa); x, t) + \frac{1}{2}\mathcal{L}[\rho](\zeta(\kappa)) \right) \rho(\kappa)\, d\kappa\,, \tag{17}$$

$$V(\zeta; z, t) := z\,\mathrm{Im}\,[\zeta] + t\,\mathrm{Im}\left[\left(\zeta - \frac{1}{2\delta}\right)^2\right] - \theta_+(\zeta)\,, \tag{18}$$

$$\mathcal{L}[\rho](\lambda) := \frac{1}{\pi}\int_0^{\kappa_{\max}} \log\left|\frac{\lambda - \zeta(\kappa)^*}{\lambda - \zeta(\kappa)}\right| \rho(\kappa)\, d\kappa\,, \tag{19}$$

ILW: Variational Conditions

Theorem (3.7 [13])

For each (x, t), the minimizing density $\rho^{\min}(\kappa; x, t) \in \mathcal{A}$ is uniquely determined by the variational conditions

$$\begin{cases} \rho^{\min} = 0 & \text{if } V(\diamond; x, t) + \mathcal{L}[\rho^{\min}] > 0, & \textit{(voids)} \\ 0 \leq \rho^{\min} \leq \rho^{\mathrm{Weyl}} & \text{if } V(\diamond; x, t) + \mathcal{L}[\rho^{\min}] = 0, & \textit{(bands)} \\ \rho^{\min} = \rho^{\mathrm{Weyl}} & \text{if } V(\diamond; x, t) + \mathcal{L}[\rho^{\min}] < 0, & \textit{(saturations)} \end{cases} . \quad (20)$$

Naïevely taking derivatives in x and t gives

$$\begin{cases} \mathcal{L}[\rho_x^{\min}](\zeta) = -\mathrm{Im}\,[\zeta] & \text{if } \zeta \text{ is in a band}\,, \\ \rho_x^{\min}(\kappa) = 0 & \text{if } \kappa \text{ is not in a band}\,, \end{cases} \quad (21)$$

$$\begin{cases} \mathcal{L}[\rho_t^{\min}](\zeta) = -\mathrm{Im}\left[\left(\zeta - \frac{1}{2\delta}\right)^2\right] & \text{if } \zeta \text{ is in a band}\,, \\ \rho_t^{\min}(\kappa) = 0 & \text{if } \kappa \text{ is not in a band}\,. \end{cases} \quad (22)$$

ILW: One Band Ansatz

Under the ansatz of a single continuous band, we have

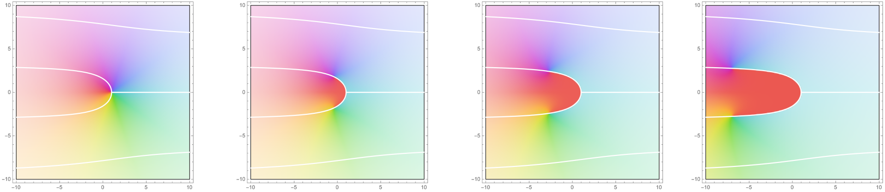
Lemma (3.8 [13])

If ρ' is Hölder continuous with power $-1/2$ and has logarithmic potential $\mathcal{L}[\rho']$ which satisfies the differentiated variational conditions (21) *or* (22)*, then the logarithmic potential $\mathcal{L}[\rho']$ is uniquely identified by the properties:*

1. *harmonic on $\mathbb{C} \setminus I$ and continuous on $\mathbb{C}$,*
2. *the value on I is dictated by the differentiated variational conditions* (21) *or* (22)*,*
3. *odd symmetry under reflection across the real axis: $\mathcal{L}[\rho'](\zeta) = -\mathcal{L}[\rho'](\zeta^*)$,*
4. *decay at infinity $\mathcal{L}[\rho'](\zeta) = \mathcal{O}\left({}^{1}/{\zeta}\right)$ as $\zeta \to \infty$.*

Additionally, the density ρ' can be extracted from $\mathcal{L}[\rho']$ [15].

$W(-ze^{-z-y})$ for $y > 0$ where for each $-z$ in a branch-region, we use the corresponding branch of Lambert W.



(a) $y = 0$ (b) $y = 2$ (c) $y = 5$ (d) $y = 10$

Figure: Complex plots of $W(-ze^{-z-y}) + 1$ for different y values. Color denotes the argument with red: 0 and cyan: $\pm\pi$.

It has a branch cut on $[\beta(y)^*, \beta(y)]$ where the boundary values are real. $W(-ze^{-z-y}) = -z - y - {}^{y}\!/_{z} + \mathcal{O}\left({}^{1}\!/_{z^2}\right)$ as $z \to \infty$.

ILW: One Band Ansatz (cont.)

For any sized interval there is a $y \leq 0$ such that the expressions

$$\mathcal{L}[\rho_x^{\min}](\zeta) = -\mathrm{Im}\left[\zeta + \frac{y}{2\delta} + \frac{1}{2\delta}W\left(-2\delta\zeta \mathrm{e}^{-2\delta\zeta - y}\right)\right], \tag{23}$$

$$\mathcal{L}[\rho_t^{\min}](\zeta) = -\mathrm{Im}\left[\left(\zeta - \frac{1}{2\delta}\right)^2 + \frac{y}{2\delta^2} - \left(\frac{y + 1 + W\left(-2\delta\zeta \mathrm{e}^{-2\delta\zeta - y}\right)}{2\delta}\right)^2\right], \tag{24}$$

solve the differentiated variational conditions (21) and (22). Consistent integration in x and t requires

$$0 = \partial_x \mathcal{L}[\rho_t^{\min}](\zeta) - \partial_t \mathcal{L}[\rho_x^{\min}](\zeta) = \left(\frac{y_t}{2\delta} + 2\frac{y_x}{2\delta}\frac{y}{2\delta}\right)\mathrm{Im}\left[\frac{1}{1 + W\left(-2\delta\zeta \mathrm{e}^{-2\delta\zeta - y}\right)}\right]. \tag{25}$$

Applying Theorem 3.5 [13], the distributional limit is

$$\mathrm{d} - \lim_{N\to\infty} u_N^{\mathrm{SSE}}(x, t) = \frac{y}{2\delta}. \tag{26}$$

Setting $y = 2\delta u^{\mathrm{B}}$ and integrating in x from $+\infty$, we show that the result solves the variational conditions (20) [13, Theorem 3.9].

ILW: The main result (cont.)

With a little bit more work, relying on theorems from [9, 10], we can upgrade the distributional limit to an L^2 one.

Theorem (1.1 [13])

Let u_0 be an admissible initial condition and $u_N^{\mathrm{SSE}}(\diamond, t)$ be the associated semiclassical soliton ensemble. Furthermore, let u^{B} denote the solution to invicid Burgers' equation (4) with initial condition $u^{\mathrm{B}}(x,0) = u_0(x)$. Recall that this is only globally well-defined up until the time of gradient catastrophe t_{c} (6). Then as $N \to +\infty$,

$$\lim_{N\to+\infty} u_N^{\mathrm{SSE}}(\diamond, t) = u^{\mathrm{B}}(\diamond, t) \tag{27}$$

converging in $L^2(\mathbb{R})$ norm uniformly for all $0 \le t < t_{\mathrm{c}}$.

Beyond break time, the equilibrium measure should have more bands. Constructing the solution is still an open problem.

Korteweg-de Vries

KdV: Exact Direct Scattering

The KdV direct scattering problem is the Schrodinger equation

$$0 = \epsilon^2 \psi_{xx} + (u_0 + \lambda)\psi \,, \tag{28}$$

The continuous spectrum occurs when $k \in \mathbb{R}$ and $\lambda = k^2$ while the discrete spectrum is for $\kappa \in \mathbb{R}$ and $\lambda = -\kappa^2$.
It is well-known that for the specific potential $u_0(x) = \operatorname{sech}^2(x)$ and the sequence $\epsilon_N = 1/\sqrt{N(N+1)}$, there are exactly N eigenvalues and connection coefficients

$$\kappa_n = \epsilon_N n \qquad \text{and} \qquad b_n = (-1)^{N-n} \,, \tag{29}$$

for $1 \leq n \leq N$ and the reflection coefficient is identically zero.
This is an exact SSE!

KdV: Riemann-Hilbert Analysis

$\boldsymbol{m}(z) \equiv \boldsymbol{m}(z; x, t, N)$ is a 2-dim row vector meromorphic in $\mathbb{C}$ with large z limit

$$\lim_{z\to\infty} \boldsymbol{m}_N(z) = [\,1\;1\,]\,. \tag{30}$$

$\boldsymbol{m}$ has simple poles at the points $\{\pm i\kappa_n\}_{n=1}^N$, and $\boldsymbol{m}_N$ satisfies the residue conditions

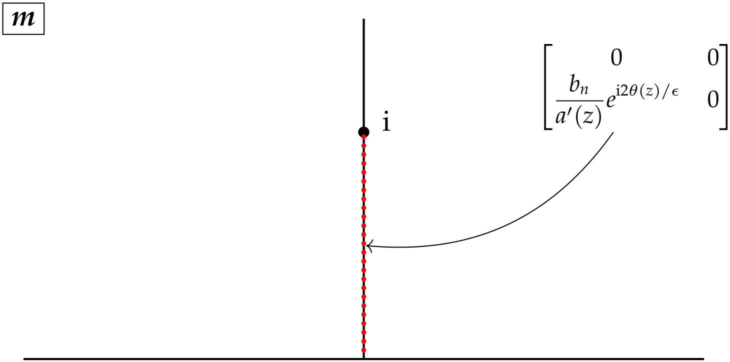
$$\operatorname*{res}_{z=i\kappa_n} \boldsymbol{m}_N(z) = \lim_{z\to i\kappa_n} \boldsymbol{m}_N(z) \begin{bmatrix} 0 & 0 \\ \dfrac{b_n}{a'(z)} e^{i2\theta(z;x,t)/\epsilon} & 0 \end{bmatrix} \tag{31}$$

$$\operatorname*{res}_{z=-i\kappa_n} \boldsymbol{m}(z) = \lim_{z\to -i\kappa_n} \boldsymbol{m}_N(z) \begin{bmatrix} 0 & \dfrac{b_n}{\overline{a}'(z)} e^{-i2\theta(z;x,t)/\epsilon} \\ 0 & 0 \end{bmatrix} \tag{32}$$

where a and $\overline{a}$ are the Blaschke factors $a(z) = \displaystyle\prod_{n=1}^{N} \frac{z - i\kappa_n}{z + i\kappa_n} = \frac{1}{\overline{a}(z)}$ and

$\theta(z; x, t) := zx + 4z^3 t/3.$

KdV: Riemann-Hilbert Analysis (cont.)



Using the following, we can multiply by meromorphic matrices to remove the poles

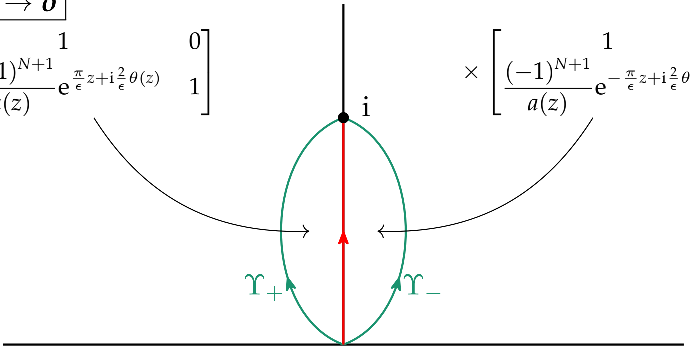
$$b_n = (-1)^{N-n} = (-1)^N \mathrm{e}^{\pm \mathrm{i}\pi n} = (-1)^N \mathrm{e}^{\pm \pi \mathrm{i} \kappa_n / \epsilon}\,. \tag{33}$$

(Everything in the lower half plane is handled by the essential symmetry $\boldsymbol{m}(-z) = \boldsymbol{m}(z)\sigma_1$.)

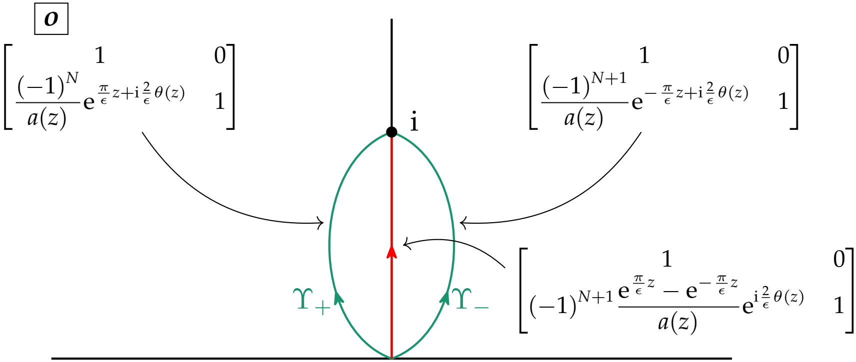
$\boxed{\boldsymbol{m} \to \boldsymbol{o}}$

$$\times \begin{bmatrix} 1 & 0 \\ \dfrac{(-1)^{N+1}}{a(z)} \mathrm{e}^{\frac{\pi}{\epsilon} z + \mathrm{i} \frac{2}{\epsilon} \theta(z)} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 \\ \dfrac{(-1)^{N+1}}{a(z)} \mathrm{e}^{-\frac{\pi}{\epsilon} z + \mathrm{i} \frac{2}{\epsilon} \theta(z)} & 1 \end{bmatrix}$$



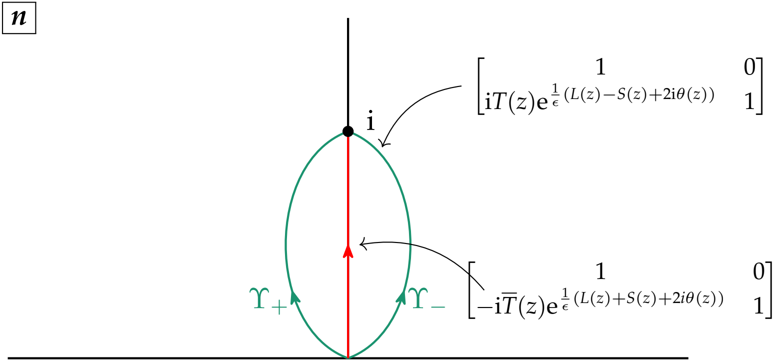
Multiplying $\boldsymbol{m}$ by the above matrices in the above regions yields the new RHP for the new sectionally-analytic unknown $\boldsymbol{o}$ with “jumps” on the boundaries of the regions.



Asymptotics of a can be computed using Γ functions both on and off axis

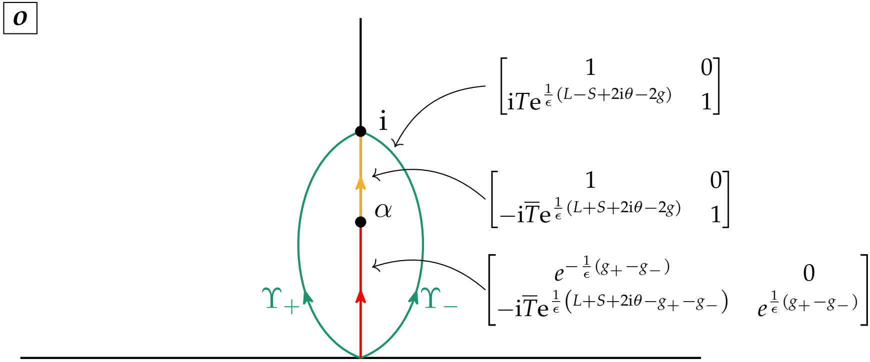
$$a(z) = \exp\left[-\frac{1}{\epsilon}L(z) + O(\epsilon)\right], \qquad L(z) = -\mathrm{i}z \log\left(\frac{1+z^2}{z^2}\right) + \log\left(\frac{-\mathrm{i}z+1}{-\mathrm{i}z-1}\right). \quad (34)$$

KdV: Riemann-Hilbert Analysis (cont.)



T and $\overline{T}$ are $1+\mathcal{O}(\epsilon)$ functions, we've defined
$S(z) := \operatorname{sgn}(\operatorname{Re}[z])\pi(z-\mathrm{i})$ and $L+S$ is real and analytic on $(-\mathrm{i},\mathrm{i})$.

KdV: Riemann-Hilbert Analysis, Genus-0



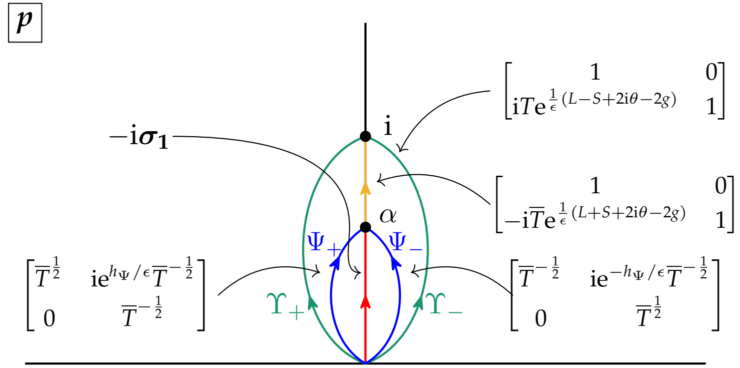
Inserting a g-function $\boldsymbol{o}(z) := \boldsymbol{n}(z)\mathrm{e}^{-\sigma_3 g(z)/\epsilon}$ such that g is cut on $(-\alpha, \alpha)$, odd, and has jump

$$g_+(z) + g_-(z) = 2\mathrm{i}\theta(z) + L(z) + S(z) \text{ for } z \in (-\alpha, \alpha)\,. \tag{35}$$

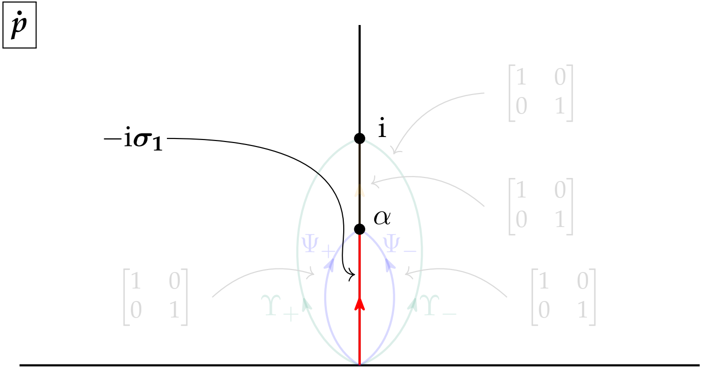
This can be solved exactly yielding condition

$$-\alpha^2 = \operatorname{sech}^2\left(x + 2\alpha^2 t\right) = u^{\mathrm{B}}(x,t)\,. \tag{36}$$

KdV: Riemann-Hilbert Analysis, Genus-0 (cont.)



Now we three-factor the jump on $(0, \alpha)$ and, multiplying by the outside analytic matrices on the left and on the right, we “open the lens” to get the RHP for another unknown $\boldsymbol{p}$. Here, $g_+ - g_-$ admits an analytic continuation h_Ψ.



Replacing every jump with its limit, we generate the "model problem" for $\dot{\boldsymbol{p}}$ which determines the leading order asymptotics of the solution to the RHP for $\boldsymbol{p}$.

KdV: Riemann-Hilbert Analysis, Genus-0 (cont.)

After throwing away the jumps that are near identity, then it satisfies

$$\dot{\boldsymbol{p}}_+(z) = \dot{\boldsymbol{p}}_-(z) \begin{cases} -\mathrm{i}\boldsymbol{\sigma_1} & \text{if } z \in (0,\alpha) \\ \mathrm{i}\boldsymbol{\sigma_1} & \text{if } z \in (-\alpha,0) \end{cases} . \tag{37}$$

The solution can be written as

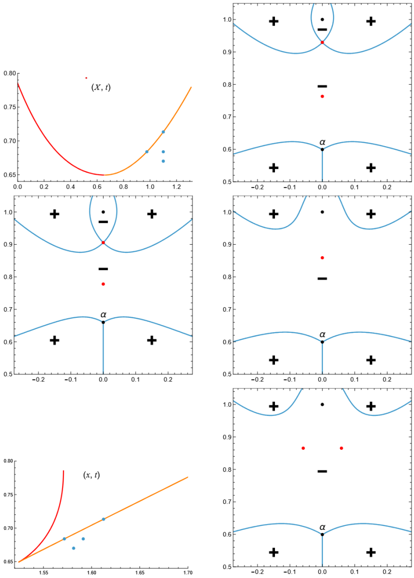
$$\dot{\boldsymbol{p}}(z) = \left(\frac{z^2}{z^2-\alpha^2}\right)^{1/4} \begin{bmatrix} 1 & 1 \end{bmatrix} . \tag{38}$$

By undoing the explicit series of transformations $\dot{\boldsymbol{p}} \approx \boldsymbol{p} \to \boldsymbol{o} \to \boldsymbol{n} \to \boldsymbol{m}$, the leading-order solution is given by the extraction formula

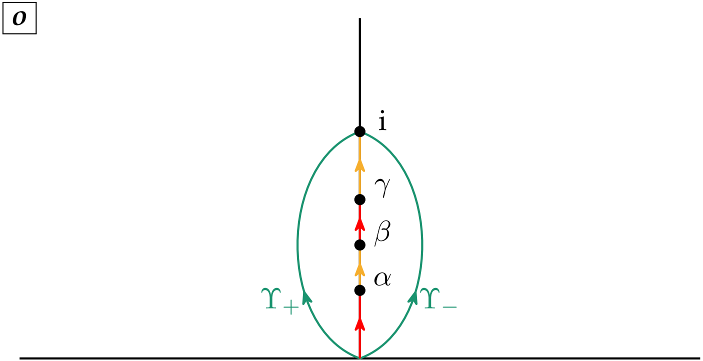
$$u(x,t) = \lim_{z\to\infty} 2z^2\big(1 - m_1(z;x,t)m_2(z;x,t)\big) \approx -\alpha^2 = u^{\mathrm{B}}(x,t) , \tag{39}$$

the solution to invicid Burgers' equation!

KdV: Riemann-Hilbert Analysis, Breaking



KdV: Riemann-Hilbert Analysis, Genus 1



Returning to $\boldsymbol{o}$ and inserting a genus-1 G-function yields the jump conditions for some constant $\omega \in \mathbb{R}$

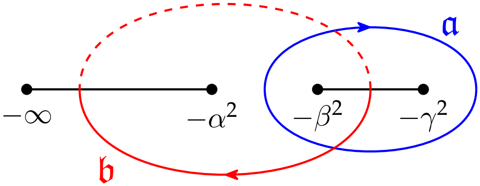
$$G_+(z) + G_-(z) = 2\mathrm{i}\theta(z) + L(z) + S(z) \qquad z \in [-\alpha, \alpha] \tag{40}$$

$$G_+(z) - G_-(z) = \mathrm{i}\omega \qquad z \in [-\beta, -\alpha] \cup [\alpha, \beta] \tag{41}$$

$$G_+(z) + G_-(z) = 2\mathrm{i}\theta(z) + L(z) + S(z) \qquad z \in [-\gamma, -\beta] \cup [\beta, \gamma], \tag{42}$$

KdV: Riemann-Hilbert Analysis, Genus-1

Due to the symmetry, we fold up the plane and get two bands, leading us to define the following genus-1 Riemann surface:



The leading order solution to KdV is extracted from the RHP model problem as

$$u(x,t) \approx \beta^2 - \alpha^2 - \gamma^2 - \frac{\alpha^2 - \gamma^2}{8K(m)^2}\left(\frac{\Theta''(0)}{\Theta(0)^2} - (\partial^2 \log \Theta)\left(\frac{\mathrm{i}\Delta}{2\pi\epsilon}\right)\right) \quad (43)$$

with $\frac{\mathrm{i}\Delta}{2\pi\epsilon} = \frac{\mathrm{i}\sqrt{\alpha^2-\gamma^2}}{2K(m)}\left(x + \frac{4}{3}(\alpha^2+\beta^2+\gamma^2)t - x_0\right)$.

KdV: Conclusion and Remaining Work

- Rigorous error analysis
  - requires the construction of a matrix solution to the model problem
  - easy in the genus-0 case
  - for genus-1, we believe it can be done following the techniques of [4]
- Work out all the details regarding “triangularity flipping” in the genus-1 region
- Demonstrate that there is no further breaking beyond genus-1

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Questions