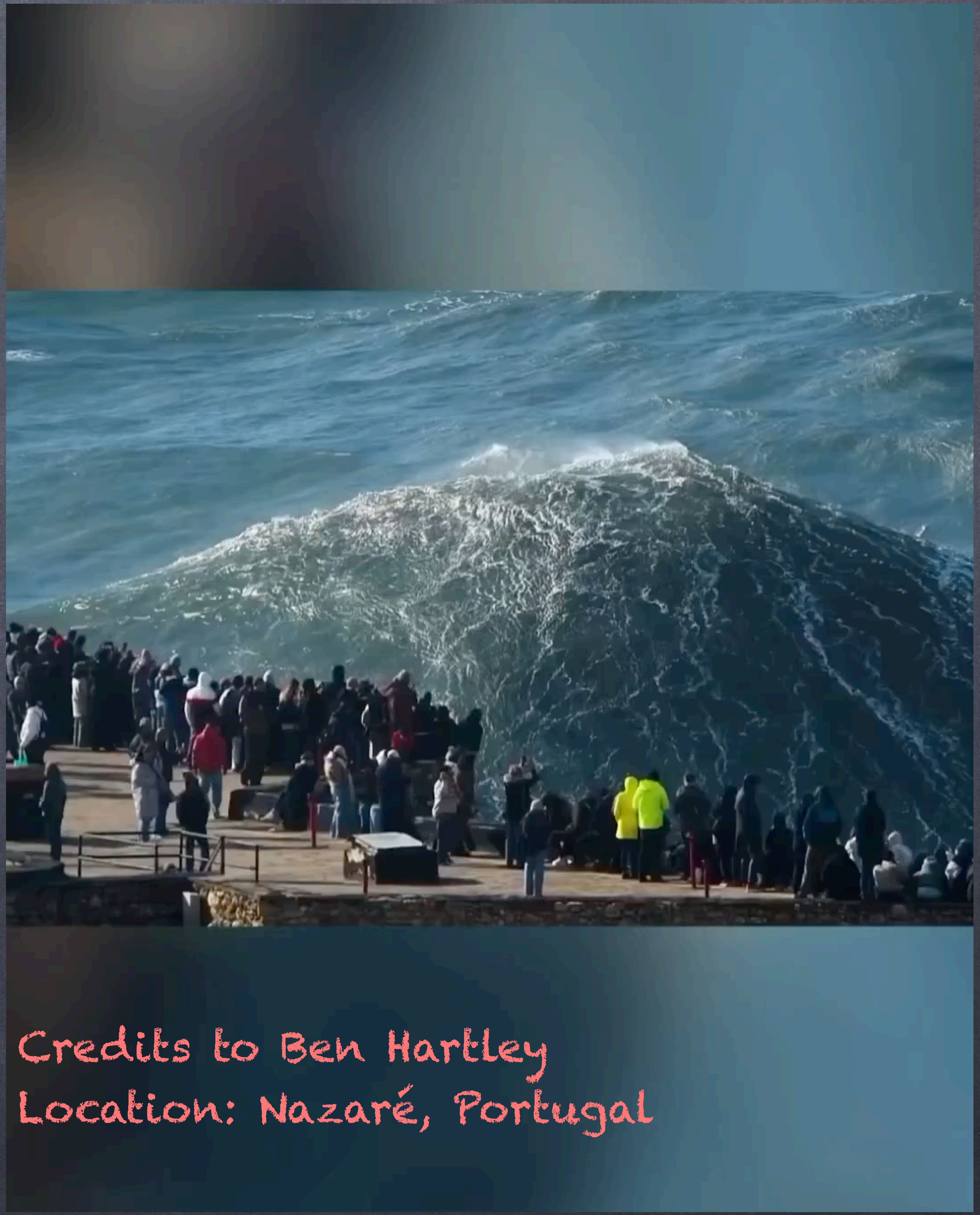
The Rogue Waves of NLS and Their Universality

Guido Mazzuca
Tulane University

Joint work with A. Gkogkou, K. D. T-R McLaughlin, T. Grava, R. Jenkins, M. Girotti and M. Yattselev

References : Arxiv:2602.05101 & DOI: 10.1088/1361-6544/ae1435 (Nonlinearity)

Rogue Waves



Rogue Waves

Credits to Ben Hartley
Location: Nazaré, Portugal

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- Freak or monster waves

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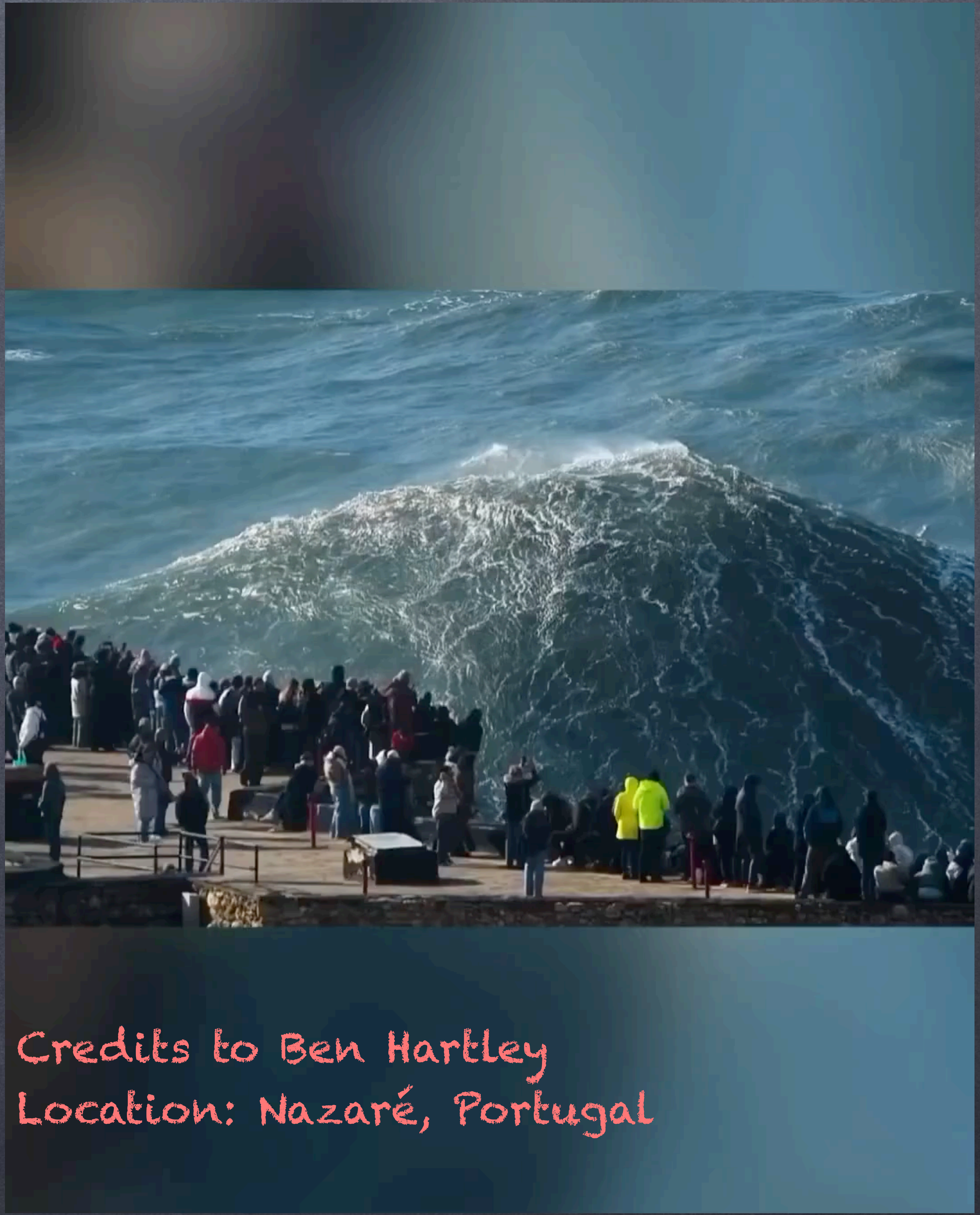
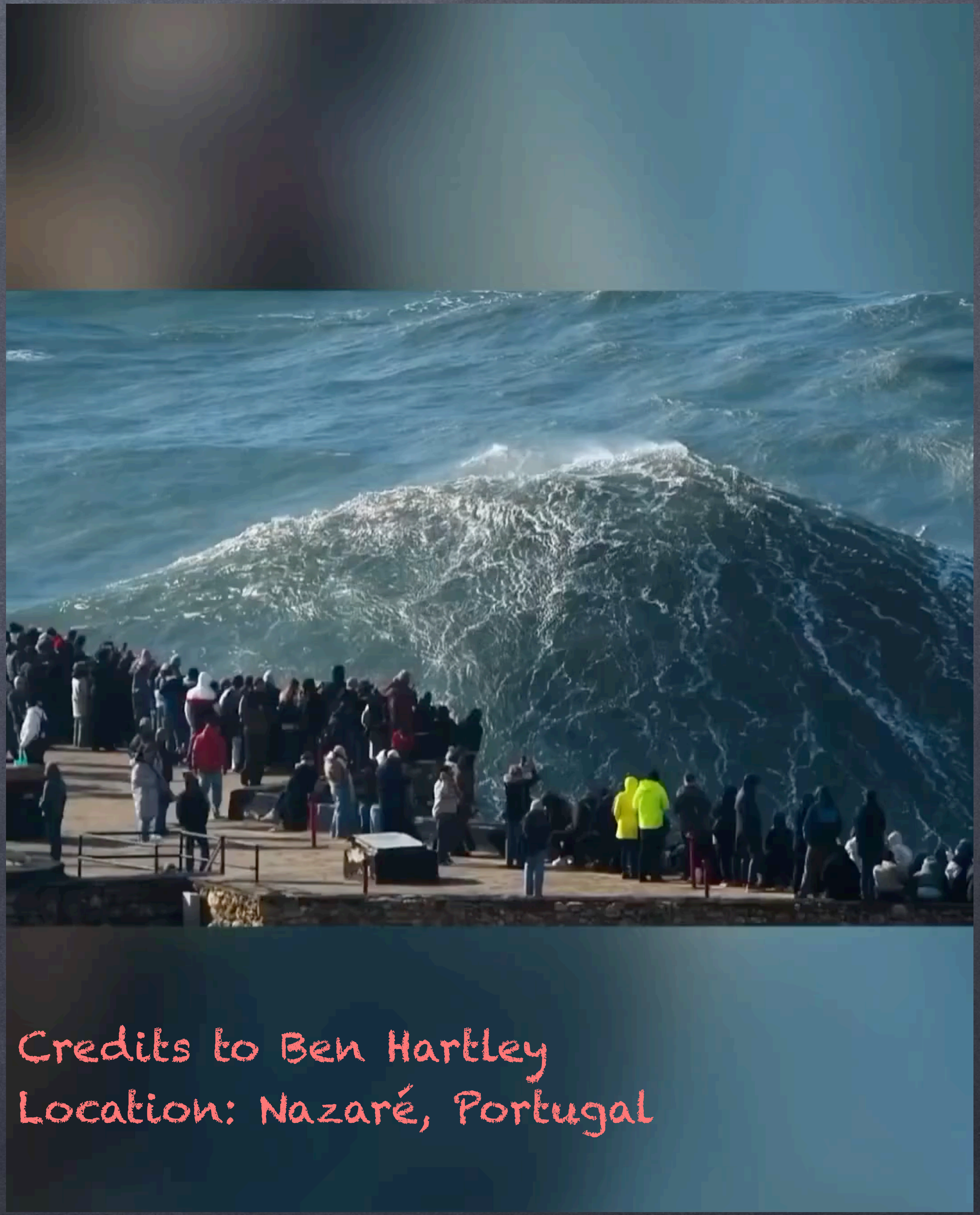
- Localized in space-time



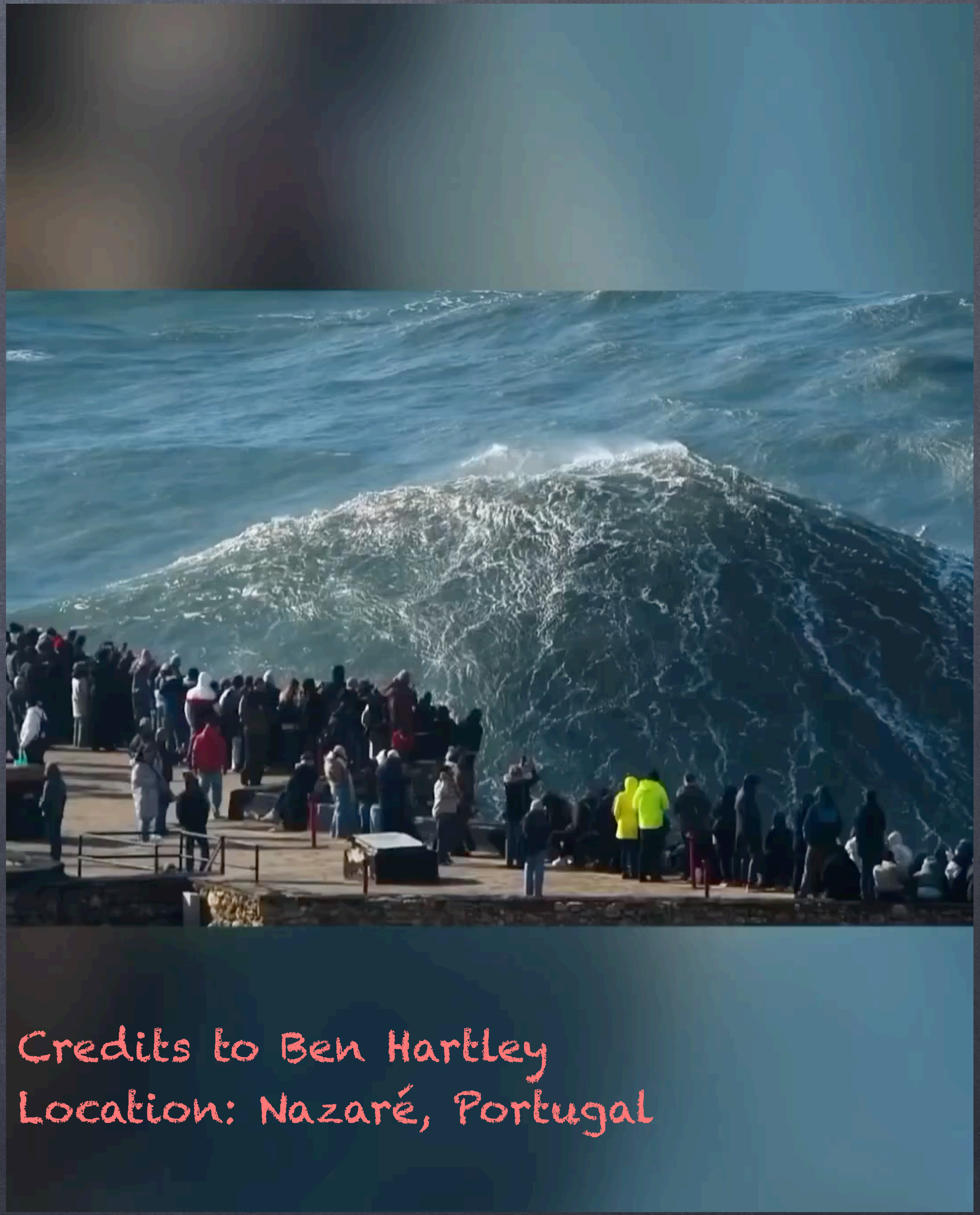
Rogue Waves

- Freak or monster waves
- Localized in space-time
- Highly disruptive (usually)



Rogue Waves

- Freak or monster waves
- Localized in space-time
- Highly disruptive (usually)
- Difficult to predict



Credits: Bright Side (Youtube)

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The NLS equation

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$$i\partial_t \psi = -\frac{1}{2}\partial_{xx}\psi - |\psi|^2 \psi$$

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Integrable

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Integrable

Fiber Optics

The NLS equation

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Integrable

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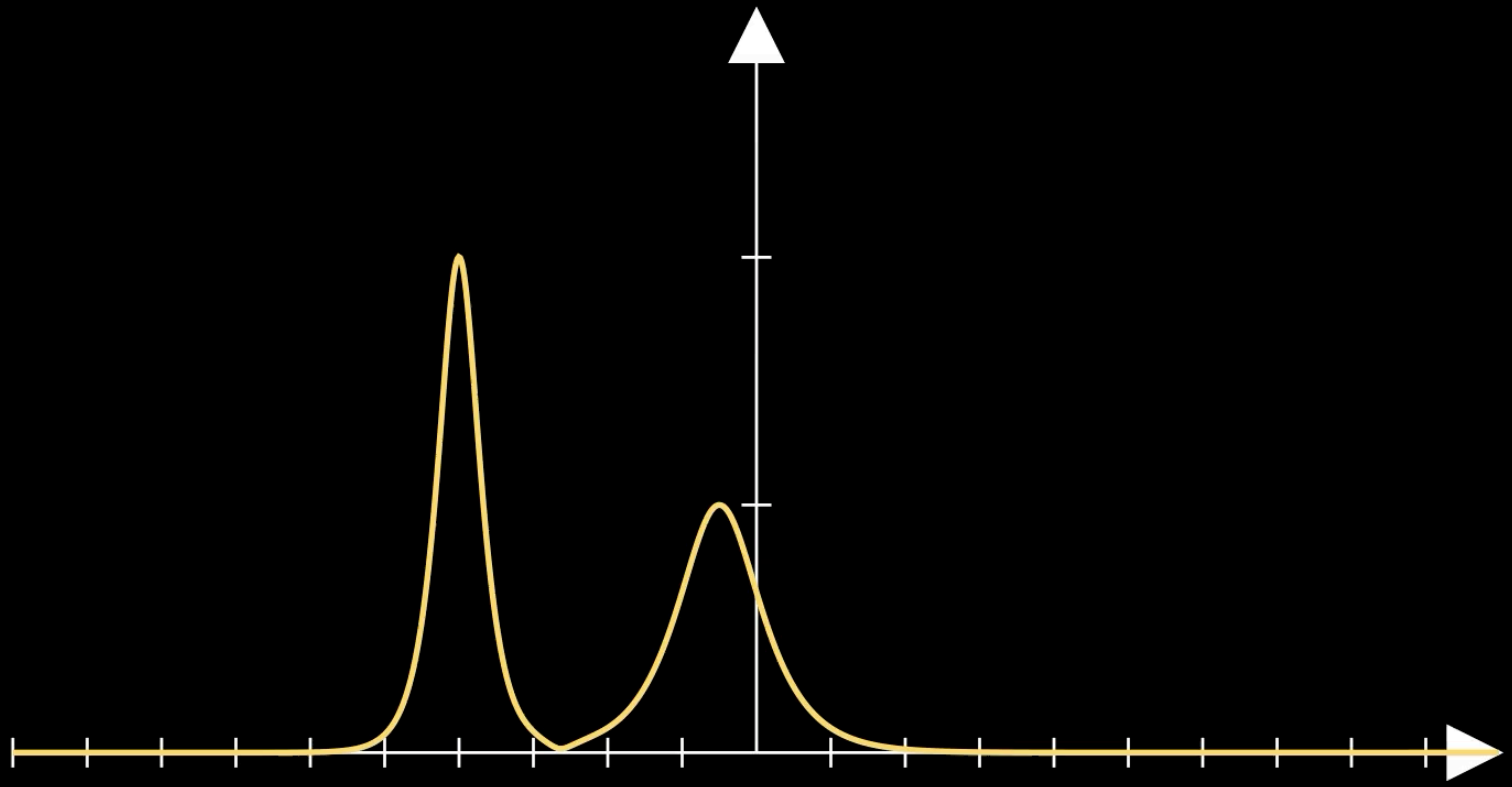
Waves

True Nonlinear Evolution

Linear Superposition



True Nonlinear Evolution

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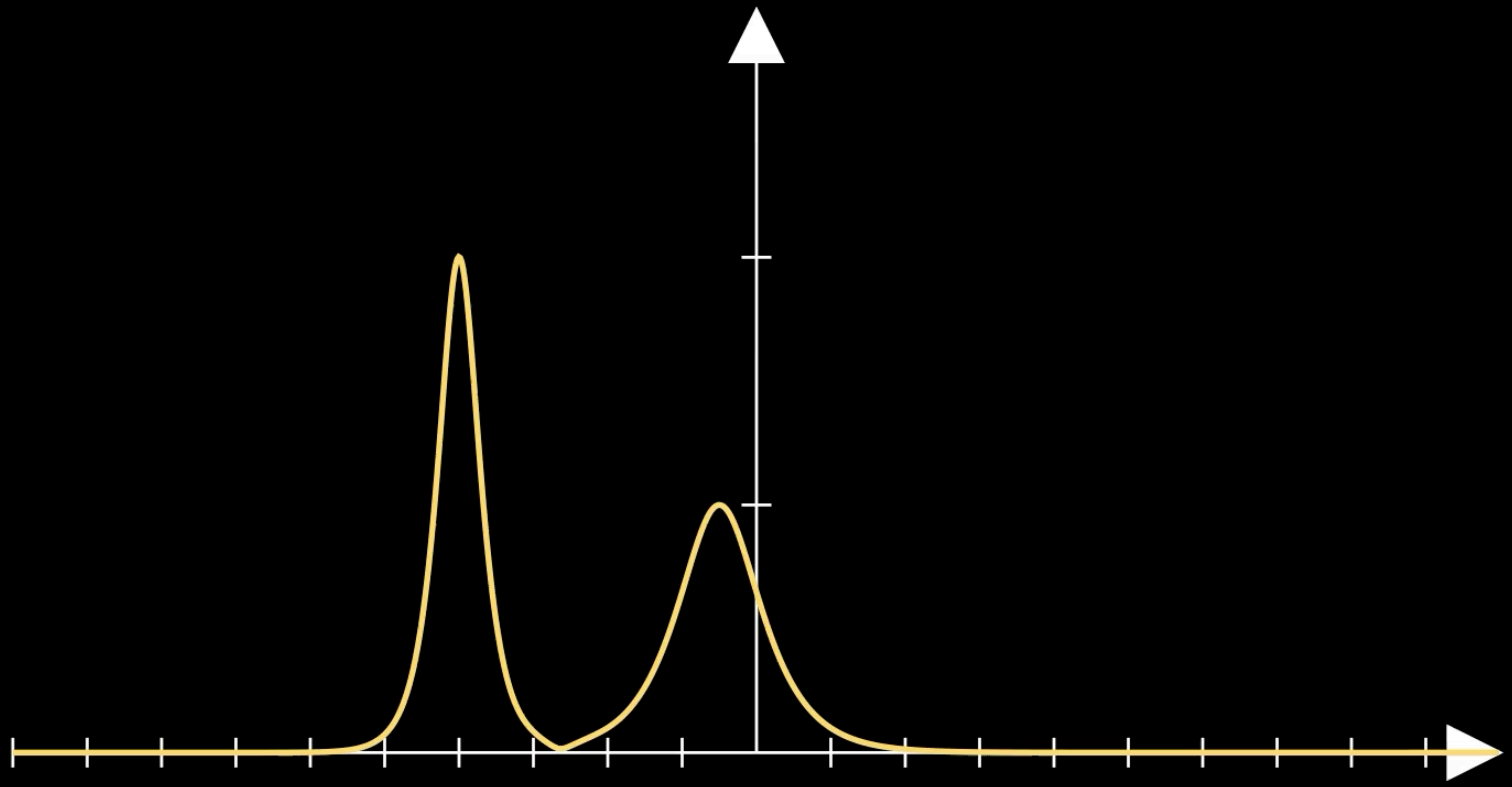

Questions

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- How likely is it?

Questions

$$i\partial_t\psi = -\frac{1}{2}\partial_{xx}\psi - |\psi|^2\psi$$

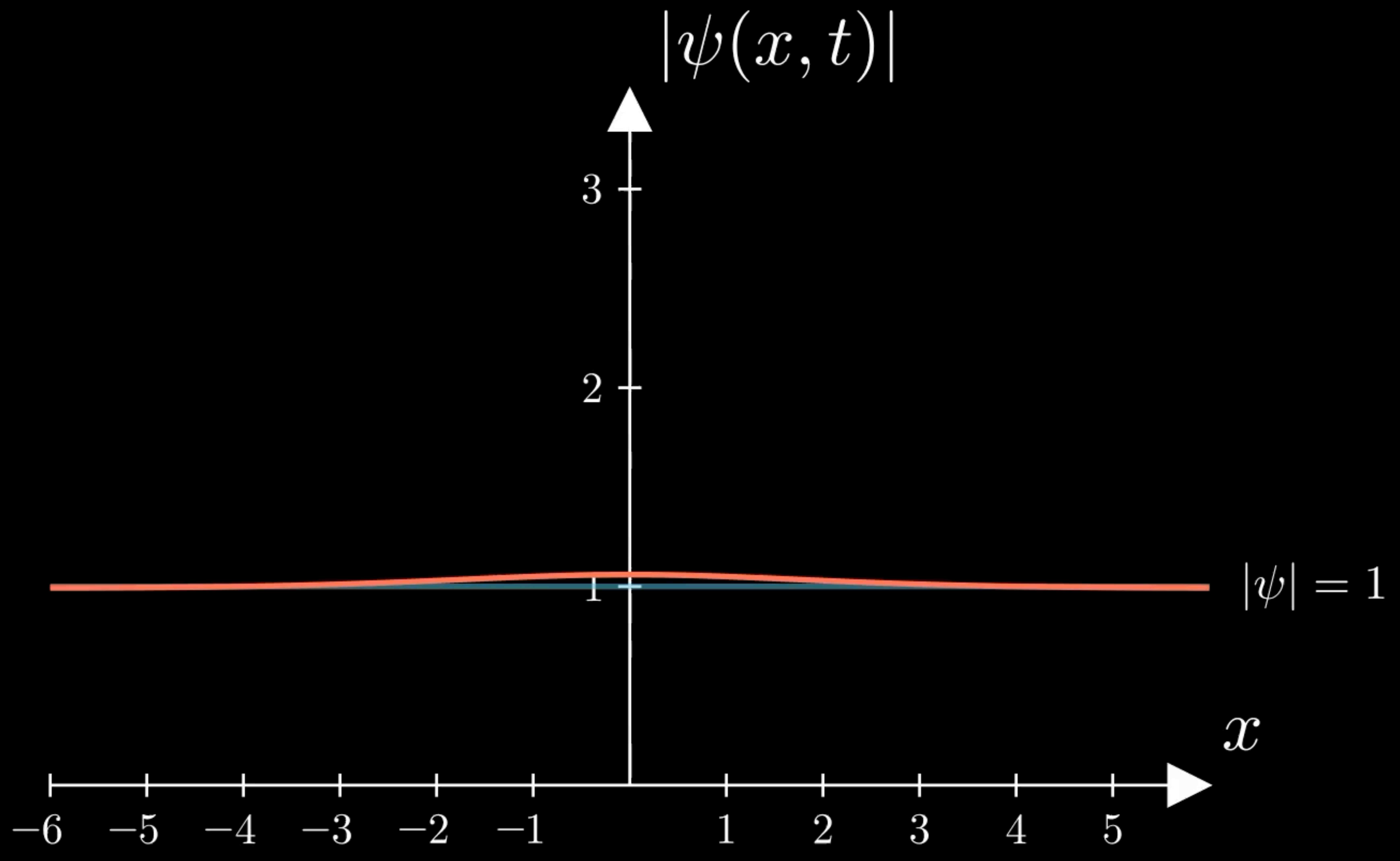
- What is a rogue wave for NLS?
- How likely is it?
- Can we describe it analytically?

Rogue Wave

$$|\psi(x,t)| \gg \mathbb{E}[\,|\psi(x,t)|\,]$$

Peregrine Breather Profile

Time (t): -4.00



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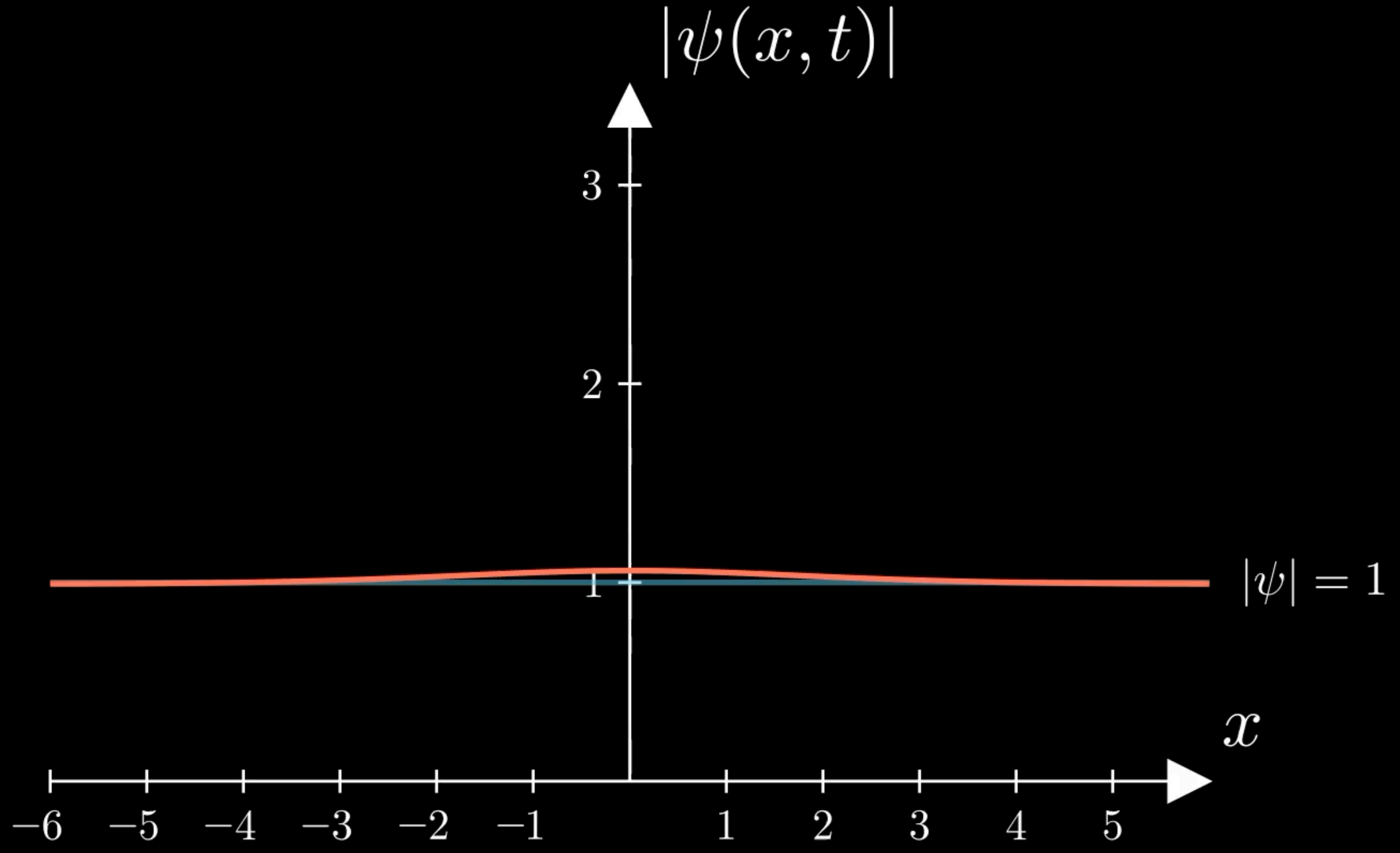

Communications on
PURE AND APPLIED MATHEMATICS

Universality for the Focusing Nonlinear Schrödinger Equation at the Gradient Catastrophe Point: Rational Breathers and Poles of the Tritronquée Solution to Painlevé I

Marco Bertola, Alexander Tovbis

First Universality result of the day

How likely is it?

How likely is it?

$$F(\delta, T) = \mathbb{P}\left(\max_{x \in (-L, L)} |\psi(x, T)| > \delta\right)$$

How likely is it?

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Rogue waves and large deviations in deep sea

Giovanni Dematteis[a,b], Tobias Grafke[a,c], and Eric Vanden-Eijnden[a,1]

[a]Courant Institute of Mathematical Sciences, New York University, New York, NY 10012; [b]Dipartimento di Scienze Matematiche, Politecnico di Torino, I-10129 Torino, Italy; and [c]Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

Edited by David A. Weitz, Harvard University, Cambridge, MA, and approved December 11, 2017 (received for review June 13, 2017)

The appearance of rogue waves in deep sea is investigated by using the modified nonlinear Schrödinger (MNLS) equation in one spatial dimension with random initial conditions that are assumed to be normally distributed, with a spectrum approximating real-

random initial data drawn from a Gaussian distribution (30). The spectrum of this field is chosen to have a width comparable to that of the Joint North Sea Wave Project (JONSWAP) spectrum (31, 32) obtained from observations in the North Sea. We cal-

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- Characterize the most likely precursor to the rogue wave

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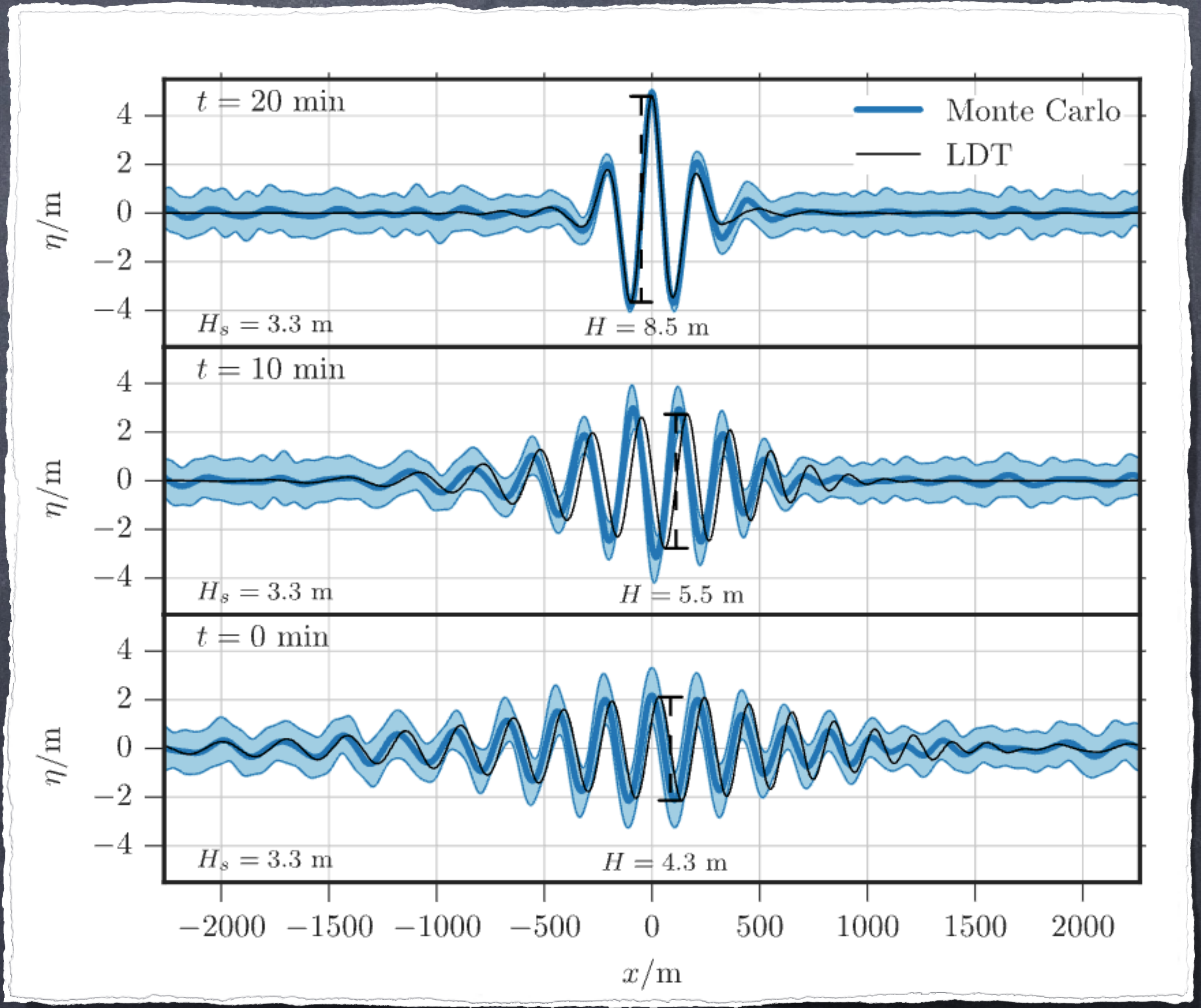

$$F(\delta, T) = \mathbb{P}\Big(\sup_{x \in (0,2\pi)} |\psi(x, T)| > \delta\Big)$$

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RESEARCH ARTICLE

Large deviations principle for the cubic NLS equation

Miguel Angel Garrido, Ricardo Grande, Kristin M. Kurianski, Gigliola Staffilani

$$\lim_{\varepsilon\to 0^+} \varepsilon \log\left(\mathbb{P}\left(\sup_{x\in(0,2\pi)} |\psi(x,T)| > \varepsilon^{-\frac{1}{2}}\delta\right)\right) = -\frac{\delta^2}{\sum_k c_k^2}$$

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$$\psi_0(x) = \sum_{k \in \mathbb{Z}} c_k \eta_k e^{ikx}$$

1.3.2. *Main ideas in Theorem 1.9.* The goal of this result is to show that the most likely way in which rogue wave arise in this weakly nonlinear context is due to phase synchronization[5]. The construction of the one-sided neighborhood $\mathcal{U}(\varepsilon)$ in (1.13) is based on two-scales: we control the moduli of $\mathcal{O}(|\log \varepsilon|)$ Fourier modes, but we force $\mathcal{O}(\varepsilon^{-1/2})$ phases to almost synchronize at a certain point in space and time.

What is the shape of the NLS rogue wave?

RHP for the NLS

Pure soliton!

RHP for the NLS

Pure soliton!

$M(z) \in \mathrm{Mat}(\mathbb{C},2)$

$M(z)$ is analytic in $\mathbb{C} \setminus \cup_{j=1}^{N} \{z_j, \overline{z}_j\}$

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$$\theta(z,x,t) = xz + tz^2$$

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$$M(z) = \mathbb{I} + \frac{1}{2iz} \begin{pmatrix} -m(x,t) & \psi(x,t) \\ \overline{\psi(x,t)} & m(x,t) \end{pmatrix} + \mathcal{O}(z^{-2})$$

RHP for the NLS

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Spectral parameters

$$M(z) = \mathbb{I} + \frac{1}{2iz} \begin{pmatrix} -m(x,t) & \psi(x,t) \\ \overline{\psi(x,t)} & m(x,t) \end{pmatrix} + \mathcal{O}(z^{-2})$$

Theoretical Max

$$\max_{x,t} |\psi(x,t)| \leq 2 \sum_{j=1}^{N} \Im(z_j)$$

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$$a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \bar{z}_j} \quad \oplus \quad c_j = \frac{1}{a'(z_j)} \longrightarrow$$

$$|\psi(0,0)| = 2 \sum_{j=1}^{N} \Im(z_j)$$

*See NN and RW papers for a proof

Solitons Synchronization

$z_j = v_j + i\mu$ $\quad v_j \in [-j\Delta, -(j-1)\Delta]$ $\quad c_j = 2i\mu e^{-2iz_j^2}$

Nonlinearity, DOI 10.1088/1361-6544/ae1435

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$$\frac{1}{N\mu}\psi\left(\frac{2X}{N\Delta}, 1 + \frac{T}{(N\Delta)^2}\right) = 2\psi_0(X,T) + \mathcal{O}\left(\max\left\{\frac{1}{N}, \frac{1}{\Delta}\right\}\right),$$

$$\psi_0(X,T) = -\int_0^1 e^{i\left(2Xs - \frac{T}{2}s^2\right)} ds\,.$$

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$$\frac{1}{N\mu}\psi_N\left(\frac{2X}{NV}, 1\right) = -2e^{iX}\frac{\sin(X)}{X} + \mathcal{O}\left(\max\left\{\frac{1}{N}, \frac{1}{\Delta}\right\}\right)$$

Nonlinearity, DOI 10.1088/1361-6544/ae1435

Solitons Synchronization

$$z_j = v_j + i\mu_j \qquad v_j = -j\Delta = -\alpha j N^{\gamma}, \gamma > \frac{1}{2} \qquad c_j = 2i\mu_j e^{-2iz_j^2}$$

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Iid sub-exp

$$\frac{1}{2N\mu_{\mathscr{D}}}\psi\left(\frac{2X}{N\Delta}, 1 + \frac{T}{(N\Delta)^2}\right) \xrightarrow{N\to\infty} \psi_0(X,T)$$

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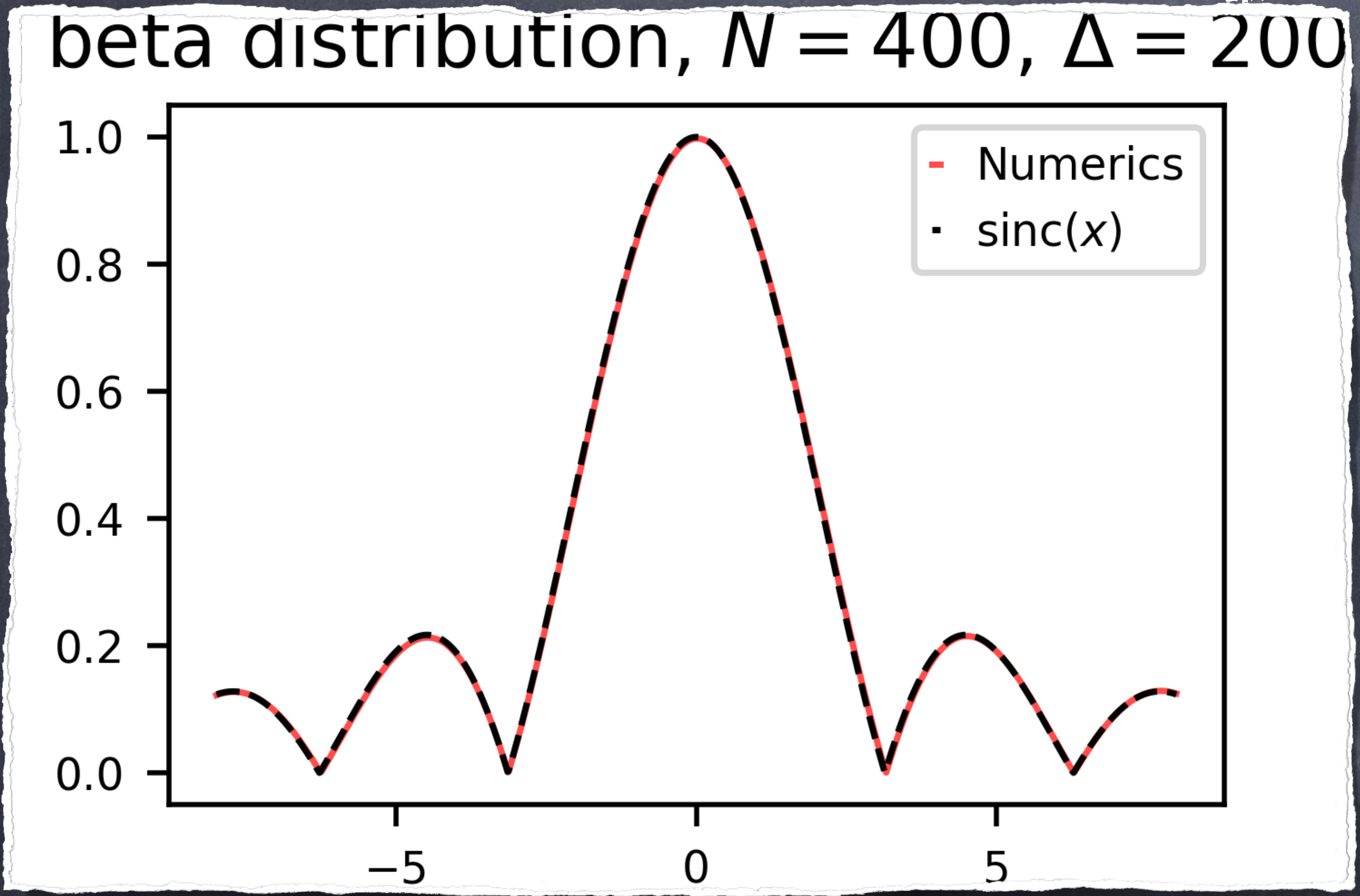
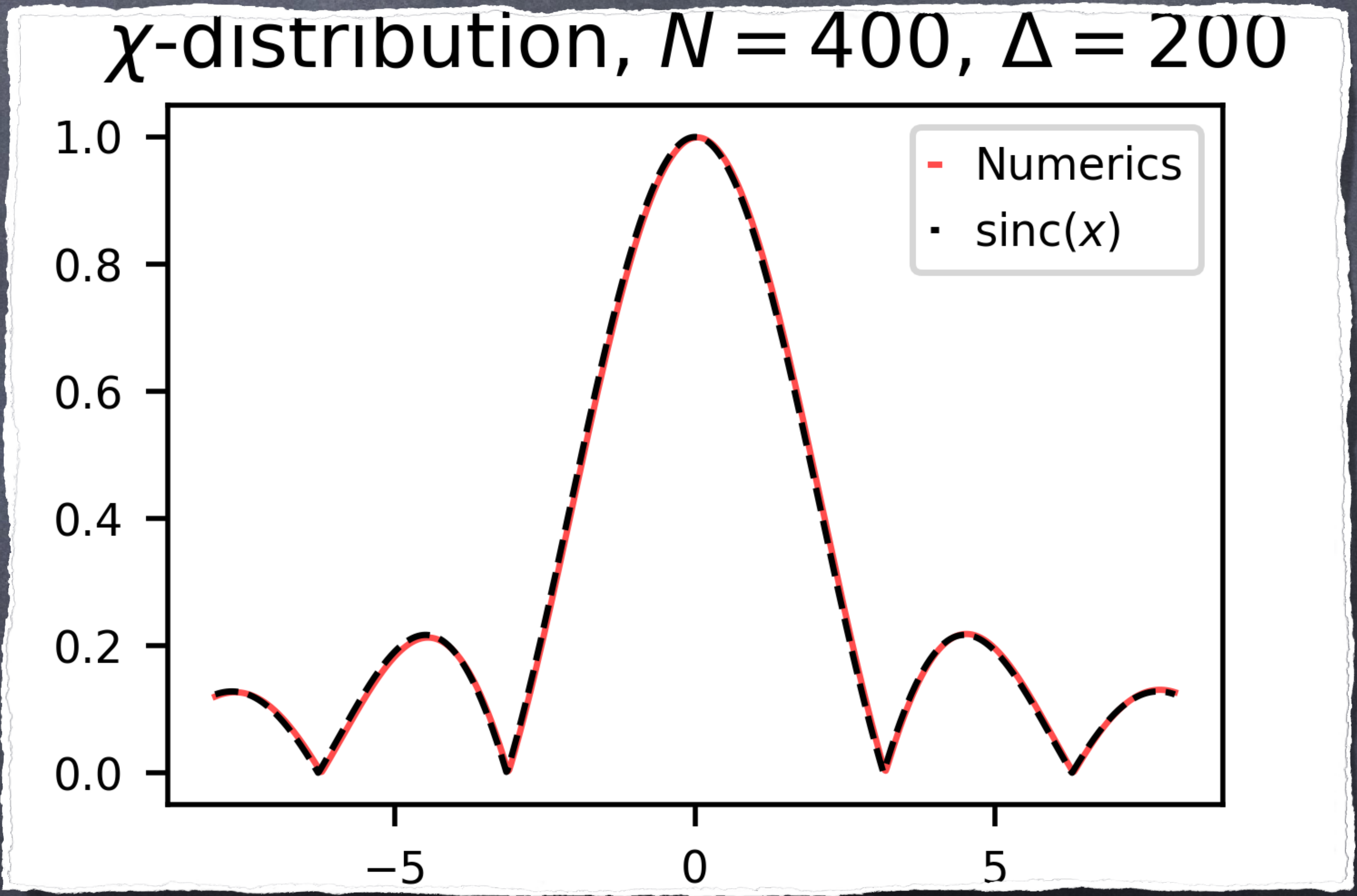
Solitons Synchronization





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RW and Painlevé

$$z_j = v_j + i\mu_j \qquad c_j = \frac{1}{a'(z_j)} \qquad a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z}_j}$$

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RW and Painlevé

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Iid sub-exp

$$\boxed{c_j = \frac{1}{a'(z_j)}}$$

Sync at the origin

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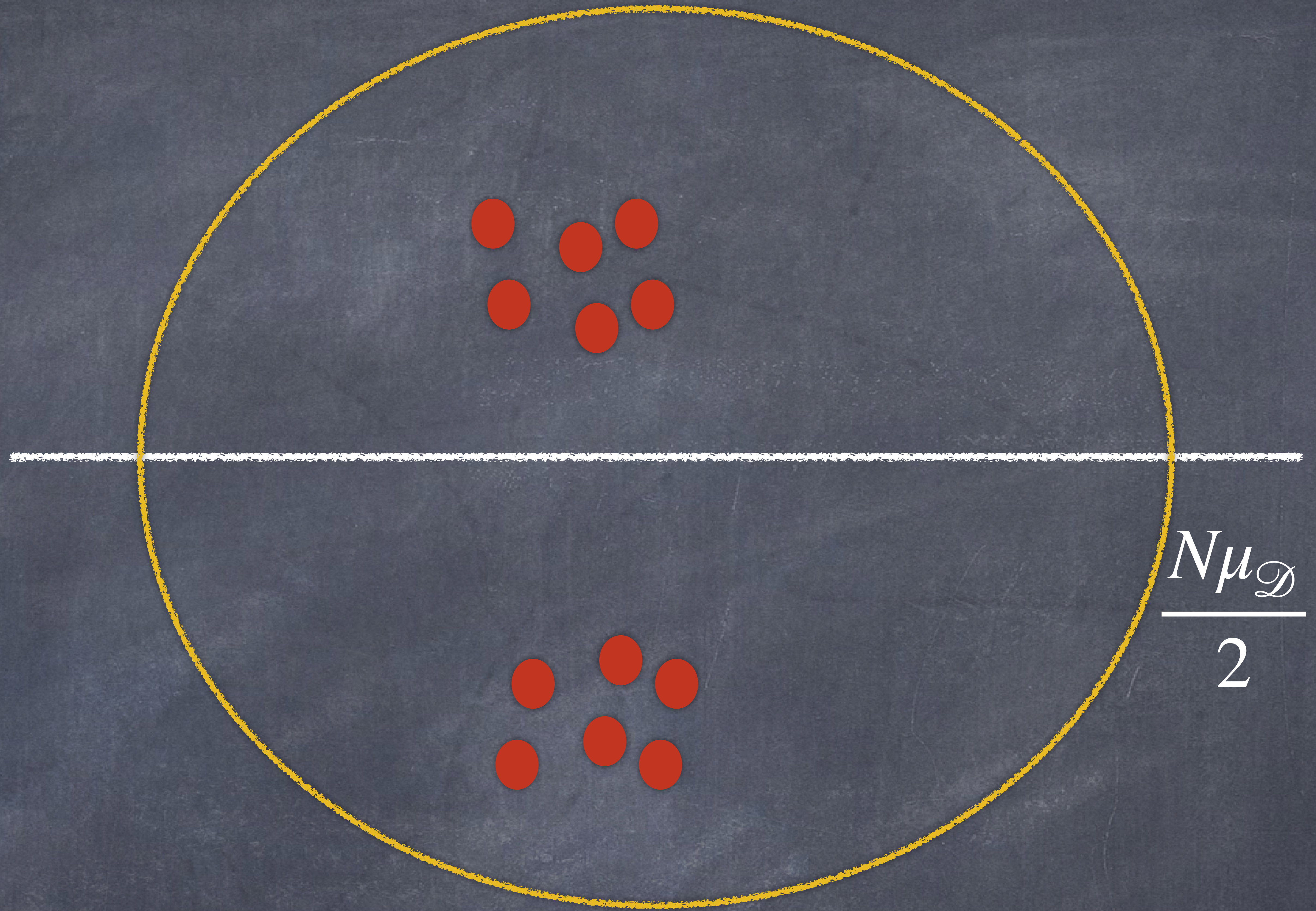

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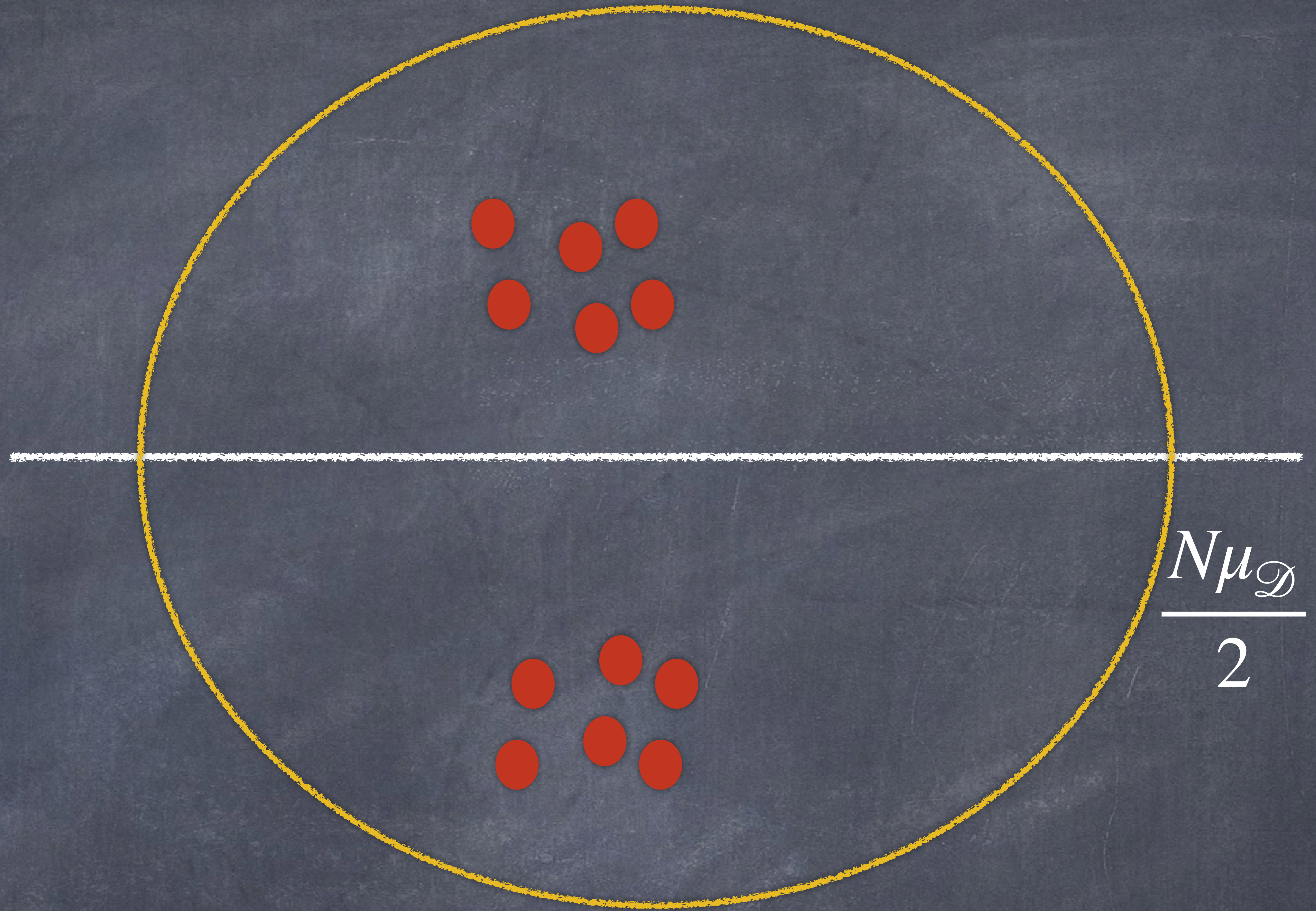

$$A = \left\{ |\mu_j| < \frac{\mu_{\mathscr{D}}}{4} N \right\} \cap \left\{ |v_j| < \frac{\mu_{\mathscr{D}}}{4} N \right\} \cap \{\text{too technical}\}$$

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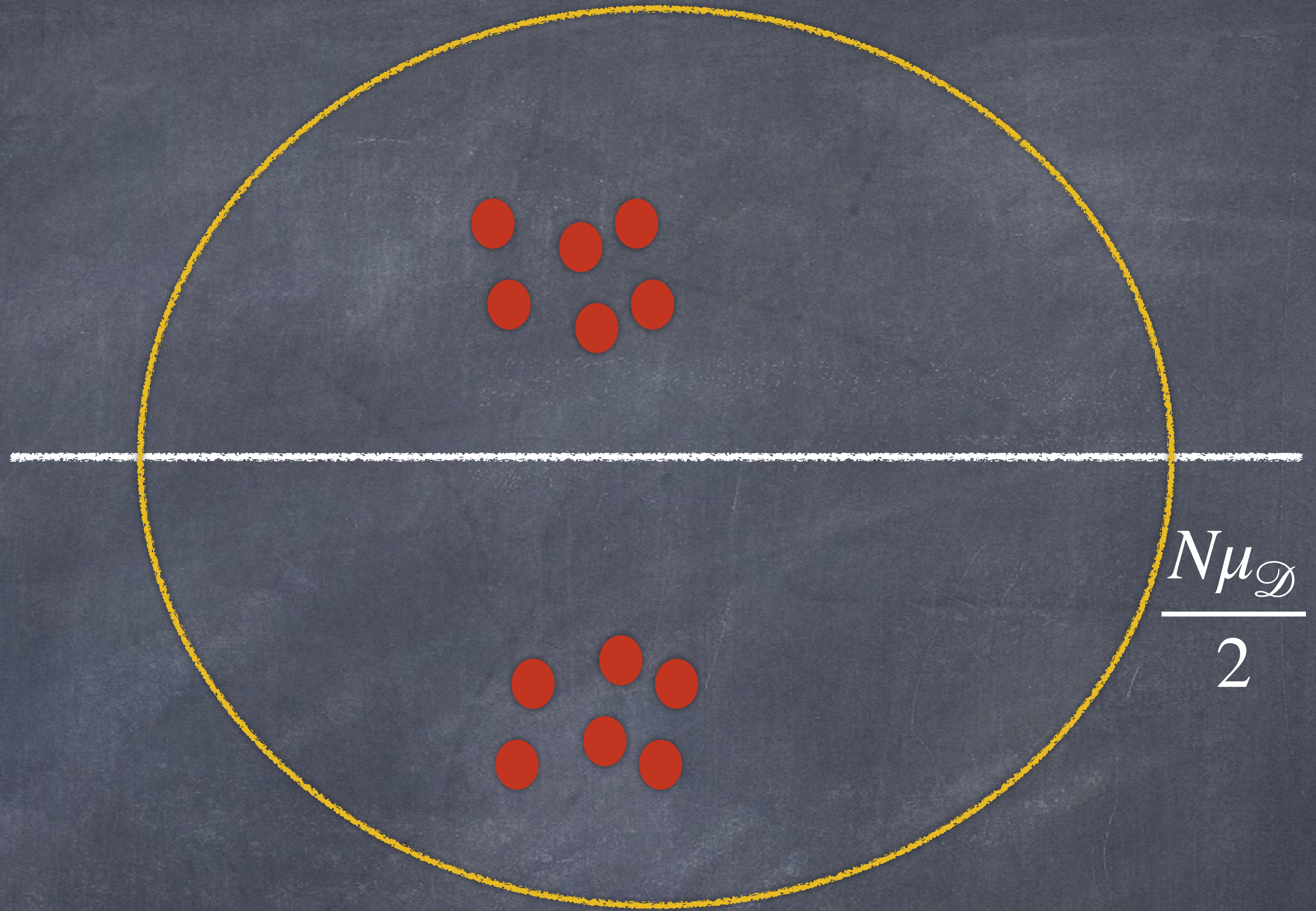

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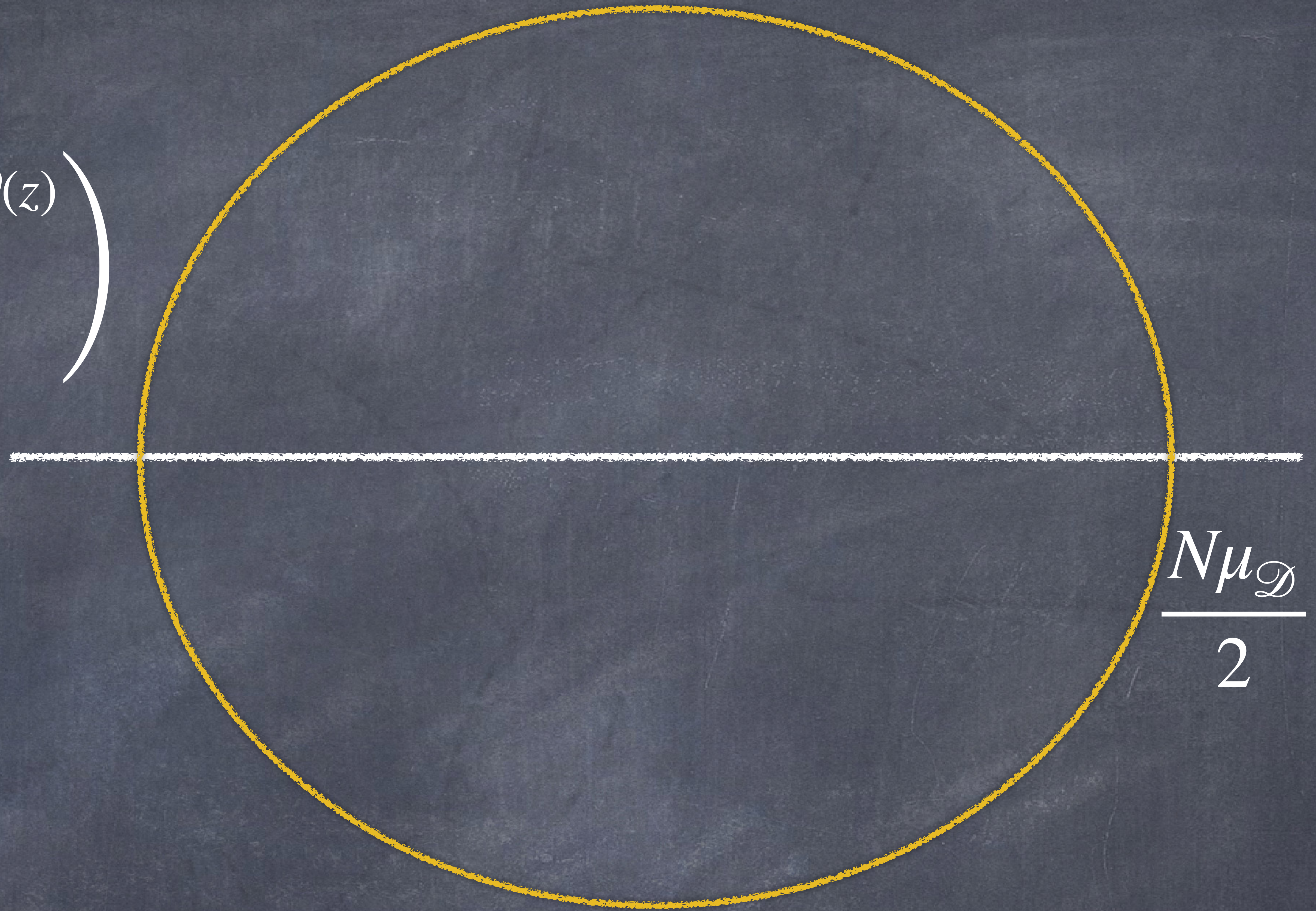
$$\mathbb{P}(A^c) < e^{-dN}$$

$$B(z)_+ = B(z)_- \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & a(z)e^{-2i\theta(z)} \\ -a^{-1}(z)e^{2i\theta(z)} & 1 \end{pmatrix}$$

$$a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z}_j}$$



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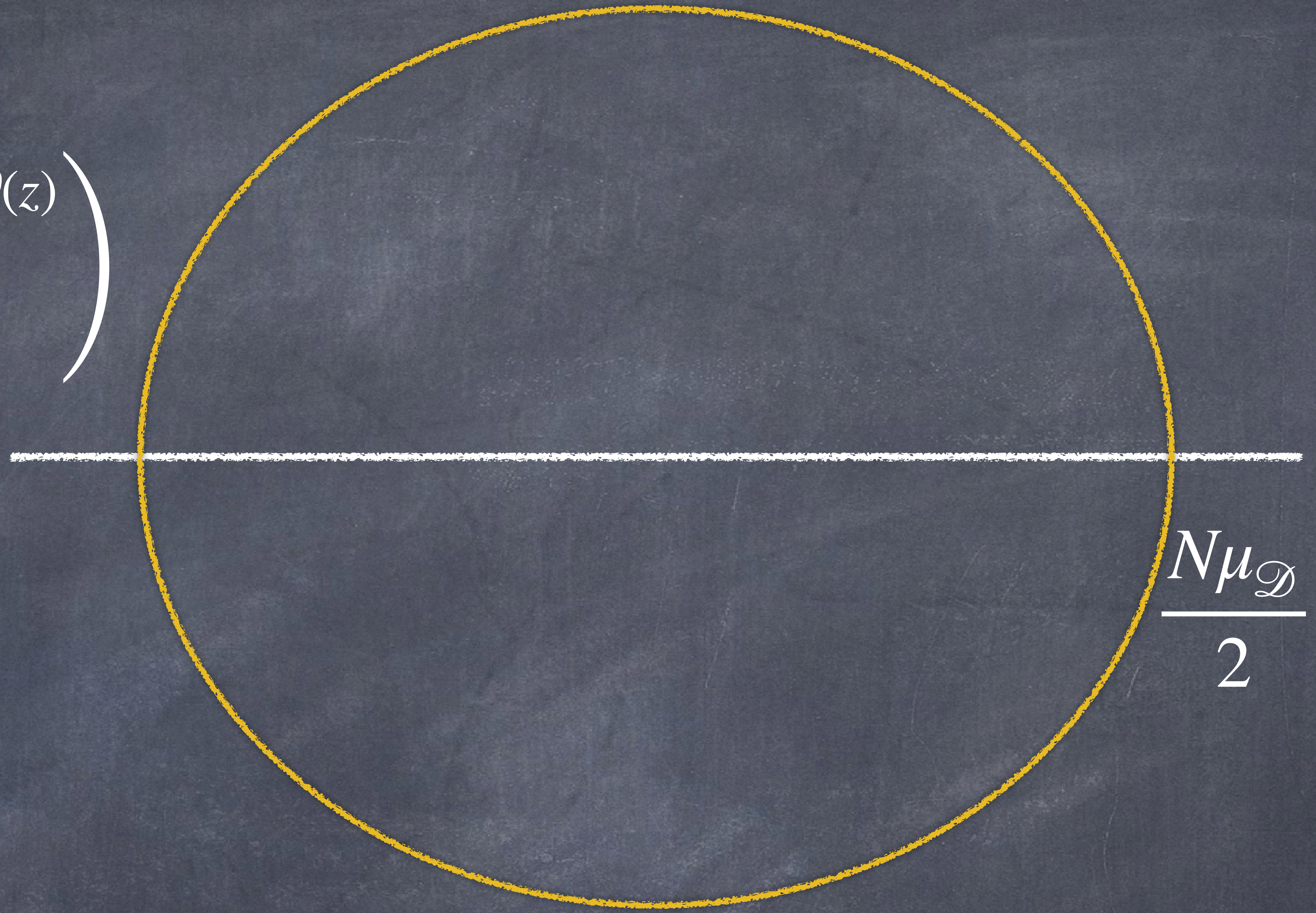

$$z = \frac{N\mu_{\mathscr{D}} Z}{2}, \quad x = \frac{2X}{N\mu_{\mathscr{D}}}, \quad t = \frac{4T}{\mu_{\mathscr{D}}^2 N^2}.$$

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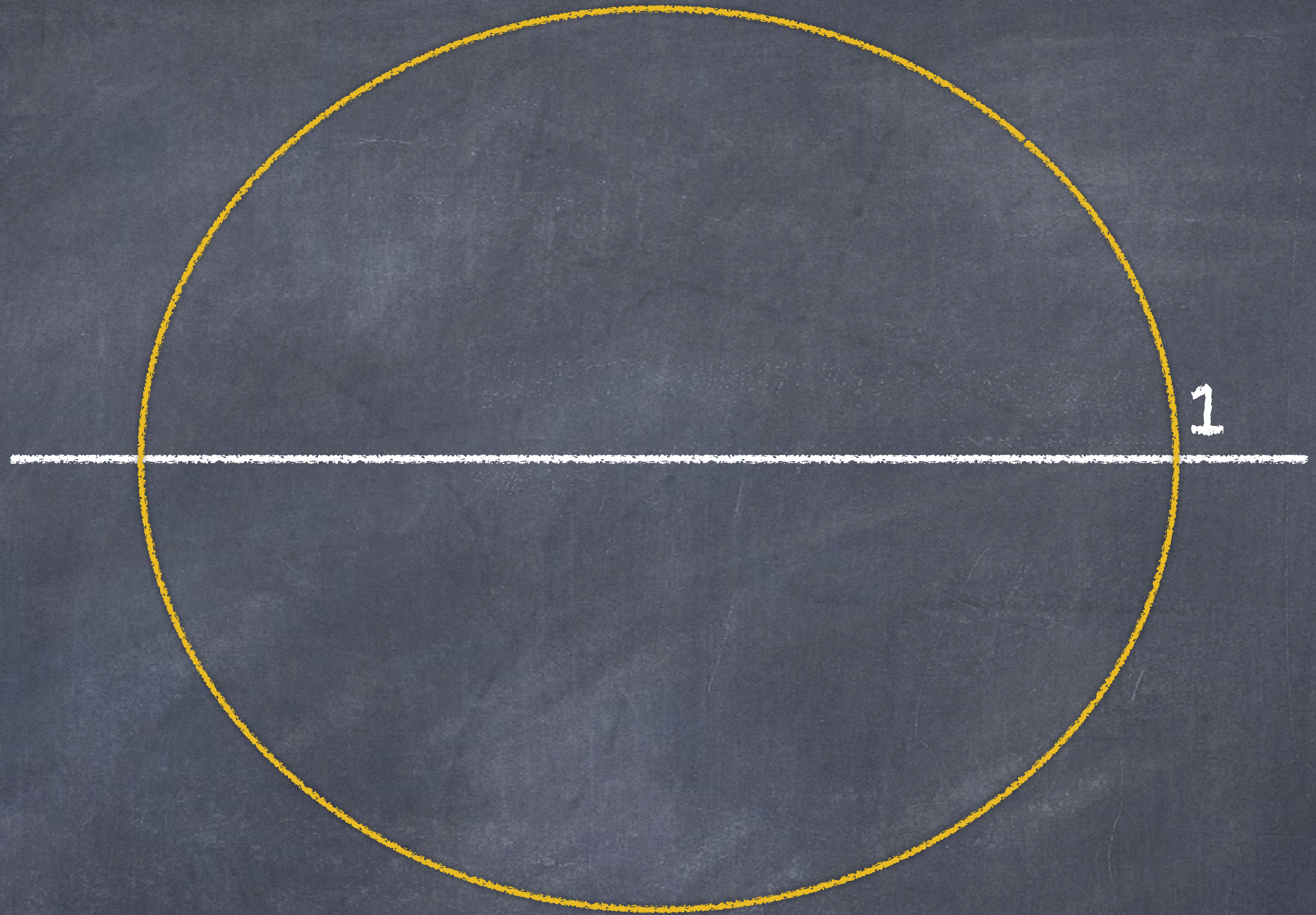
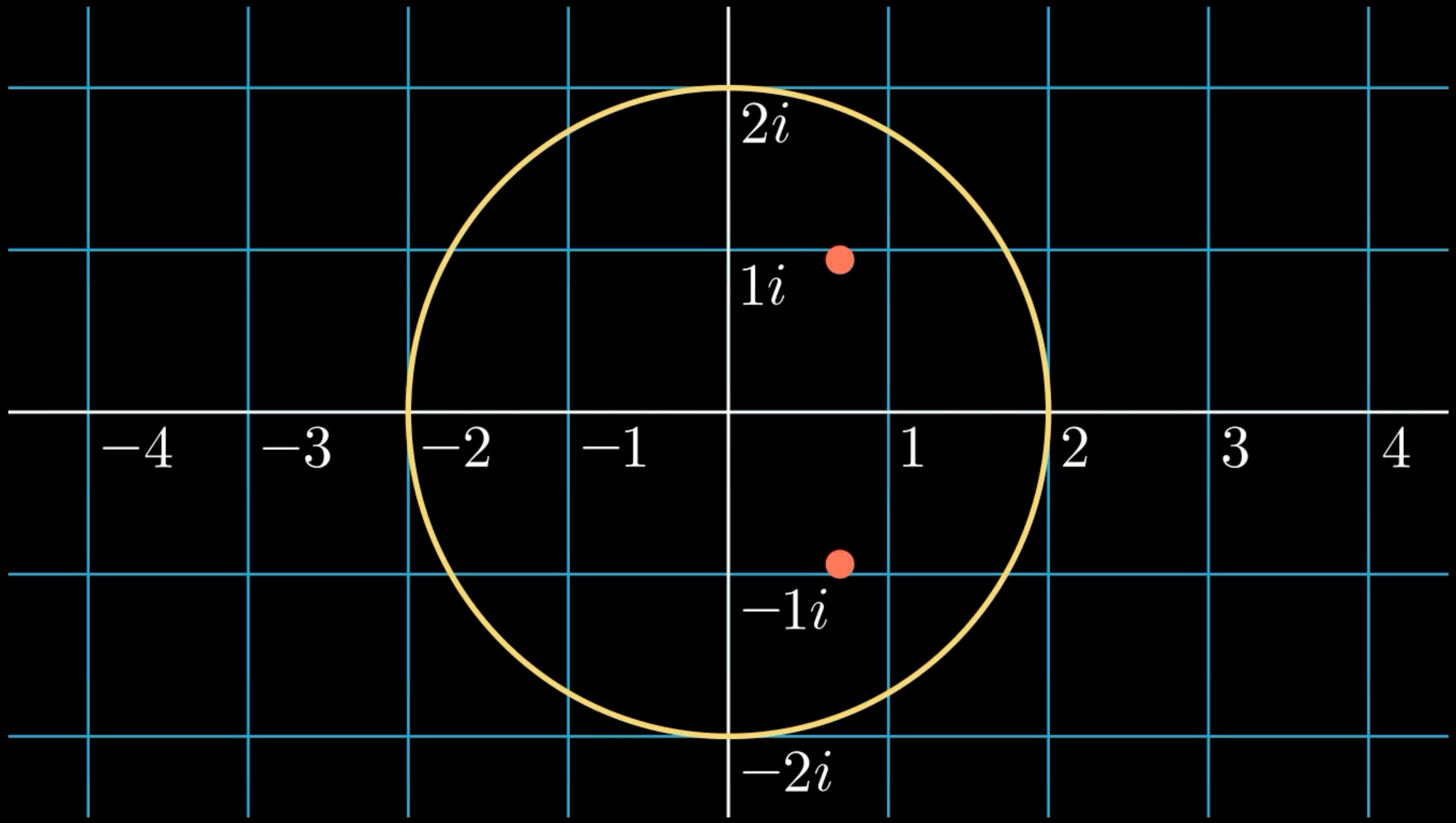


$$R_+(Z)=R\_(Z)\frac{1}{\sqrt{2}}e^{-i(XZ+TZ^2+2Z^{-1})\sigma_3}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}e^{i(XZ+TZ^2+2Z^{-1})\sigma_3}$$

Number of Poles: 1

Order of the pole: 1



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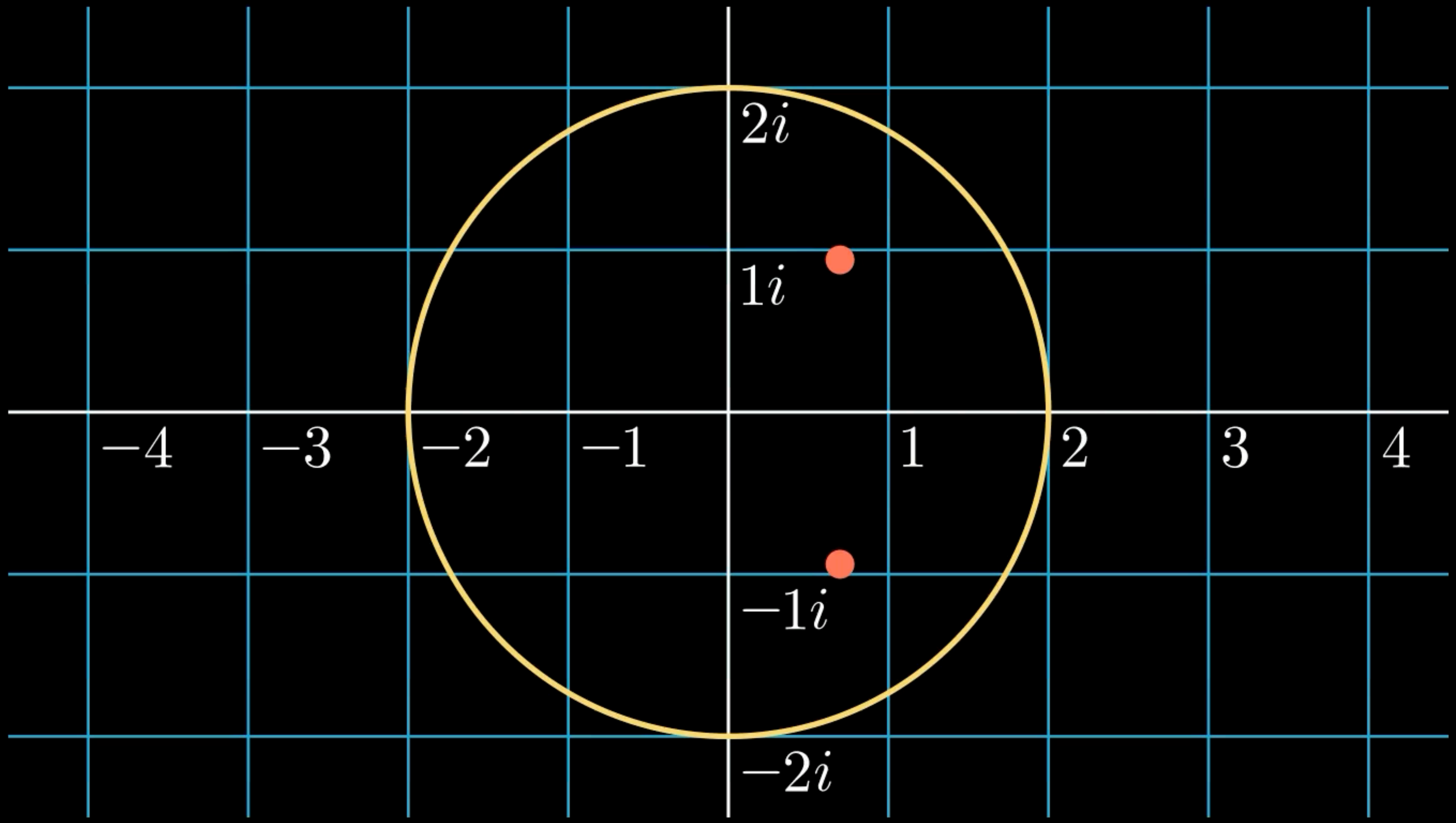

We got lucky

$$R_+(Z) = R_-(Z)\frac{1}{\sqrt{2}}e^{-i(XZ+TZ^2+2Z^{-1})\sigma_3}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}e^{i(XZ+TZ^2+2Z^{-1})\sigma_3}$$

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15 March 2020

Extreme superposition: Rogue waves of infinite order and the Painlevé-III hierarchy

Deniz Bilman, Liming Ling, Peter D. Miller

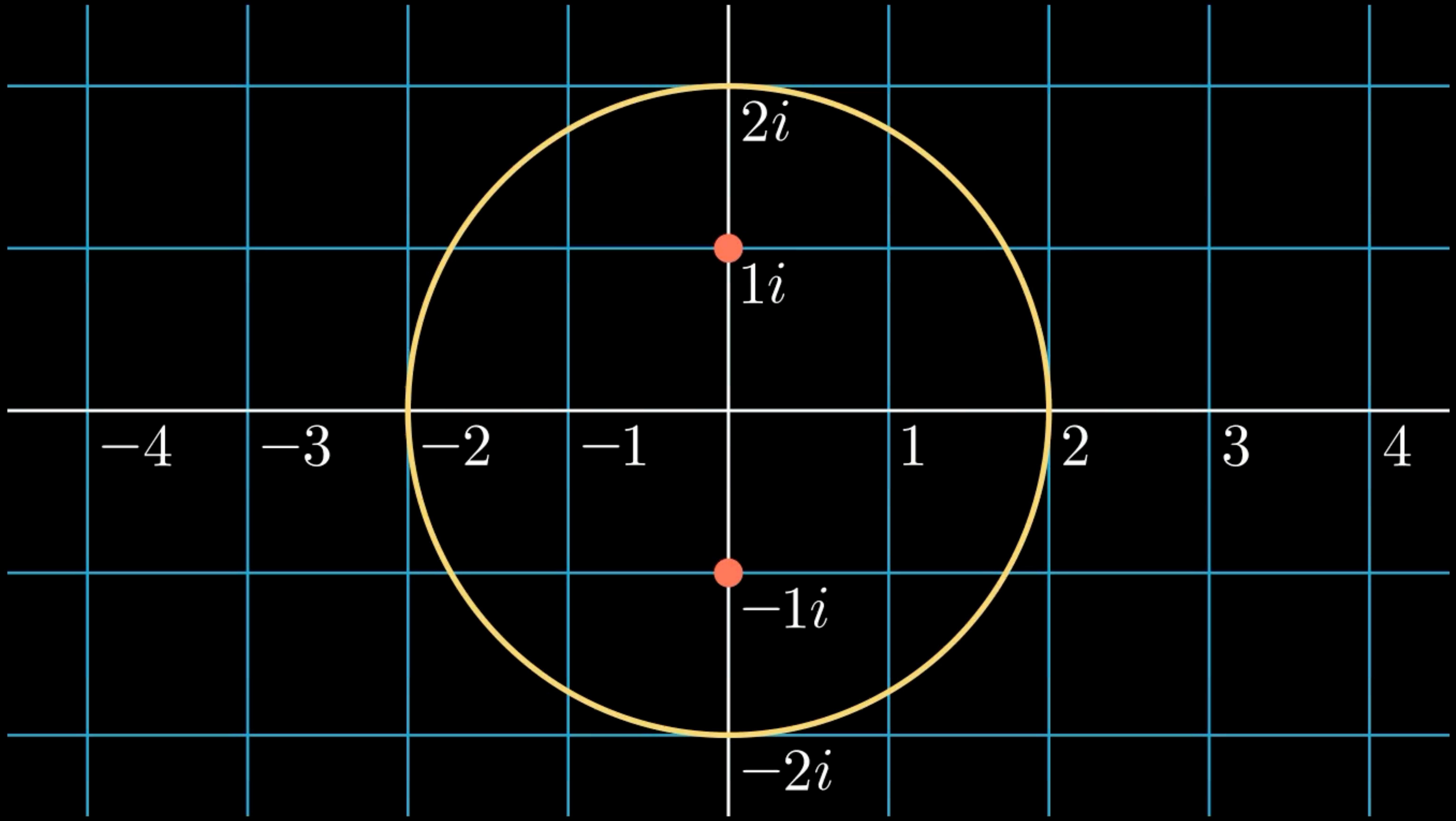
Duke Math. J. 169(4): 671-760 (15 March 2020). DOI: 10.1215/00127094-2019-0066

$$\Psi_{III}\left(-\frac{x^2}{8},0\right) = \frac{\kappa}{x^2}\exp\left(\int_1^x \frac{2}{u(s)}\,ds\right)$$

The RWIO

Number of Poles: 1

Order of the pole: 1



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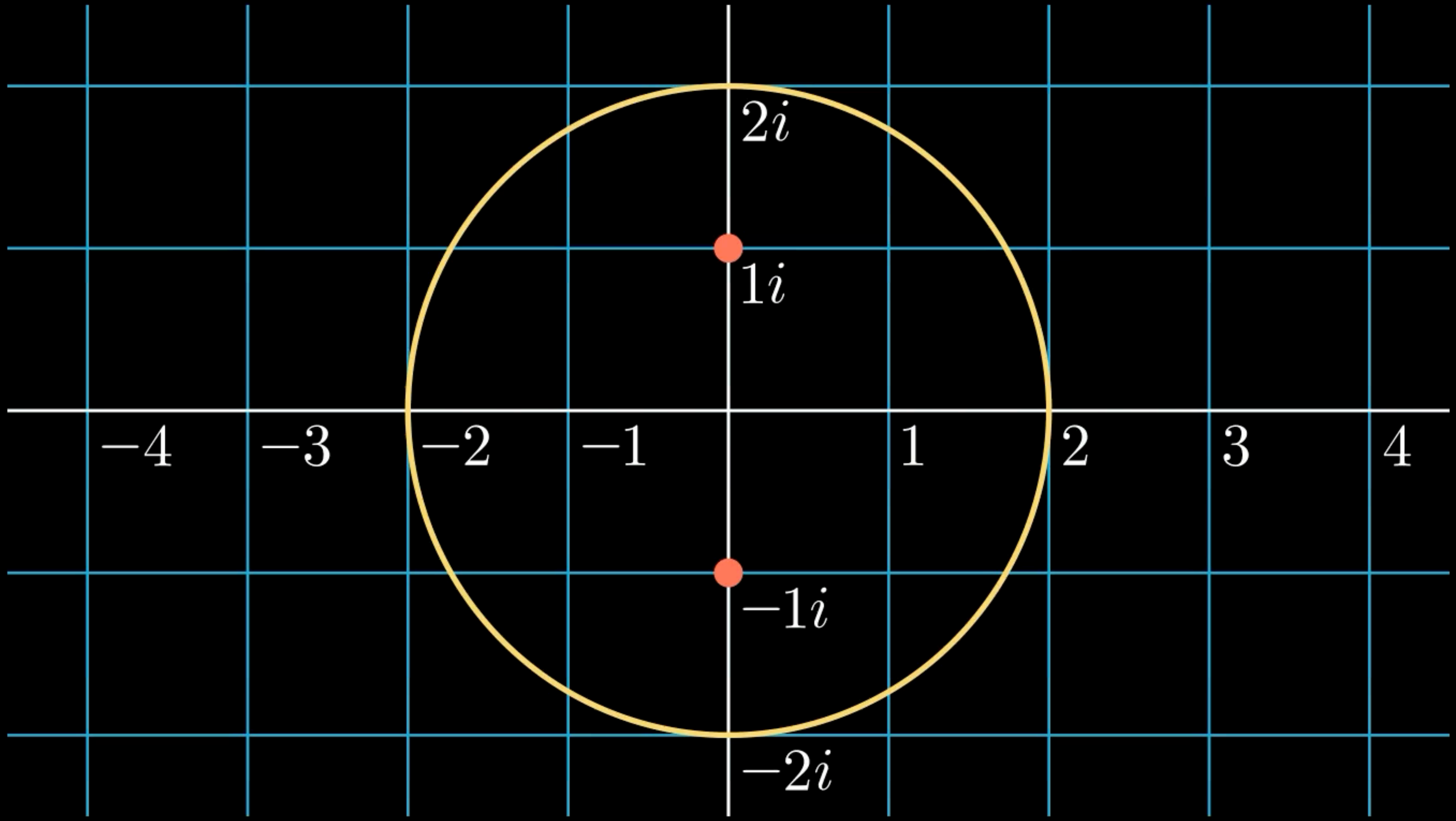
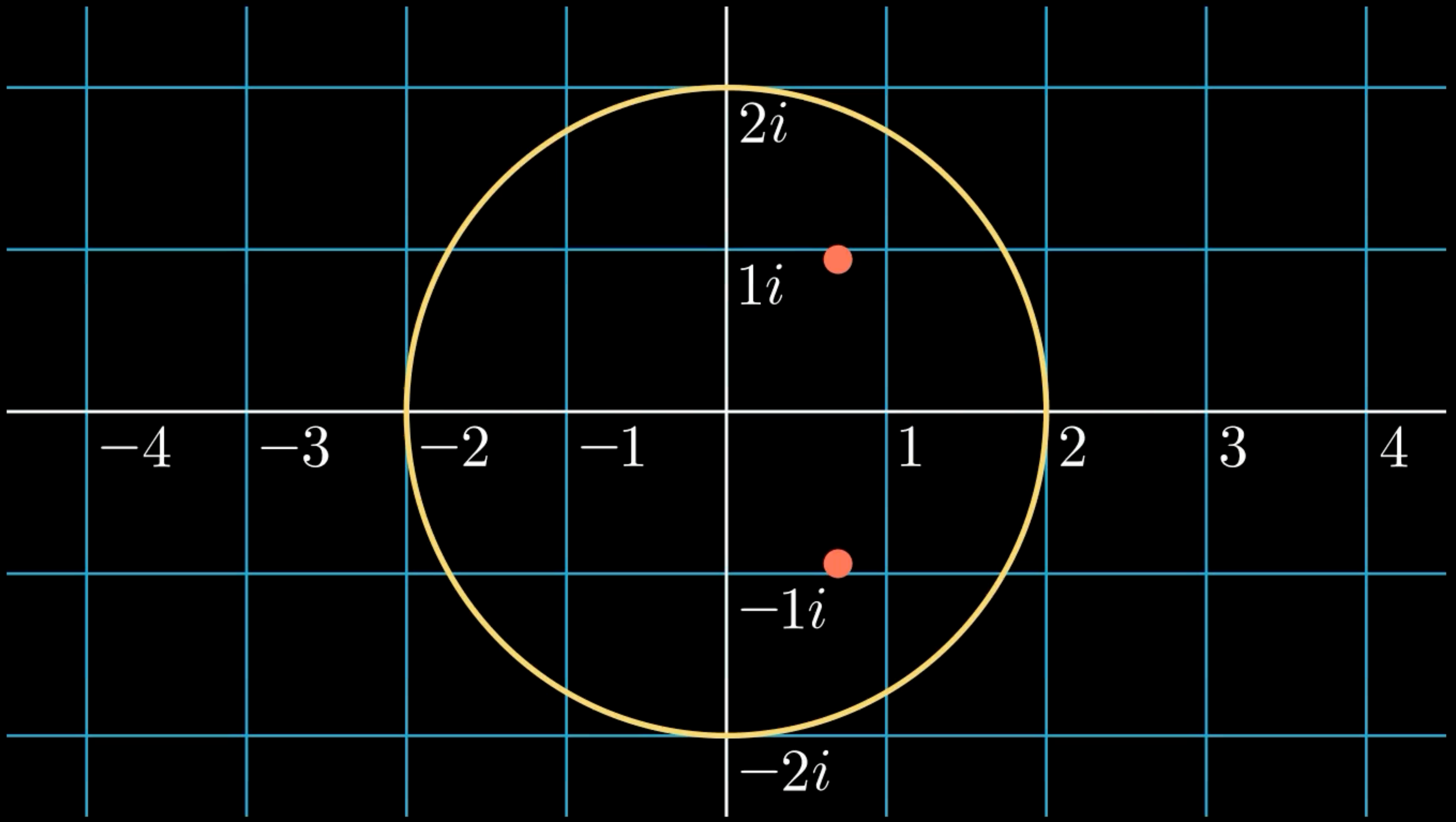


The PIII-Rogue wave

Number of Poles: 1

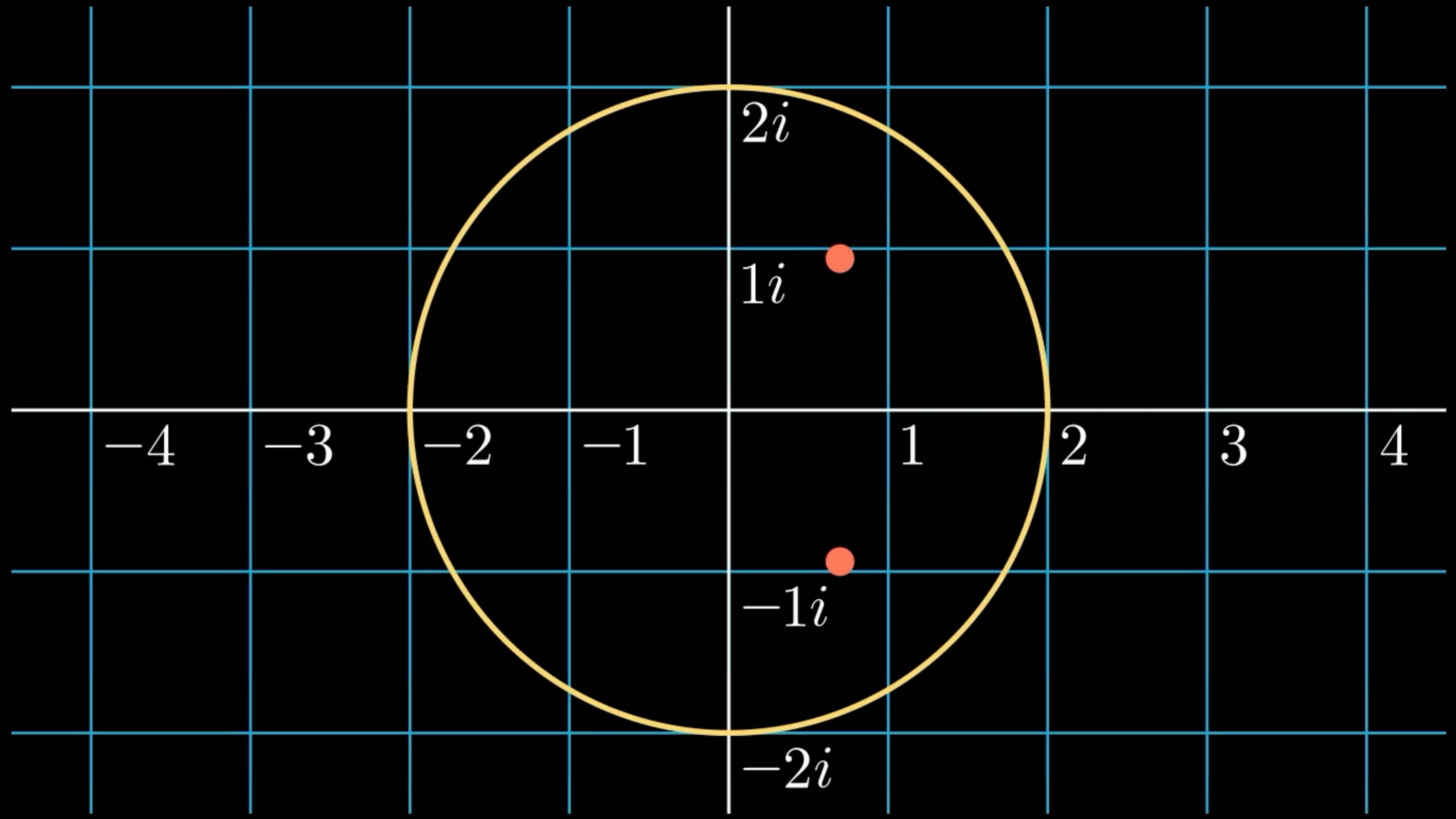
Order of the pole: 1



The PIII-Rogue wave

Number of Poles: 1

Order of the pole: 1



$$A = \left\{ |\mu_j| < \frac{\mu_{\mathscr{D}}}{4} N \right\} \cap \left\{ |v_j| < \frac{\mu_{\mathscr{D}}}{4} N \right\} \cap \{\text{too technical}\}$$

$$\frac{2}{N\mu_{\mathscr{D}}} \psi \left(\frac{2X}{N\mu_{\mathscr{D}}}, \frac{4T}{\mu_{\mathscr{D}}^2 N^2} \right) - \Psi_{III}(X, T) \xrightarrow{N \to \infty} 0$$

$$A = \left\{ |\mu_j| < \frac{\mu_{\mathscr{D}}}{4} N \right\} \cap \left\{ |\nu_j| < \frac{\mu_{\mathscr{D}}}{4} N \right\} \cap \{\text{too technical}\}$$

$$\frac{2}{N\mu_{\mathscr{D}}} \psi\left(\frac{2X}{N\mu_{\mathscr{D}}}, \frac{4T}{\mu_{\mathscr{D}}^2 N^2} \right) - \Psi_{III}(X, T) \xrightarrow{N\to\infty} 0$$

$$\mathbb{E}\left[\left| \frac{2}{N\mu_{\mathscr{D}}} \psi\left(\frac{2X}{N\mu_{\mathscr{D}}}, \frac{4T}{\mu_{\mathscr{D}}^2 N^2} \right) - \Psi_{III}(X, T) \right| \right] \xrightarrow{N\to\infty} 0$$

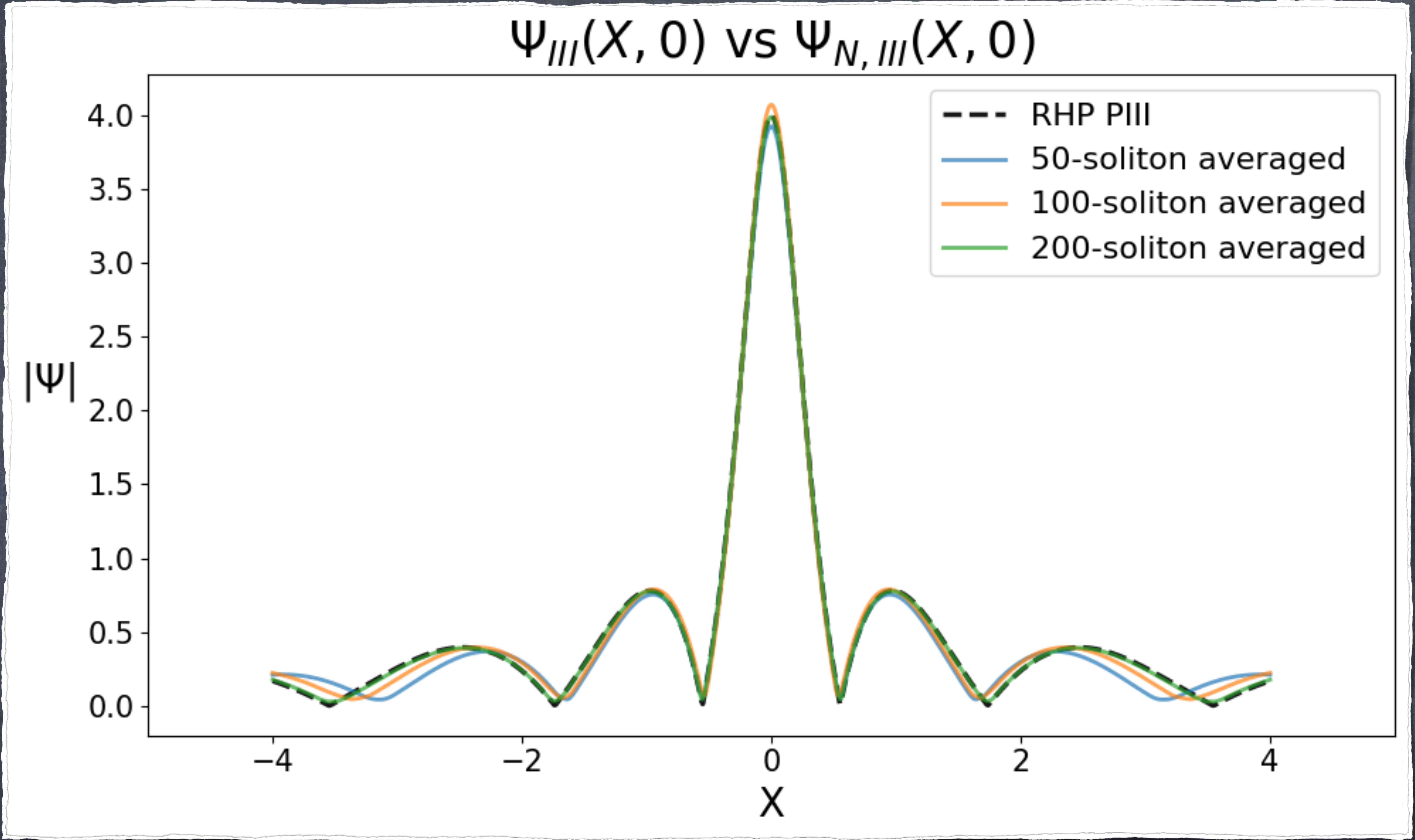
$$z_j = v_j + i\mu_j$$

Iid sub-exp

$$c_j = \frac{1}{a'(z_j)}$$

$$a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z_j}}$$

$$\mathbb{E}\left[\left|\frac{2}{N\mu_{\mathscr{D}}}\psi\left(\frac{2X}{N\mu_{\mathscr{D}}}, \frac{4T}{\mu_{\mathscr{D}}^2 N^2}\right) - \Psi_{III}(X, T)\right|\right] \xrightarrow{N\to\infty} 0$$



$$\frac{2}{N\mu_{\mathscr{D}}}\psi\left(\frac{2X}{N\mu_{\mathscr{D}}},\frac{4T}{\mu_{\mathscr{D}}^2N^2}\right),\Psi_{III}(X,T)$$

More exotic RW

$$z_j = -\zeta j + v_j + i\mu_j \qquad c_j = \frac{1}{a'(z_j)} \qquad a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z}_j}$$

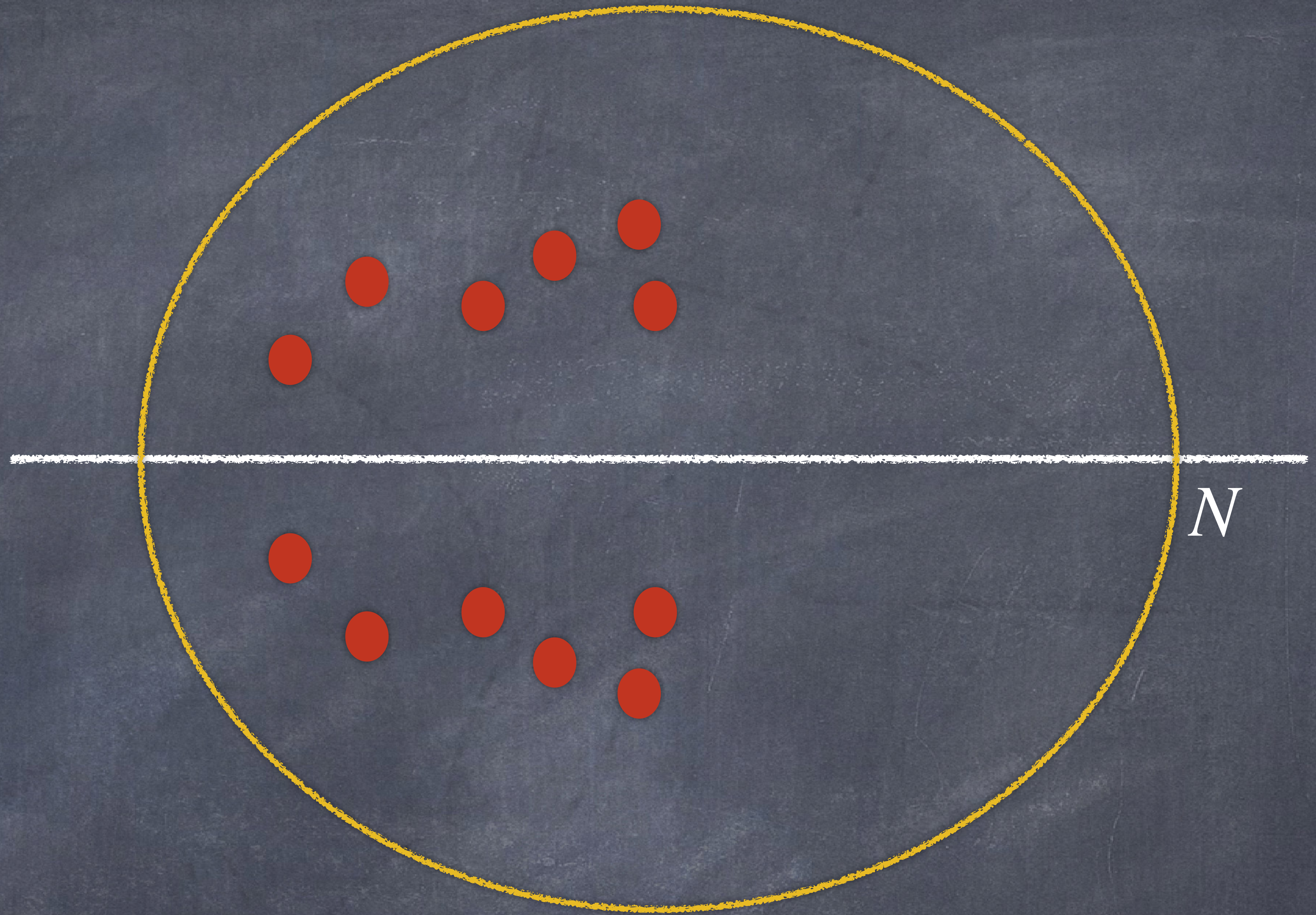
More exotic RW

$$z_j = -\zeta j + v_j + i\mu_j \qquad c_j = \frac{1}{a'(z_j)} \qquad a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z}_j}$$

Iid sub-exp

$$\mathrm{Res}_{z_j} M(z) = \lim_{z \to z_j} c_j e^{2i\theta(z,x,t)} M(z) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{Res}_{\overline{z}_j} M(z) = \lim_{z \to \overline{z}_j} -\overline{c}_j e^{-2i\theta(z,x,t)} M(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



$$\text{Res}_{z_j} M(z) = \lim_{z \to z_j} c_j e^{2i\theta(z,x,t)} M(z) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

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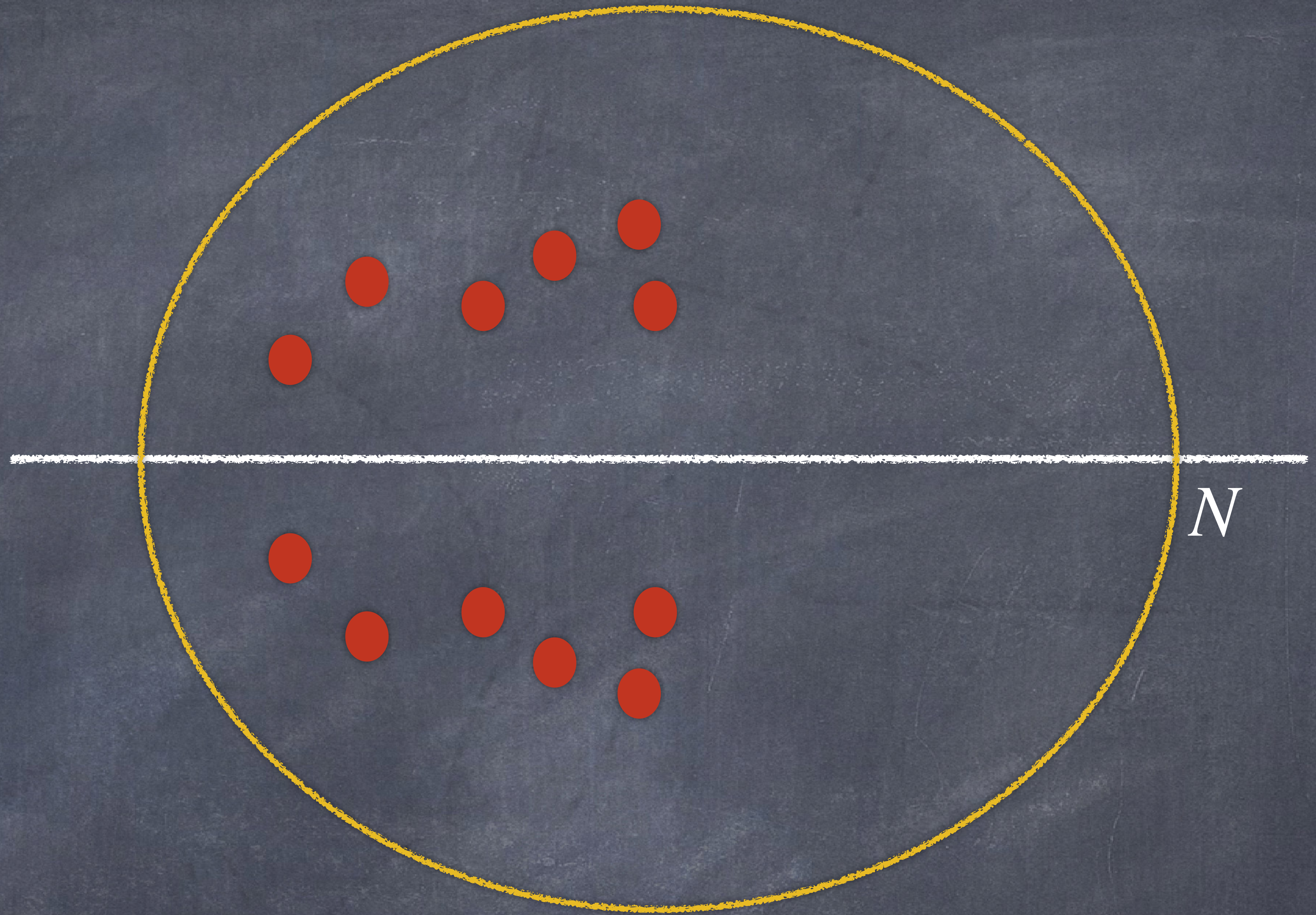

$$A = \{\, |\mu_j| < N \} \cap \{\, |v_j| < N \} \cap \{\text{too technical}\}$$

$$\mathrm{Res}_{z_j} M(z) = \lim_{z \to z_j} c_j e^{2i\theta(z,x,t)} M(z) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

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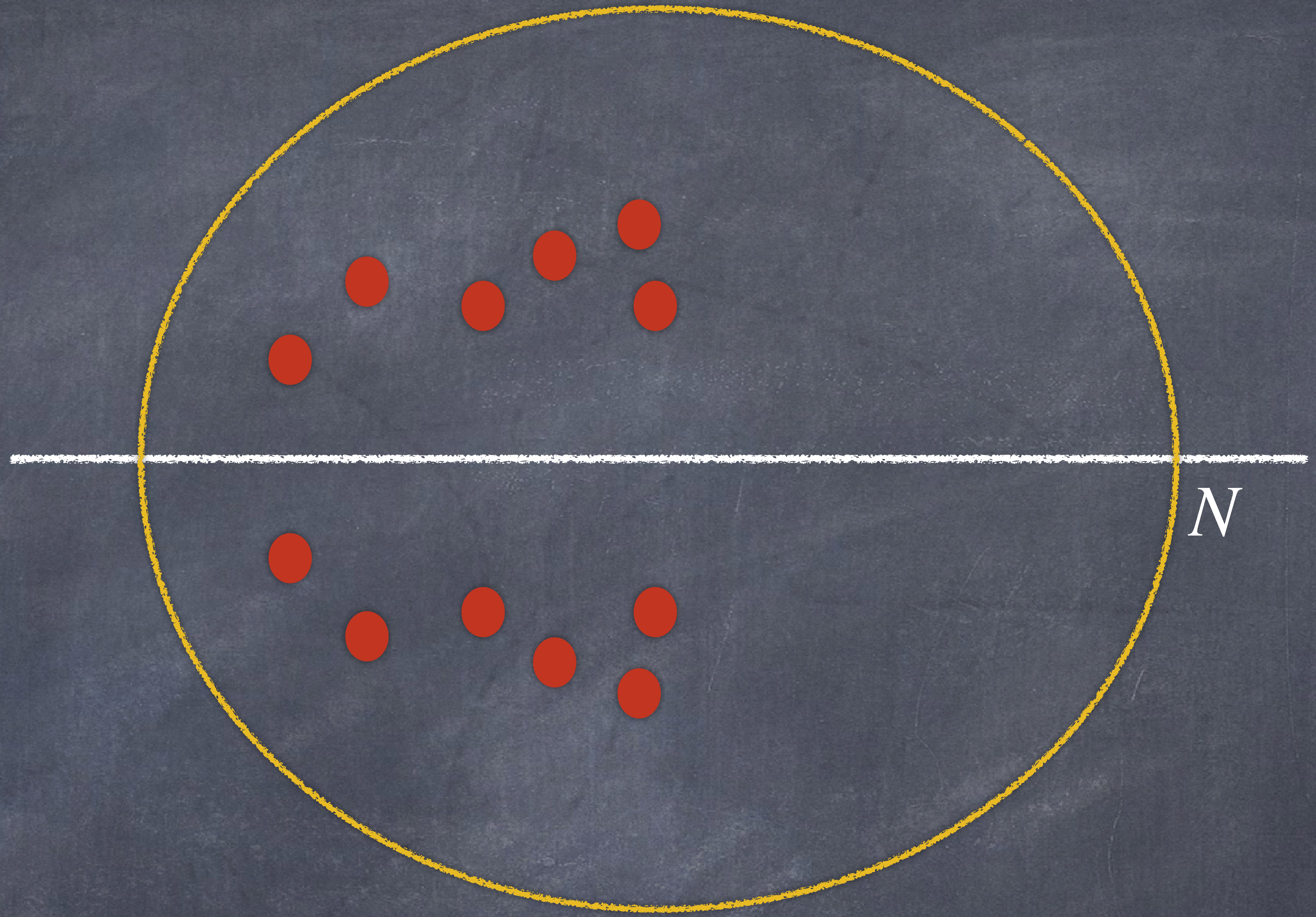
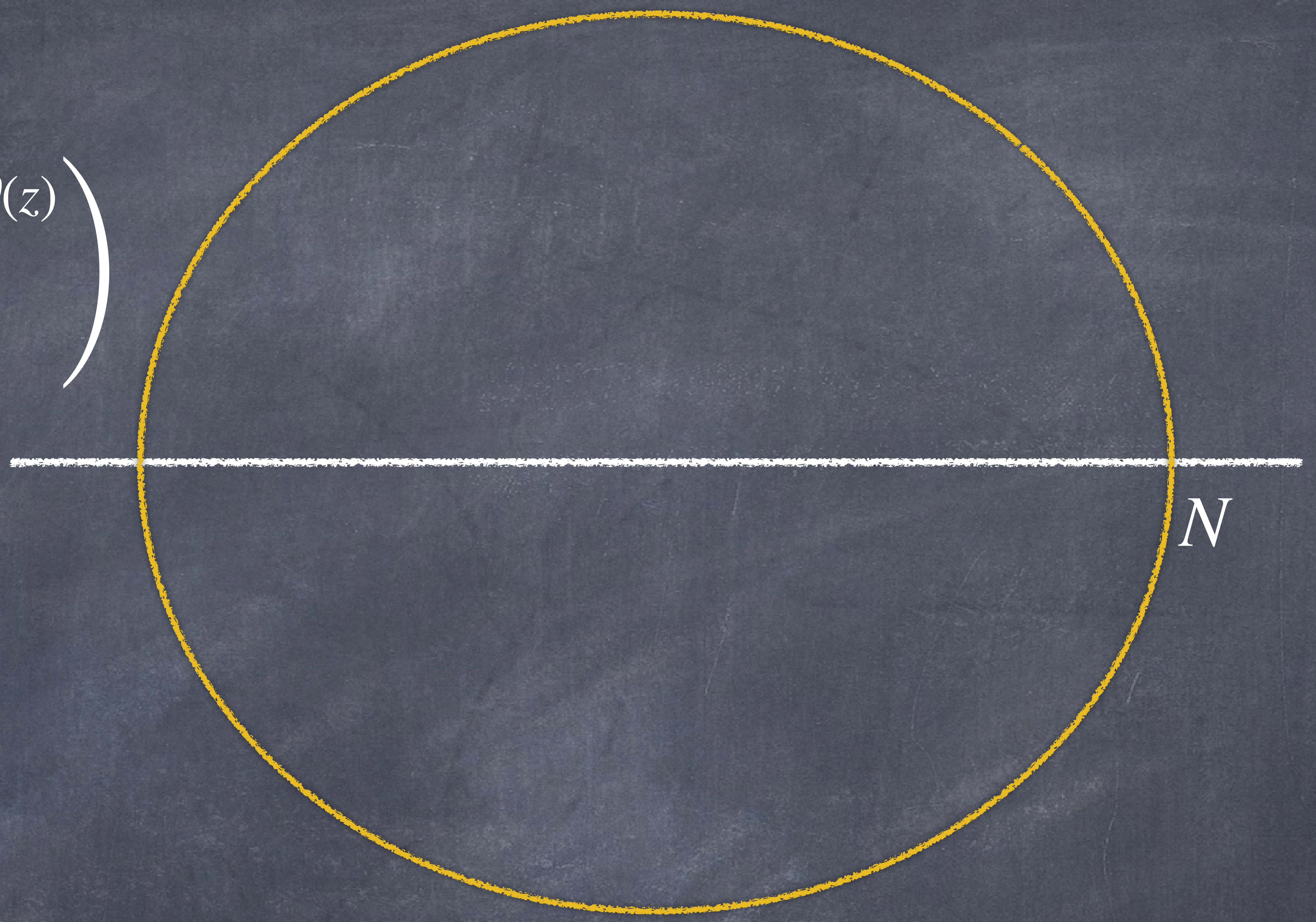


$$A = \{\, |\mu_j| < N \} \cap \{\, |v_j| < N \} \cap \{\text{too technical}\}$$

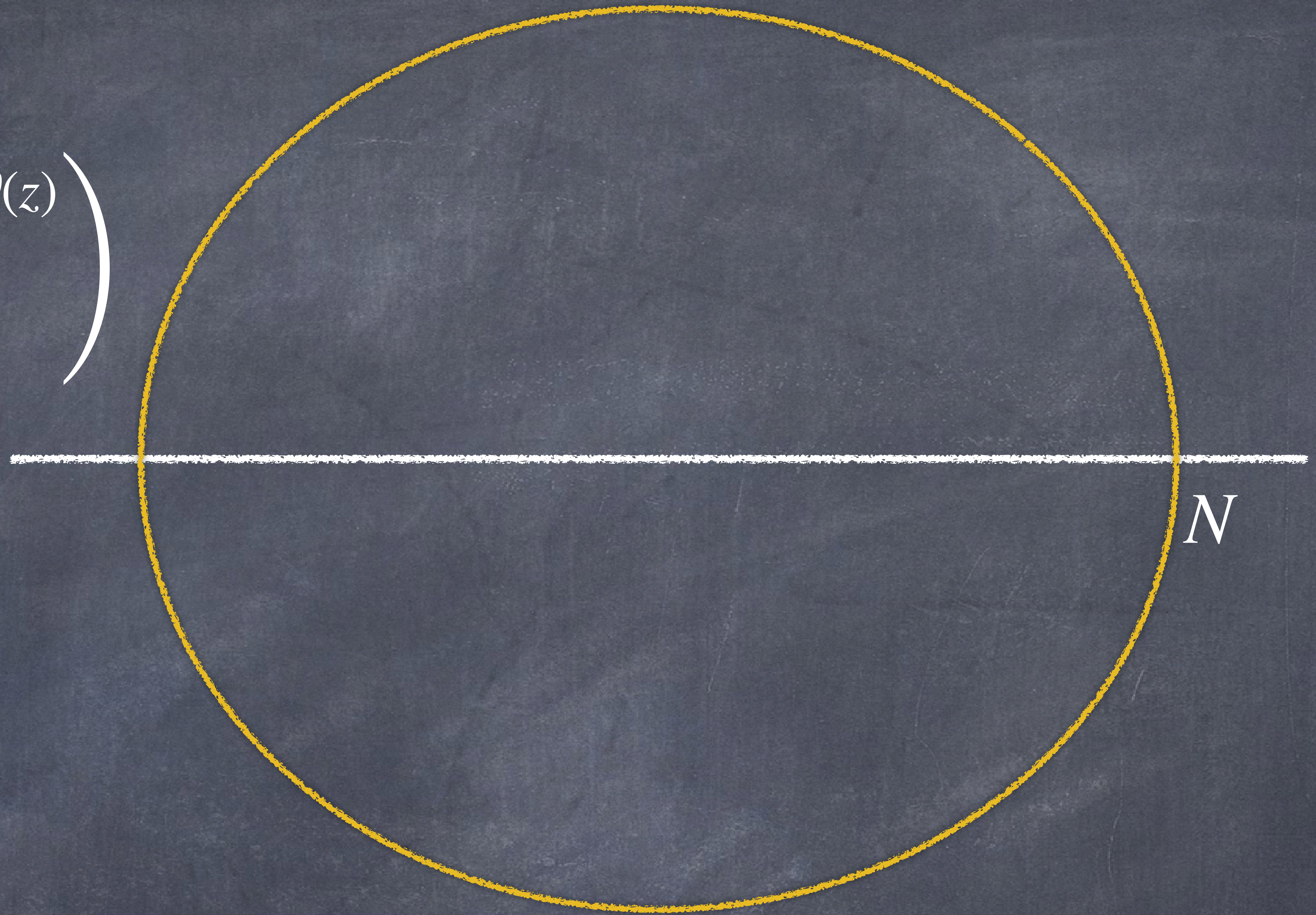
$$\mathbb{P}(A^c) < e^{-dN}$$

$$B(z)_+ = B(z)_- \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & a(z)e^{-2i\theta(z)} \\ -a^{-1}(z)e^{2i\theta(z)} & 1 \end{pmatrix}$$



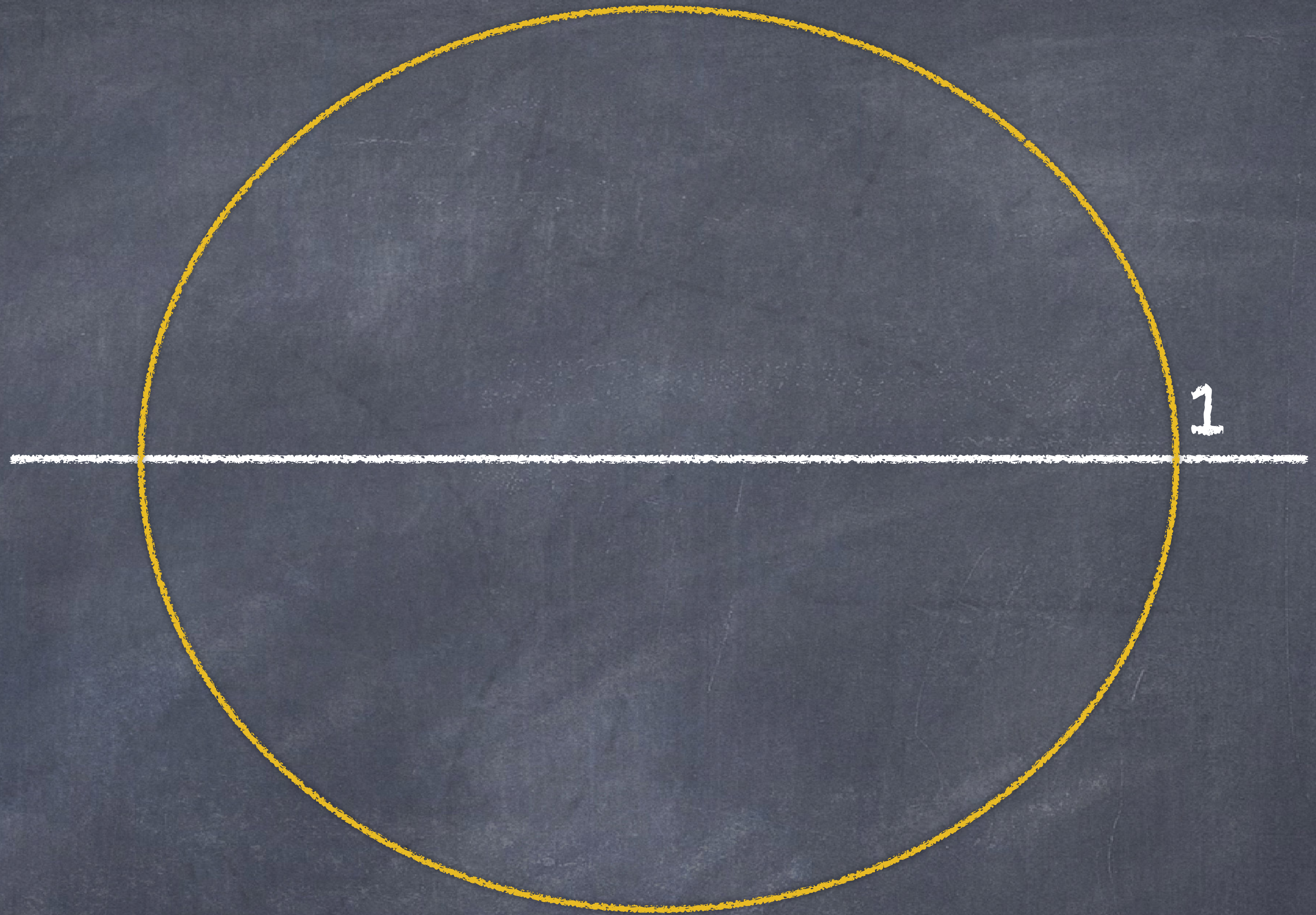
$$a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z}_j}$$

$$B(z)_+ = B(z)_- \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & a(z)e^{-2i\theta(z)} \\ -a^{-1}(z)e^{2i\theta(z)} & 1 \end{pmatrix}$$



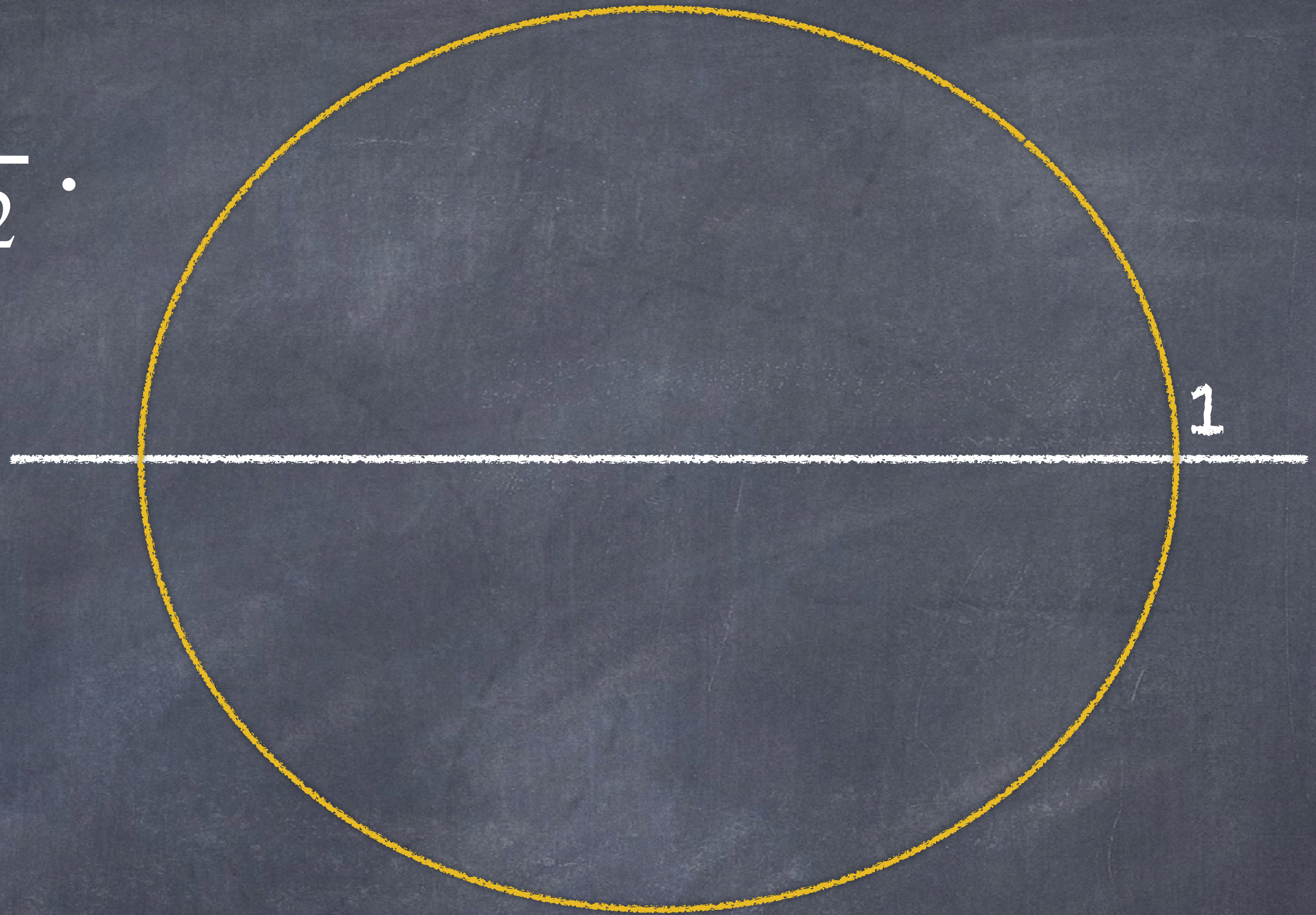
$$a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z}_j}$$

$$z = NZ, \quad x = \frac{X}{N}, \quad t = \frac{T}{N^2}.$$



$$R_+(Z) = R_-(Z)\frac{1}{\sqrt{2}}e^{-i\left(XZ + TZ^2 + \frac{\mu_{\mathscr{D}}}{\zeta}\ln\left(\frac{Z+\zeta}{Z}\right)\right)\sigma_3}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{i\left(XZ + TZ^2 + \frac{\mu_{\mathscr{D}}}{\zeta}\ln\left(\frac{Z+\zeta}{Z}\right)\right)\sigma_3}$$

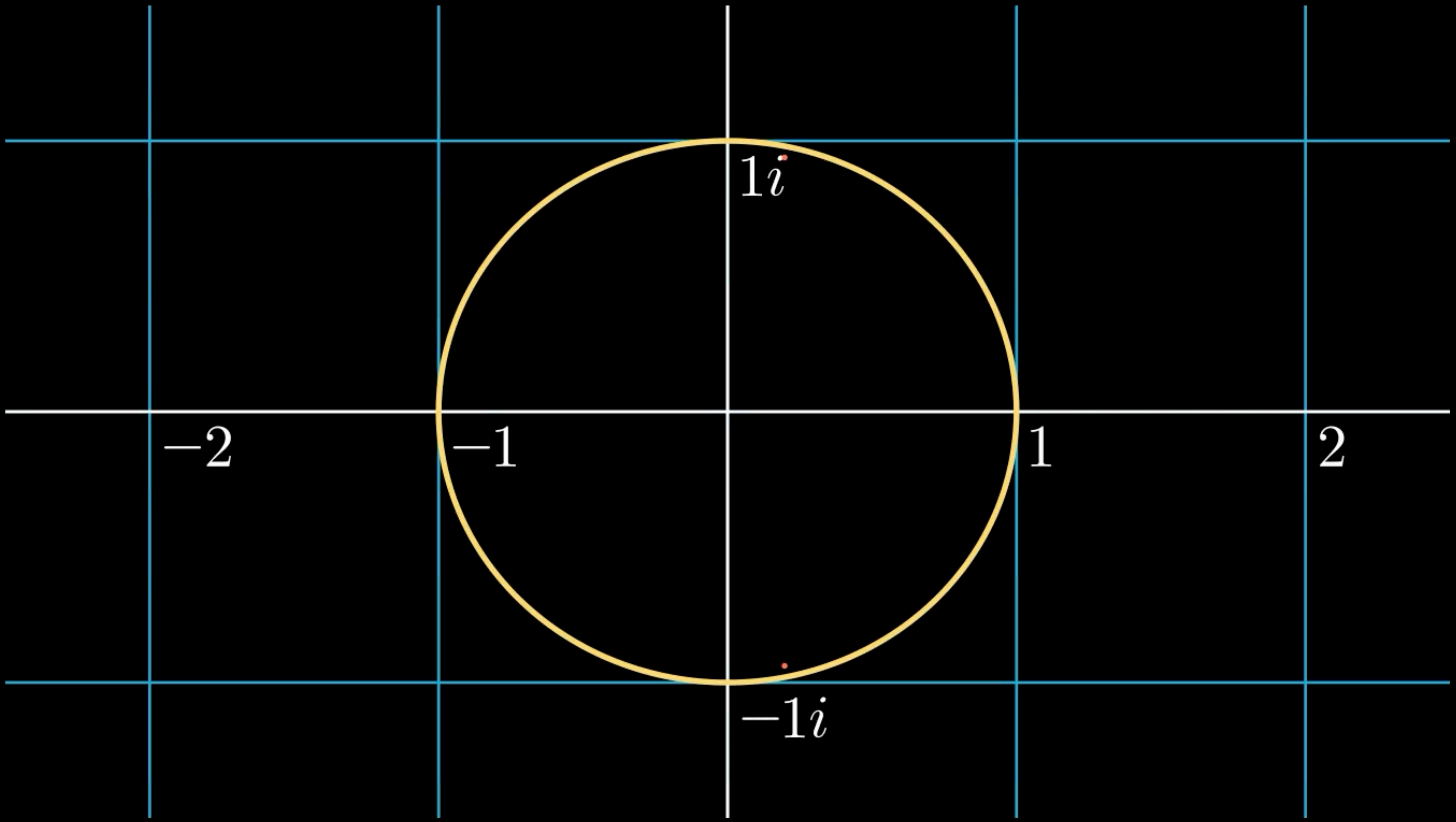
$$z = NZ\,, \quad x = \frac{X}{N}\,, \quad t = \frac{T}{N^2}\,.$$



$$R_+(Z) = R_-(Z)\frac{1}{\sqrt{2}}e^{-i\left(XZ + TZ^2 + \frac{\mu_{\mathscr{D}}}{\zeta}\ln\left(\frac{Z+\zeta}{Z}\right)\right)\sigma_3}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{i\left(XZ + TZ^2 + \frac{\mu_{\mathscr{D}}}{\zeta}\ln\left(\frac{Z+\zeta}{Z}\right)\right)\sigma_3}$$

Number of poles: 1

Order of the poles: 1



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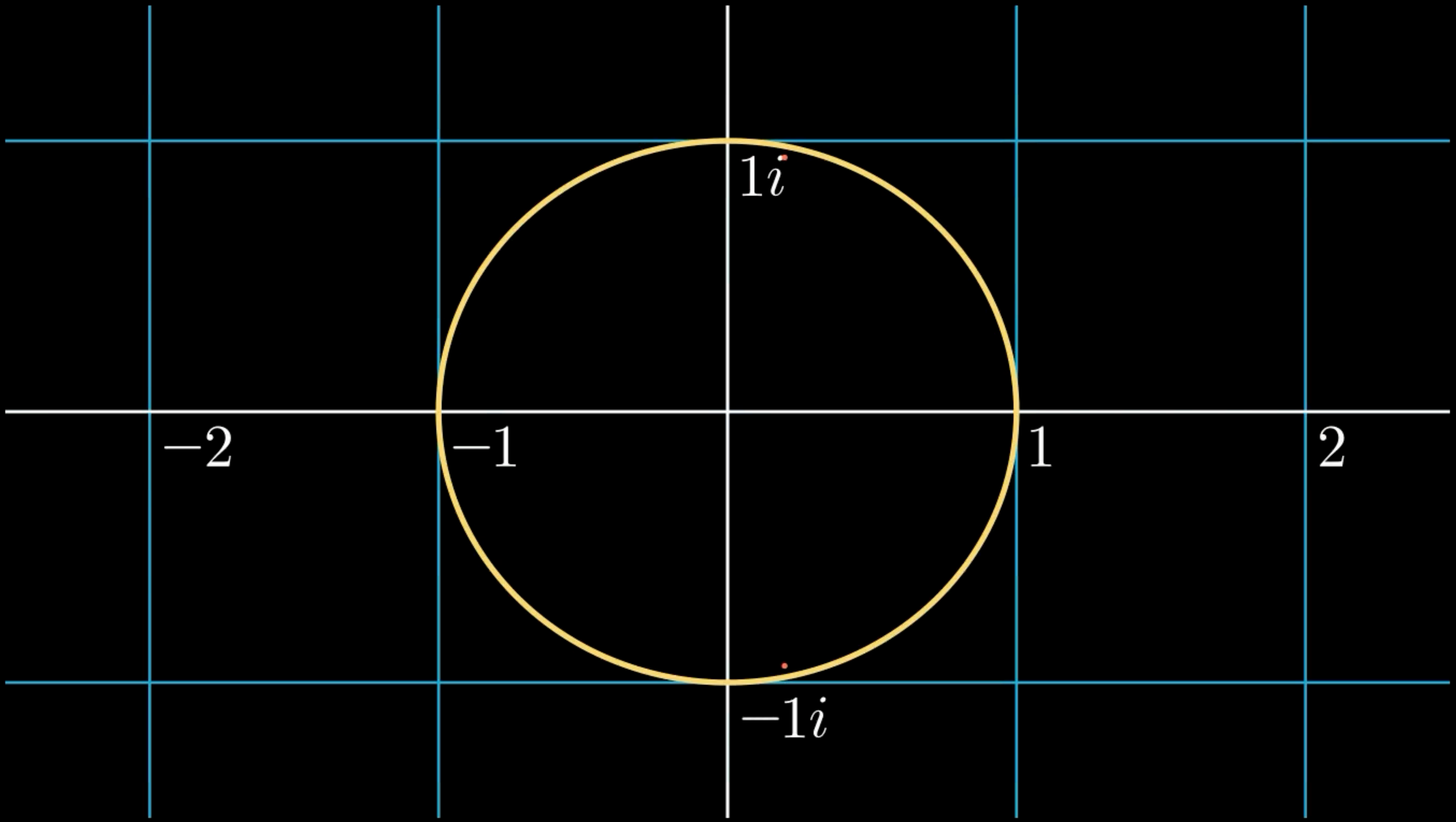

$$z_j = -\zeta j + v_j + i\mu_j \qquad c_j = \frac{1}{a'(z_j)} \qquad a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z}_j}$$

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"Explicit" solution in terms of PV

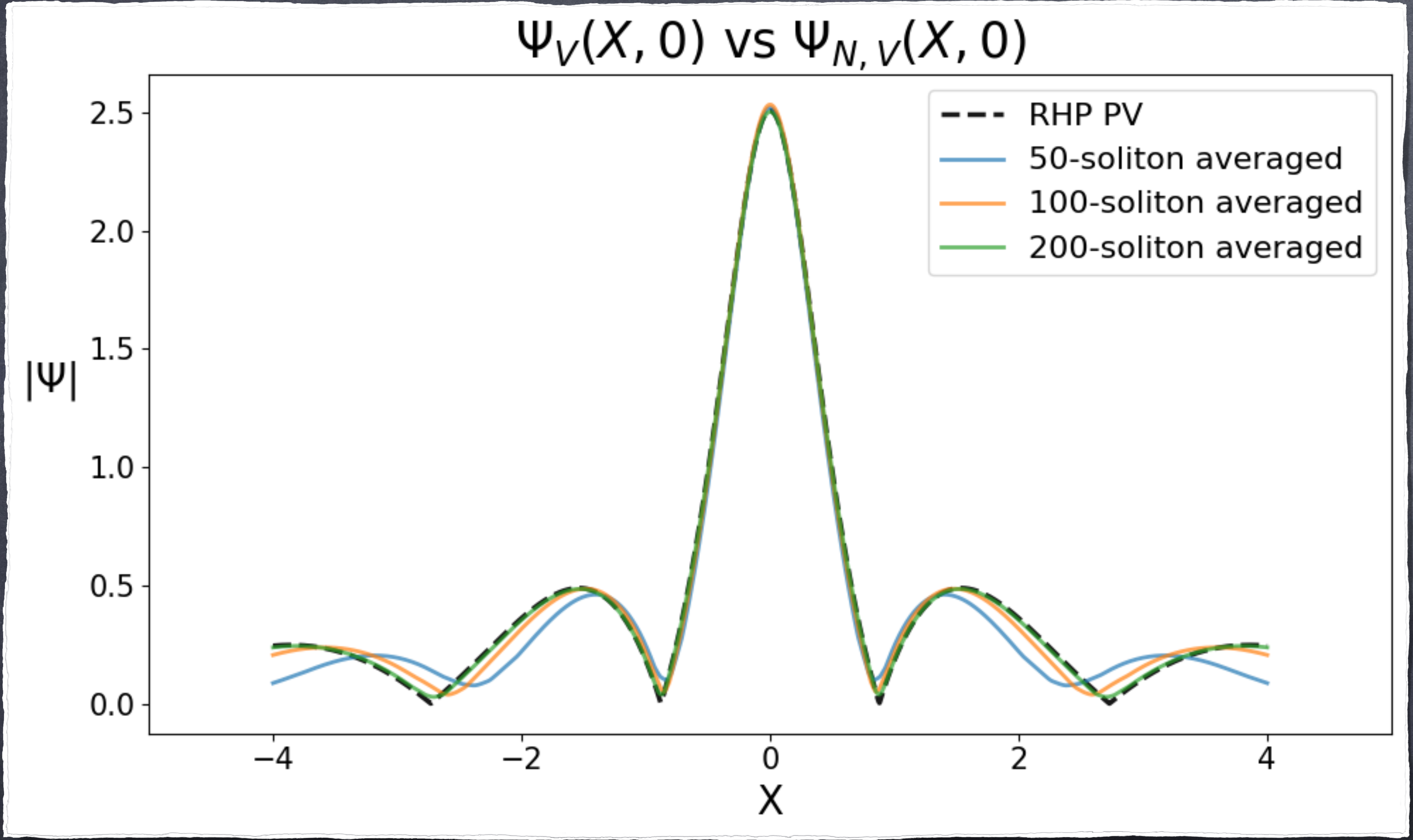
$$z_j = -\zeta j + v_j + i\mu_j \qquad c_j = \frac{1}{a'(z_j)} \qquad a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z_j}}$$

$$\mathbb{E}\left[\left|\frac{1}{N}\psi\left(\frac{X}{N}, \frac{T}{N^2}\right) - \Psi_V(X, T)\right|\right] \xrightarrow{N\to\infty} 0$$

$$z_j = -\zeta j + v_j + i\mu_j \qquad c_j = \frac{1}{a'(z_j)} \qquad a(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \overline{z_j}}$$

$$\mathbb{E}\left[\left|\frac{1}{N}\psi\left(\frac{X}{N}, \frac{T}{N^2}\right) - \Psi_V(X,T)\right|\right] \xrightarrow{N\to\infty} 0$$

$$\Psi_V(X,0) = \frac{\kappa}{X} \exp\left(-\frac{1}{2i\zeta}\int_1^{2i\zeta X} \frac{u(s)}{1-u(s)} ds\right)$$


$\Psi_V(X,0)$ vs $\Psi_{N,V}(X,0)$
RHP PV
50-soliton averaged
100-soliton averaged
200-soliton averaged
$|\Psi|$
0.0
0.5
1.0
1.5
2.0
2.5
−4
−2
0
2
4
X

Something to try

Something to try

- Describe an order 1 open set

Something to try

- Describe an order 1 open set
- Random norming constants

Something to try

- Describe an order 1 open set
- Random norming constants
- Large Deviation Principle

Something to try

- Describe an order 1 open set
- Random norming constants
- Large Deviation Principle
- True Probabilistic RHP analysis

Thanks!