Infinite peakon solutions of the Camassa–Holm equation

Aleksey Kostenko

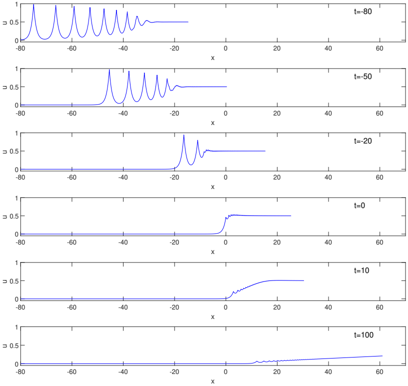
joint work with **Xiang-Ke Chang** (Beijing) and **Jonathan Eckhardt** (Loughborough)

Dispersive integrable equations: Pathfinders in Hamiltonian PDE

Institut Henri Poincaré
June 25, 2026

Snapshots of infinite peakon solutions

The goal of this talk is to explain two pictures (see also animation):



Snapshots
*of the first 100 peakons**
associated with the
Laguerre polynomials

(the details are at the very end of the talk; see Theorem I)

* Zoom in to see peakons in the tail!

Animation: doi:10.17028/rd.lboro.29110385.

Snapshots of infinite peakon solutions

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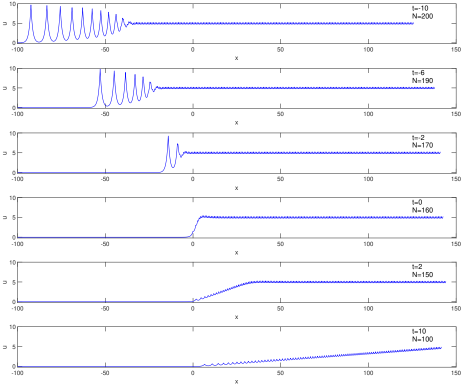

Snapshots
of the first N peakons associated with the Legendre polynomials

(the details are at the very end of the talk; see Theorem II)

Animation: doi:10.17028/rd.lboro.29110385.

The Camassa–Holm Equation

$$u_t - u_{xxt} + 2\kappa u_x = 2u_x u_{xx} - 3uu_x + uu_{xxx} \qquad \text{(CH)}$$

First appearance within a family of **bi-Hamiltonian** PDEs in

B. Fuchssteiner & A. S. Fokas// Phys. D **4** (1982)

with Hamiltonians given by

$$H_1 = \int u^2 + u_x^2\, dx, \qquad H_2 = \frac{1}{2}\int u^3 + uu_x^2 - \kappa u^2\, dx.$$

Denoting $m = u - u_{xx} + \kappa$, (CH) can be written as

$$m_t = -(m\partial_x + \partial_x m)\frac{\delta H_1}{\delta m} = -(\partial_x - \partial_x^3)\frac{\delta H_2}{\delta m},$$

which then gives recursively an infinite number of conserved quantities $\{H_n\}_{n\in\mathbb{Z}}$

$$(m\partial_x + \partial_x m)\frac{\delta H_{n-1}}{\delta m} = (\partial_x - \partial_x^3)\frac{\delta H_n}{\delta m}.$$

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A model for shallow water waves,
(u is the fluid velocity in x direction, $\kappa > 0$ is the critical wave speed)

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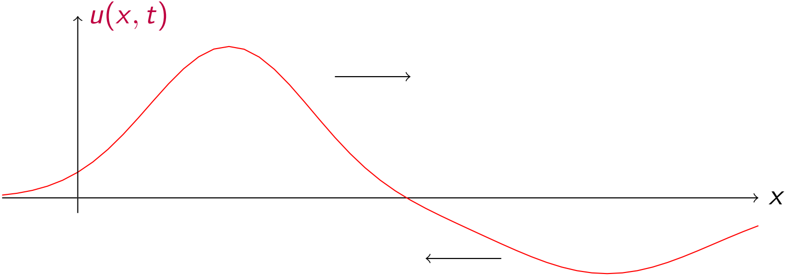
*If $u(t,x)$ solves (CH), then $u(t,x-\kappa t)+\kappa$ solves (CH) with $\kappa = 0$

Weak formulation (with $\kappa = 0$):

$$u_t + uu_x + P_x = 0, \qquad P = \frac{1}{2}\mathrm{e}^{-|x|} * \left(u^2 + \frac{1}{2}u_x^2\right) \qquad \text{(CHweak)}$$

Camassa–Holm equation: Wave breaking

The Camassa–Holm equation models wave breaking:



- Smooth initial data may develop singularities in finite time:

  Solutions stay bounded but their slope may become vertical.

- Wave breaking only happens when $\omega = u - u_{xx}$ changes sign!

(Smallness or smoothness of initial data are irrelevant, only the form of an initial profile matters.)

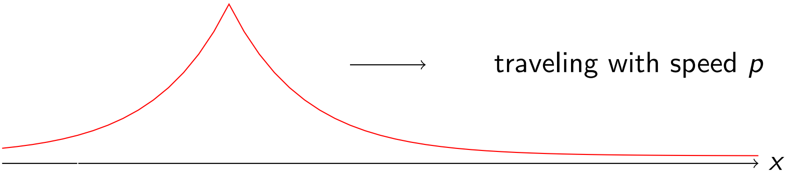
Camassa–Holm equation: Peakon solutions ($\kappa = 0$)

The Camassa–Holm equation (weak formulation)

$$u_t + uu_x + P_x = 0, \qquad P = \frac{1}{2}\mathrm{e}^{-|x|} * \Big(u^2 + \frac{1}{2}u_x^2\Big), \qquad \text{(CHweak)}$$

has traveling wave solutions called peakons ($\sim$ *peaked solitons*)

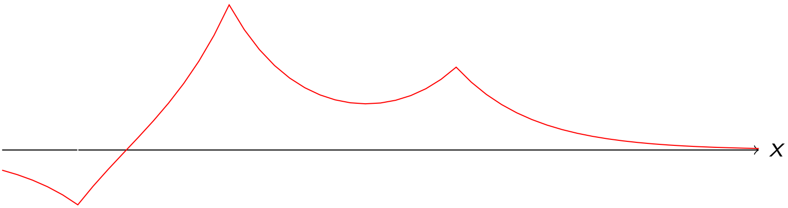
$$u(x,t) = p\,\mathrm{e}^{-|x-pt-c|} = \frac{1}{2}\mathrm{e}^{-|x|} * 2p\,\delta_{pt+c}, \quad x,\ t \in \mathbb{R}$$



Camassa–Holm equation: Multi-peakon solutions

More generally a multi-peakon is given by

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{N} p_n(t)\mathrm{e}^{-|x-q_n(t)|}, \quad x,\ t \in \mathbb{R},$$



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where p, q are solutions of the following Hamiltonian system

$$\dot{q}_n = \frac{\partial H(p,q)}{\partial p_n}, \qquad \dot{p}_n = -\frac{\partial H(p,q)}{\partial q_n},$$

with the Hamiltonian

$$H(p,q) = \frac{1}{4}\sum_{n,k=1}^{N} p_n p_k \mathrm{e}^{-|q_n-q_k|} = 2\|u\|^2_{H^1(\mathbb{R})} = 2H_1[u].$$

R. Beals, D. H. Sattinger & J. Szmigielski, *Multipeakons and the classical moment problem*, Adv. Math. **154** (2000).

R. Beals, D. H. Sattinger & J. Szmigielski, *Peakons, strings, and the finite Toda lattice*, Comm. Pure Appl. Math. **54** (2001).

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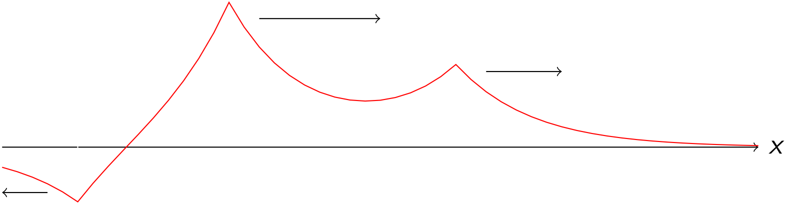

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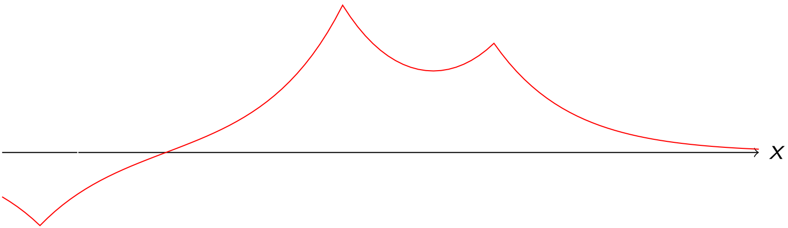

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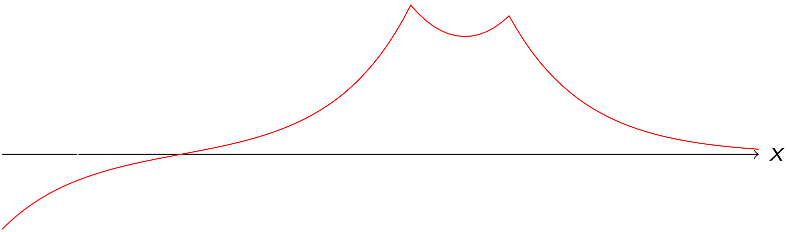

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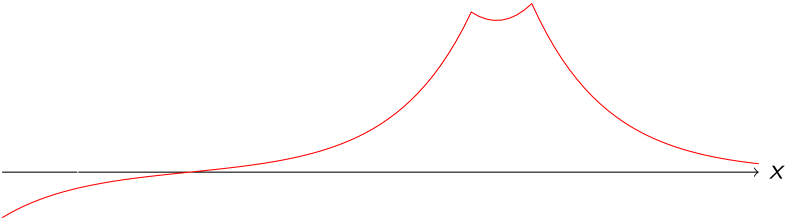

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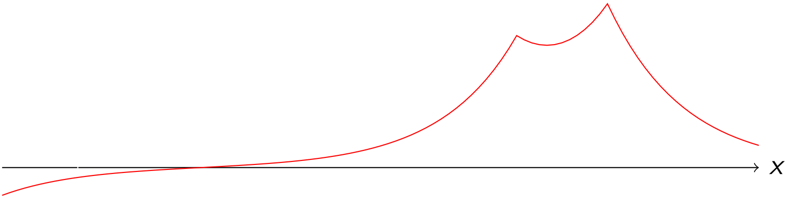

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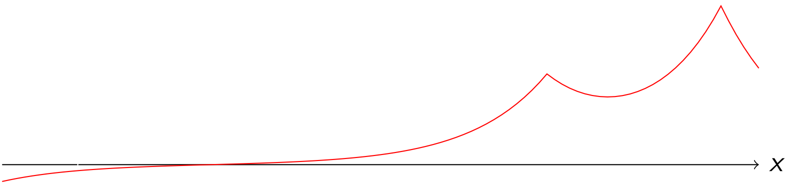

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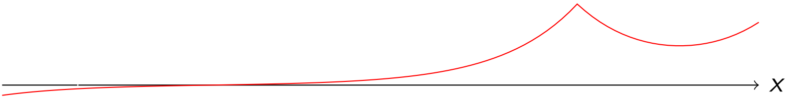

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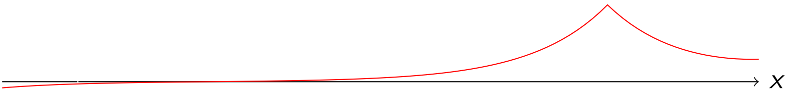
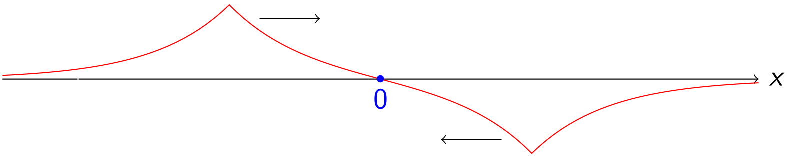


Peakon–Antipeakon Collision

$$u(x,t) = \frac{1}{2}p(t)\mathrm{e}^{-|x-q(t)|} - \frac{1}{2}p(t)\mathrm{e}^{-|x+q(t)|}; \qquad p(0) > 0, \quad q(0) < 0.$$

R. Camassa, D. Holm and J. Hyman, *A new integrable shallow water equation*, Adv. Appl. Mech. **31** (1994).



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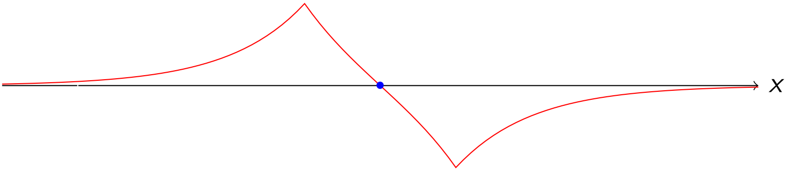

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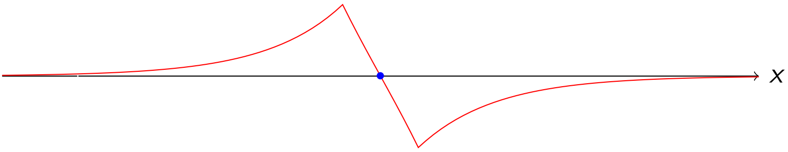

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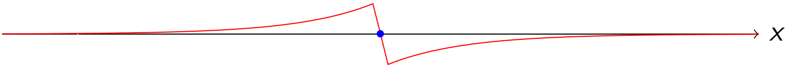

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$$\longrightarrow u(x,t^{\times}) \equiv 0$$

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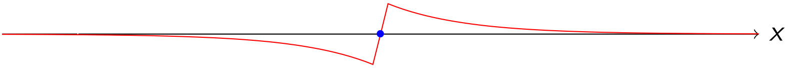

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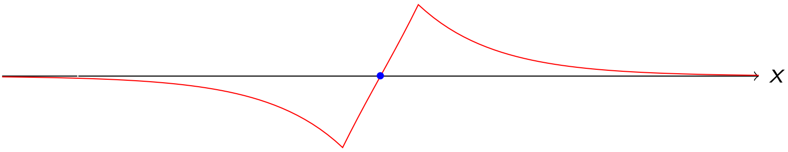

Inverse Spectral/Scattering Transform

Setting

$$\omega(x,t) := u(x,t) - u_{xx}(x,t)$$

consider the family of Sturm–Liouville problems ("Lax operators" for (CH))

$$-f'' + \frac{1}{4}f = z\,\omega(\,\cdot\,,t)f \quad \text{on} \quad \mathbb{R} \tag{Iso}$$

Lax pair structure

Equation (CH) follows from the compatibility condition $\psi_{xxt} = \psi_{txx}$ for the system

$$-\psi_{xx} + \frac{1}{4}\psi = z\omega(x,t)\psi,$$
$$\psi_t = -\Big(\frac{1}{2z} + u(x,t)\Big)\psi_x + \frac{1}{2}u_x\psi.$$

NOTE that $\omega \in H^{-1}_{\mathrm{loc}}$ if $u \in H^1_{\mathrm{loc}}$.

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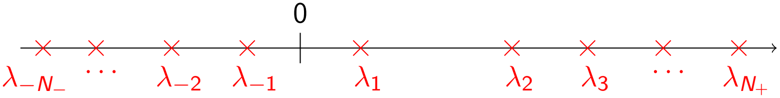
- Spectral picture for multi-peakons: $\omega = \sum_{n=1}^{N} p_n(t)\delta_{q_n(t)}$



Spectral data $S = (\sigma, \gamma)$

λ belongs to the spectrum σ $\Leftrightarrow$ there is a solution $0 \neq f \in H^1(\mathbb{R})$ to (Iso).
For $\lambda \in \sigma$, the norming constant $\gamma_\lambda := \|f_\lambda\|^{-1}$ with $f_\lambda = \mathrm{e}^{x/2}$ near $-\infty$.

- $N_\pm = \#\{n \in \{1,\ldots,N\} \mid \pm p_n(0) > 0\}$

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consider the family of Sturm–Liouville problems ("Lax operators" for (CH))

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- The inverse spectral/scattering transform

$$\begin{tikzcd}
\mathbf{S}(0) \arrow[rr, "\text{linear flow}"'] & & \mathbf{S}(t) \arrow[d, "\Psi^{-1}"] \\
u(\cdot,0) \arrow[u, "\Psi"] \arrow[rr, dashed, "\text{Potential weak solution}"'] & & u(\cdot,t)
\end{tikzcd}$$

$\boxed{\text{Time evolution: } \sigma(t) = \sigma(0) \text{ and } \gamma_\lambda(t) = \mathrm{e}^{-t/2\lambda}\gamma_\lambda(0)}$

Inverse Spectral/Scattering Transform

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$$\begin{array}{ccc} \mathbf{S}(0) & \xrightarrow[\text{linear flow}]{\hspace{6em}} & \mathbf{S}(t) \\ \Psi \uparrow & & \downarrow \Psi^{-1} \\ u(\cdot,0) & \xrightarrow[\text{Potential weak solution}]{\hspace{6em}} & u(\cdot,t) \end{array}$$

Requires to solve the inverse spectral problem for (Iso): $(\sigma, \{\gamma_\lambda\}_{\lambda\in\sigma}) \mapsto u$

- Uniqueness
- Existence
- Stability
- Reconstruction

Inverse Spectral/Scattering Transform

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Spectral Problem

$$-f'' + \frac{1}{4} f = z\,\omega(\,\cdot\,,t) f \quad \text{on} \quad \mathbb{R} \tag{Iso}$$

It would be convenient to make the following simplification: change of a spatial variable

$$s\colon x \mapsto \mathrm{e}^{x}$$

which maps $\mathbb{R}$ to $(0,\infty)$, then $g = f \circ s$ solves

$$-g'' = z\,\tilde{\omega}(\,\cdot\,,t) g \quad \text{on} \quad [0,\infty) \tag{Iso}$$

with

$$\int_0^x \tilde{\omega} = -\frac{u(\log x) + u'(\log x)}{x}.$$

In particular, in the multi-peakon case

$$\tilde{\omega} = p_0 \delta_0 + \sum_{n=1}^{N} \frac{p_n}{\mathrm{e}^{q_n}} \delta_{\mathrm{e}^{q_n}}\,, \qquad p_0 := -\sum_{n=1}^{N} \frac{p_n}{\mathrm{e}^{q_n}}$$

Interlude: Krein and Stieltjes strings

$$-f'' = z\,\omega f \quad \text{on} \quad [0, L) \tag{S1}$$

...with $L \in (0, \infty]$ and ω a positive Borel measure on $[0, L)$.

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...with $L \in (0, \infty]$ and ω a positive Borel measure on $[0, L)$.

How to understand this equation?

(S1) is equivalent to the integral equation

$$f(z,x) = a + bx - z\int_{[0,x)} (x-\xi) f(z,\xi)\,\omega(d\xi)$$

- Existence & uniqueness,
- Analyticity w.r.t spectral parameter,
- Variation of parameters formula, ...
- Fundamental system of solutions: $c(z,x)$ and $s(z,x)$,

$$c(z,0) = s'(z,-0) = 1, \quad c'(z,-0) = s(z,0) = 0$$

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The Weyl–Titchmarsh function

$$m(z) = \lim_{x \uparrow L} -\frac{c(z,x)}{z\, s(z,x)}, \quad z \in \mathbb{C} \backslash [0, \infty)$$

... also called *a coefficient of a dynamical compliance of a string*

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- m is a Stieltjes function

$$m(z) = \omega(\{0\}) - \frac{1}{Lz} + \int_{(0,\infty)} \frac{1}{\lambda - z} d\rho(\lambda), \quad z \in \mathbb{C} \backslash [0, \infty).$$

- ρ is a spectral measure, which satisfies $\int_{(0,\infty)} \frac{d\rho(\lambda)}{1+\lambda} < \infty$.

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- m is a Stieltjes function: for multi-peakons
$$m(z) = p_0 + \sum_{\lambda\in\sigma} \frac{\gamma_\lambda}{\lambda - z}.$$
- ρ is a spectral measure, which satisfies $\int_{(0,\infty)} \frac{d\rho(\lambda)}{1+\lambda} < \infty$.

The inverse spectral problem: Stieltjes strings

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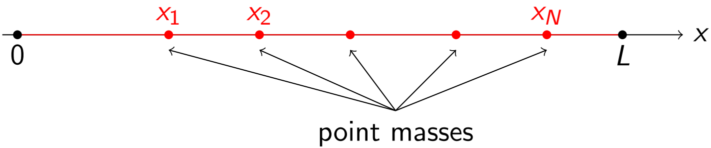

Stieltjes strings

$$\omega = \sum_{n=0}^{N} \omega_n \delta_{x_n}$$

The inverse spectral problem: Stieltjes strings

$$-f'' = z\,\omega f \quad \text{on} \quad [0, L) \tag{S1}$$

...with ω a positive Borel measure on $[0, L)$.

The corresponding integral equation

$$y(z,x) = a + bx - z\sum_{x_n<x}(x - x_n)y(z,x_n)\omega_n,$$

which turns to be the finite-difference equation:

$$\begin{aligned} y(z,x_{n+1}) - y(z,x_n) &= (x_{n+1} - x_n)y'(z,x_n+), \\ y'(z,x_n+) - y'(z,x_n-) &= -z\omega_n y(z,x_n), \end{aligned}$$

where we also used $y(z,x_n-) = y(z,x_n+) =: y(z,x_n)$.

The inverse spectral problem: Stieltjes strings

$$-f'' = z\,\omega f \quad \text{on} \quad [0, L) \tag{S1}$$

...with ω a positive Borel measure on $[0, L)$.

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Hence $c(z,x)$ and $s(z,x)$ are polynomials in z of order at most N and therefore $m(z)$ is a rational Stieltjes function.

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Weyl–Titchmarsh function of Stieltjes strings

$$m(z) = \omega_0 + \cfrac{1}{-l_0 z + \cfrac{1}{\omega_1 + \cfrac{1}{\ddots + \cfrac{1}{-l_{N-1}z + \cfrac{1}{\omega_N - \cfrac{1}{l_N z}}}}}}$$

$$l_n = x_n - x_{n-1}, \quad x_0 = 0, \quad x_N = L$$

The inverse spectral problem: Continued fractions

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Given a **rational** Stieltjes function m, we can write

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$\rightarrow$ construct a Stieltjes string (L, ω) with W–T function m

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...with ω a positive Borel measure on $[0, L)$.

T. Stieltjes, *Recherches sur les Fractions Continues* (1894)

$$l_n = \frac{\Delta_{1,n}^2}{\Delta_{0,n}\Delta_{0,n+1}}, \qquad \omega_n = \frac{\Delta_{0,n}^2}{\Delta_{1,n-1}\Delta_{1,n}}$$

$$\Delta_{0,k} = \left|(s_{i+j})_{i,j=0}^{k-1}\right|, \quad \Delta_{1,k} = \left|(s_{i+j+1})_{i,j=0}^{k-1}\right|$$

$$s_k = \int_{\mathbb{R}} \lambda^k d\rho(\lambda), \qquad m(z) = -\sum_{k=1}^{\infty} \frac{s_{k-1}}{z^k}, \quad |z| \to \infty$$

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Stieltjes formulas work for signed measures ω, however, there are Herglotz functions not having the above continued fraction expansion!

Rational Herglotz functions and Krein–Langer strings

However, every **rational Herglotz function** admits the expansion

$$m(z) = \omega_0 + \upsilon_0 z + \cfrac{1}{-l_0\, z + \cfrac{1}{\omega_1 + \upsilon_1 z + \cfrac{1}{\ddots + \cfrac{1}{-l_n\, z}}}}.$$

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It is the Weyl–Titchmarsh function of the **Krein–Langer string**

$$-f'' = z\,\omega f + z^2\,\upsilon f, \qquad x \in [0, L),$$

with $\omega = \sum \omega_k \delta_{x_k}$, $\upsilon = \sum \upsilon_k \delta_{x_k}$, $x_k - x_{k-1} := l_k$ and $L = \sum l_k$.

M. G. Krein & H. Langer, *On some extension problems which are closely connected with the theory of Hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems*, Beiträge Anal. **14**, 25–40 (1979); **15**, 27–45 (1980).

More general class of spectral problems!

$$-f'' + \frac{1}{4}f = z\,\omega(\,\cdot\,,t)f + z^2\upsilon(\,\cdot\,,t)f \quad \text{on} \quad \mathbb{R} \tag{Iso}$$

M. G. Krein and H. Langer, ... *Indefinite analogues of the Hamburger and Stieltjes moment problems*, Beiträge Anal. **14**, (1979).

- The inverse spectral/scattering transform

$$\begin{array}{ccc} \mathbf{S}(0) & \xrightarrow[\text{linear flow}]{} & \mathbf{S}(t) \\ \Big\uparrow \Psi & & \Big\downarrow \Psi^{-1} \\ u(\cdot,0) & \overset{}{\dashrightarrow} & (u(\cdot,t),\upsilon(\cdot,t)) \\ & \text{Potential weak solution} & \end{array}$$

$\boxed{\text{Requires to solve the inverse spectral problem for (Iso): } (\sigma,\{\gamma_\lambda\}_{\lambda\in\sigma}) \mapsto (u,\upsilon)}$

- Uniqueness • Existence • Stability • Reconstruction

Inverse Spectral/Scattering Transform

More general class of spectral problems!

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- The inverse spectral/scattering transform

$$\begin{array}{ccc}
\mathbf{S}(0) & \xrightarrow[\text{linear flow}]{\hspace{6em}} & \mathbf{S}(t) \\
\uparrow \Psi & & \downarrow \Psi^{-1} \\
u(\cdot,0) & \xrightarrow[\text{Weak solution}]{\hspace{6em}} & (u(\cdot,t),\upsilon(\cdot,t))
\end{array}$$

J. Eckhardt and A. Kostenko, *An isospectral problem for global conservative multi-peakon solutions of the Camassa–Holm equation*, Comm. Math. Phys. **329** (2014).

Peakon–Antipeakon Interaction

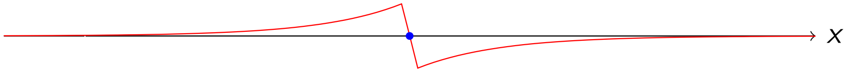
$$u(x,t) = \frac{1}{2}p(t)\mathrm{e}^{-|x-q(t)|} - \frac{1}{2}p(t)\mathrm{e}^{-|x+q(t)|}; \qquad p(0)>0, \quad q(0)<0.$$

A. Bressan and A. Constantin, *Global conservative solutions of the Camassa–Holm equation*, Arch. Ration. Mech. Anal. **183** (2007)

- $u(x,t) \to 0$ as $t \uparrow t^{\times}$ for all $x \in \mathbb{R}$, and
- $u_x(t,0) = -p(t)\mathrm{e}^{q(t)} \downarrow -\infty$, as $t \uparrow t^{\times}$.

However,

$$\int_{-q(t)}^{q(t)} (u^2+u_x^2)dx = \frac{1}{4}H_0^2(1+\mathrm{e}^{2q(t)}) \to \frac{1}{2}H_0^2 = \frac{1}{2}H(p,q) = \|u_0\|_{H^1}^2, \quad t\uparrow t^{\times}.$$



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Conservative solutions: (u,μ) with $\mu = \upsilon + u^2 + u_x^2$

additional quantity υ measuring the loss of energy at the times of blowup...
For multi-peakons, $\upsilon(\cdot,t) \not\equiv 0$ only at the blowup times, supported at collisions and measures the energy losses of colliding peakons

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Theorem (Eckhardt & AK// Comm. Math. Phys. (2014))

Global conservative multi-peakon solutions are integrable by the IST.

- Collision times are the zeros of $\Delta_{1,n}(t)$, $n = 1, \ldots, N$.

Infinite multi-peakons

What happens if $N = \infty$?

Can we study solutions of the form

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} p_n(t)\mathrm{e}^{-|x-q_n(t)|}? \qquad \text{(MP)}$$

Clearly, one needs to be more specific regarding the above sum.

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Clearly, one needs to be more specific regarding the above sum.

Suppose $q_1 < q_2 < \dots$ at $t = 0$ and $\lim_{n\to\infty} q_n(0) = \mathcal{L}(0)$.

- Does u have this form for all t?
- If $\mathcal{L}(0) < \infty$, is it true that $\mathcal{L}(t) < \infty$ for all t?
- Does (p, q) satisfy an (infinite) system of ODEs?
- Can we integrate such solutions?
- ...

L.-C. Li, *Long time behaviour for a class of low-regularity solutions of the Camassa–Holm equation*, Comm. Math. Phys. (2009).

The CH flow with step-like initial data

Definition: $\mathcal{D}_{\mathrm{loc}}$ is the set of pairs $(u,\mu) \in H^1_{\mathrm{loc}}(\mathbb{R}) \times \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R})$ such that

$$\int_{-\infty}^{0} \mathrm{e}^{-s}\mu(ds) < \infty, \qquad \mu := \upsilon + u^2 + u_x^2 \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}). \tag{Dloc}$$

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For $(u, \mu) \in \mathcal{D}_{\mathrm{loc}}$, the ODE

$$-f'' + \frac{1}{4}f = z\,\omega f + z^2\upsilon f, \tag{IsoCH}$$

where $\omega = u - u_{xx} \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$, admits a **fundamental system of solutions**

$$\begin{aligned} s(z,x) &\sim \mathrm{e}^{x/2}, & c(z,x) &\sim \mathrm{e}^{-x/2}, & & \\ s^{[1]}(z,x) &\sim \mathrm{e}^{x/2}, & c^{[1]}(z,x) &\sim o(\mathrm{e}^{x/2}), & x &\to -\infty \end{aligned}$$

(quasi-derivative: $f^{[1]} = f' + \frac{1}{2}f - z(u + u')f$)

The CH flow with step-like initial data

Then define the **Weyl–Titchmarsh function**

$$m(z) = -\lim_{x\to\infty} \frac{c(z,x)}{z\, s(z,x)}$$

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$$m(z) = -\lim_{x\to\infty} \frac{c(z,x)}{z\, s(z,x)} = \int_{\mathbb{R}} \frac{z}{\lambda(\lambda - z)} d\rho(\lambda), \qquad \int_{\mathbb{R}} \frac{\rho(d\lambda)}{1+\lambda^2} < \infty. \tag{SpM}$$

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Next, let $\mathcal{D} = \{(u,\upsilon) \in \mathcal{D}_{\mathrm{loc}} \,|\, E(u,\upsilon) < \infty\}$,

$$E(u,\upsilon) = \sup_{x\in\mathbb{R}} \mathrm{e}^{x} \left(\int_x^\infty \mathrm{e}^{-s} (u(s) + u'(s))^2 ds + \int_{[x,\infty)} \mathrm{e}^{-s} d\upsilon(s) \right).$$

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For any $(u,\upsilon)\in\mathcal{D}$,

$$\frac{1}{4\,\lambda_0(\sigma)} \le \sqrt{E(u,\upsilon)} \le \frac{2}{\lambda_0(\sigma)}, \qquad \lambda_0(\sigma) := \inf\{|\lambda|\colon \lambda\in\operatorname{supp}(\rho)\}.$$

Theorem (Eckhardt & AK// Proc. London Math. Sos. (2025))

(SpM) is a bijection between $\mathcal{D}$ and spectral measures with $\lambda_0(\sigma)>0$.

The CH flow with step-like initial data

Multi-peakon profiles (i.e., $\mathrm{supp}(\rho)$ is a finite set) are dense in $\mathcal{D}$ and this enables us to extend by continuity the conservative CH flow to the whole phase space $\mathcal{D}$. Namely, *the conservative Camassa–Holm flow* Φ on $\mathcal{D}$:

$$\begin{tikzcd}
\rho_0(d\lambda) \arrow[r, "\rho(d\lambda;t) = \mathrm{e}^{-t/2\lambda}\rho_0(d\lambda)"] & \rho(d\lambda, t) \arrow[d, "\mathrm{SpM}^{-1}"] \\
(u_0, \mu_0) \arrow[u, "\mathrm{SpM}"] \arrow[r, dashed, "\Phi\colon \mathcal{D}\times\mathbb{R}\to\mathcal{D}"'] & (u(t), \mu(t))
\end{tikzcd}$$

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$$\begin{array}{ccc} \rho_0(d\lambda) & \xrightarrow{\rho(d\lambda;t) = \mathrm{e}^{-t/2\lambda}\rho_0(d\lambda)} & \rho(d\lambda, t) \\ \mathrm{SpM}\uparrow & & \downarrow \mathrm{SpM}^{-1} \\ (u_0, \mu_0) & \xrightarrow[\Phi\colon \mathcal{D}\times\mathbb{R}\to\mathcal{D}]{} & (u(t), \mu(t)) \end{array}$$

Theorem (Eckhardt & AK// Proc. London Math. Sos. (2025))

The integral curve $t \mapsto \Phi^t(u_0, \mu_0)$ is a weak solution to the conservative CH system.

The CH flow with step-like initial data

Corollary (Eckhardt & AK// Proc. London Math. Sos. (2025))

$(u, \mu) \in \mathcal{D}_{\mathrm{loc}} \cap H^1(\mathbb{R}) \times \mathcal{M}^+(\mathbb{R}) \Leftrightarrow \rho = \sum_{\lambda\in\sigma} \gamma_\lambda \delta_\lambda$ with

$$\sum_{\lambda\in\sigma} \frac{1}{\lambda^2} < \infty, \qquad \sum_{\lambda\in\sigma} \frac{\gamma_\lambda}{\lambda^2} < \infty.$$

Isospectral sets

$$\mathrm{Iso}(\sigma) = \{(u, \mu) \in \mathcal{D}_{\mathrm{loc}} \cap H^1(\mathbb{R}) \times \mathcal{M}^+(\mathbb{R}) \,|\, \mathrm{supp}\,(\rho) = \sigma\}.$$

- $\mathrm{Iso}(\sigma) \simeq \mathbb{R}^{\#\sigma}$ by means of the map $(u, \mu) \mapsto \{\log \gamma_\lambda\}_{\lambda\in\sigma}$.
- The flow $\dot{\gamma}_\lambda = -\frac{1}{2\lambda}\gamma_\lambda$ is continuous on $\mathrm{Iso}(\sigma)$ $\rightarrow$ global conservative solution.
- σ is finite $\rightarrow$ conservative multi-peakons.
- Soliton resolution: solution splits into an infinite train of peakons.

Under stronger decay at $+\infty$ in

J. Eckhardt// Arch. Ration. Mech. Anal. **224** (2017).

Infinite multi-peakons

Theorem (Chang, Eckhardt & AK// arXiv:2509.23826)

Let ρ be a spectral measure of $(u, \mu) \in \mathcal{D}$ and let $K \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then the moments of ρ exist up to order $2K$ if and only if there is $N \in \mathbb{N}$, increasing points $x_1, \dots, x_N$ in $\mathbb{R}$, real weights $\omega_1, \dots, \omega_{N-1}$ and non-negative weights $\upsilon_1, \dots, \upsilon_N$ with $\upsilon_n + |\omega_n| > 0$ for all $n \leq N-1$ and

$$N - 1 + \#\{n \in \{1, \dots, N\} \,|\, \upsilon_n \neq 0\} \geq K$$

such that the pair (u, μ) has the form

$$\omega|_{(-\infty, x_N)} = \sum_{n=1}^{N-1} \omega_n \delta_{x_n}, \qquad \upsilon|_{(-\infty, x_N]} = \sum_{n=1}^{N} \upsilon_n \delta_{x_n}.$$

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In particular, all moments of ρ are finite $\Leftrightarrow$ $N = \infty$. Moreover, in this case $x_n \to \infty \Leftrightarrow$ the corresponding moment problem is **determinate**.

- Formulae of Stieltjes, Krein and Langer allow to recover ω, υ on $(-\infty, x_N)$.

Infinite multi-peakons

Corollary (Chang, Eckhardt & AK// arXiv:2509.23826)

If the pair $(u_0, \mu_0) \in \mathcal{D}$ is such that $u_0|_{(-\infty,x)}$ and $v_0|_{(-\infty,x)}$ are of a multi-peakon form, then the corresponding solution $\Phi^t(u_0, \mu_0)$ is of this form for all $t \in \mathbb{R}$, that is, for all t there exists $x(t)$ such that $u|_{(-\infty,x(t))}$ and $v|_{(-\infty,x(t))}$ are of a multi-peakon form.

Proof: $\rho(\cdot; t) = \mathrm{e}^{-t/2\lambda}\rho_0$ has the same number of finite moment as ρ_0 since $0 \notin \operatorname{supp}(\rho_0)$.

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Corollary (Finite propagation speed// arXiv:2509.23826)

If $\omega_0|_{(-\infty,x)} \equiv 0$ and $\upsilon_0|_{(-\infty,x)} \equiv 0$ for some $x \in \mathbb{R}$, then $\omega|_{(-\infty,x_1(t))} \equiv 0$ and $\upsilon|_{(-\infty,x_1(t))} \equiv 0$ for all $t \in \mathbb{R}$ where

$$x_1(t) = -\log s_0(t) = -\log\left(\int_{\mathbb{R}} \mathrm{e}^{-t/2\lambda}\rho_0(d\lambda)\right).$$

- For classical solutions, A. Constantin, *Finite propagation speed for the Camassa–Holm equation*, J. Math. Phys. (2005)

Laguerre infinite peakons: $\rho_0(d\lambda) = 1_{[1/2,\infty)}(\lambda)\mathrm{e}^{1/2-\lambda}d\lambda$

The moment problem is determinate, hence $u_0(x) = \frac{1}{2}\sum_{n\geq 1}\omega_n \mathrm{e}^{-|x-x_n|}$. Moreover, in this case $u_0 - 1/2 \in H^1[0,\infty)$ and

$$\omega_n(0) = \sqrt{1/2n} + \mathcal{O}(1/n), \quad x_n(0) = 2\sqrt{2n} - \log(4\pi\sqrt{2\mathrm{e}}) + \mathcal{O}(1/\sqrt{n}).$$

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Theorem I (Chang, Eckhardt & AK//arXiv:2509.23826)

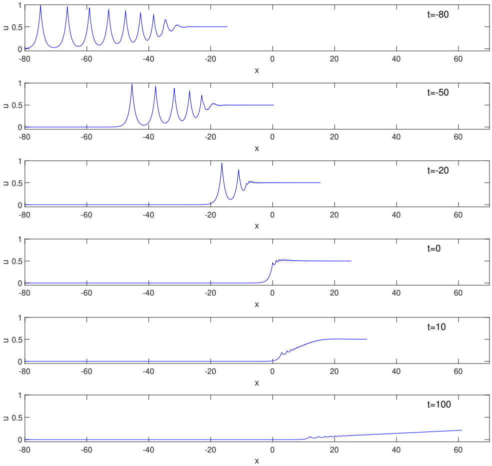
$u(t)$ has the form (MP), *$u(\cdot,t) - 1/2 \in H^1[0,\infty)$ for all $t \in \mathbb{R}$, and*

- $\omega_n(t) \sim \sqrt{2/t}, \quad x_n(t) \sim \sqrt{2t} + (n - 3/2)\log\sqrt{2t}, \qquad t \to +\infty,$
- $\omega_n(t) \sim 2, \qquad x_n(t) \sim t + (2n-1)\log(|t|) - \log(\Gamma(n)^2/2), \quad t \to -\infty.$

The proof is based on Stieltjes' formulas and the analysis of Hankel determinants

$$\Delta_{k,n}(t) = \frac{\mathrm{e}^{n/4}}{n!}\int_{(1/4,\infty)^n} \boldsymbol{\lambda}^k \prod_{1\leq i<j\leq n} |\lambda_i - \lambda_j|^2 \mathrm{e}^{-\sum_{i=1}^n \lambda_i + \frac{t}{2\lambda_i}}\, d\boldsymbol{\lambda}.$$

Laguerre infinite peakons: $\rho_0(d\lambda) = 1_{[1/2,\infty)}(\lambda)e^{1/2-\lambda}d\lambda$



Animation: doi:10.17028/rd.lboro.29110385.

Legendre infinite peakons: $\rho_0(d\lambda) = 1_{[\alpha,1+\alpha]}(\lambda)d\lambda,\ \alpha = 1/20$

The moment problem is determinate, hence $u_0(x) = \frac{1}{2}\sum_{n\geq 1}\omega_n \mathrm{e}^{-|x-x_n|}$.
In this case u_0 is close to be eventually periodic:

$$\omega_n(0) = 20/\sqrt{21} + o(1), \quad x_n(0) = 2n\log(21+\sqrt{21}/20) - \log(42\pi/5) + o(1).$$

Theorem II (Chang, Eckhardt & AK//arXiv:2509.23826)

For all $t \in \mathbb{R}$, $u(t)$ has the form (MP). *Moreover,*

- *As $t \to +\infty$,*

$$\omega_n(t) \sim 20/21, \qquad x_n(t) \sim 10t/21 + (2n-1)\log(10t/21) - \log(j_n^+).$$

- *As $t \to -\infty$,*

$$\omega_n(t) \sim 20, \qquad x_n(t) \sim 10t + (2n-1)\log(10|t|) - \log(j_n^-).$$

Here $j_n^+\ j_n^- \in (0,\infty)$ are explicit, but slightly cumbersome.

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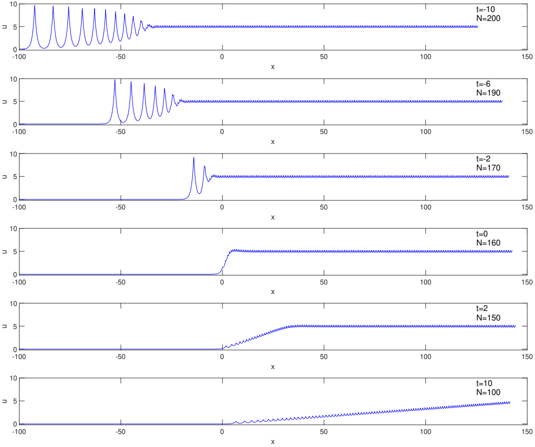

Animation: doi:10.17028/rd.lboro.29110385 Thank you for your attention!

Addendum #1: Global conservative solutions

A. Bressan and A. Constantin, *Global conservative solutions of the Camassa–Holm equation*, Arch. Ration. Mech. Anal. **183** (2007)

H. Holden and X. Raynaud, *Global conservative solutions of the Camassa–Holm equation—a Lagrangian point of view*, Comm. PDE (2007).

- Transformation to Lagrangian coordinates

$$\begin{tikzcd}
\mathcal{L}(0) \arrow[rr, "\text{quasilinear}"'] & & \mathcal{L}(t) \arrow[d, "?"] \\
u(\cdot,0) \arrow[u] & & u(\cdot,t)
\end{tikzcd}$$

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Global conservative solutions: $u \to (u,\mu)$ with $\mu = u^2 + u_x^2 + \upsilon$

$$u_t + uu_x + P_x = 0, \qquad P - P_{xx} = \frac{u^2+\mu}{2},$$
$$\mu_t + (u\mu)_x = (u^3 - 2Pu)_x.$$

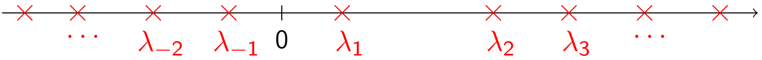
Addendum #2: Conservative Camassa–Holm flow via IST

Spectral problem: $\omega = u - u_{xx}$ and $\mu = (u^2 + u_x^2)dx + \upsilon$

$$-f'' + \frac{1}{4}f = z\omega f + z^2 \upsilon f \qquad \text{on } \mathbb{R}.$$

Usually, the Cauchy problem is posed for:

- Dispersionless CH ($\kappa = 0$) with decaying data on the line, the phase space $(u, \mu) \in H^1(\mathbb{R}) \times \mathcal{M}_+(\mathbb{R})$,



Trace formula

$$\frac{1}{2}\sum_{\lambda\in\sigma}\frac{1}{\lambda^2} = \int_{\mathbb{R}} u^2 + u_x^2\, dx + \int_{\mathbb{R}} d\upsilon = \mu(\mathbb{R}).$$

However, the inverse spectral problem is **open**!

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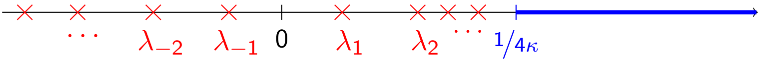

Direct and inverse spectral problem:

J. Eckhardt & A. Kostenko// Invent. Math. **329** (2024).

$\boxed{\text{However, to apply IST one needs \textbf{scattering}!}}$ (similar to KdV with L^2 data)

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Periodic multi-peakon solutions:

R. Beals, D. Sattinger, & J. Szmigielski// J. Inst. Math. Jussieu **4** (2005).

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Positive smooth algebro-geometric solutons:

Constantin & McKean// Comm. Pure Appl. Math. **52** (1999)

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Direct and inverse spectral theory: The literature is rather scarce...

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- Periodic initial data (CH on the circle), the phase space $(u, \mu) \in H^1(\mathbb{T}) \times \mathcal{M}_+(\mathbb{T})$,
- Two-component Camassa–Holm equation ...

Well-posedness of the conservative CH flow: mainly by K. Grunert, H. Holden and X. Raynaud.

Addendum #3: IST with step-like initial data on the line

For $\kappa > 0$, $\mathcal{D}_\kappa := \{(u,\mu) \in \mathcal{D}_{\mathrm{loc}} \,|\, u - \kappa \in H^1(0,\infty),\ \upsilon \in \mathcal{M}^+(\mathbb{R}_{>0})\}$.

Theorem 1 (Eckhardt & AK// *Invent. Math.* **238** (2024))

$(u,\mu) \in \mathcal{D}_\kappa \Leftrightarrow$ *the spectral measure* ρ *satisfies*

(i) $\{0\} \notin \sigma = \mathrm{supp}(\rho)$,

(ii) $\sum_{\lambda\in\sigma\cap(-\infty,0)} |\lambda|^{-3/2} + \sum_{\lambda\in\sigma\cap(0,1/4\kappa)} (1/4 - \lambda)^{3/2} < \infty$,

(iii)

$$\int_{1/4\kappa}^{\infty} \frac{\sqrt{\lambda - 1/4\kappa}}{\lambda^3} \log\left(\frac{d\rho(\lambda)}{d\lambda}\right) d\lambda > -\infty.$$

Corollary

$\mathcal{D}_\kappa$ is invariant under the conservative CH flow.

Example: Laguerre infinite multi-peakons,

Take (u_0,μ_0) such that $\rho_0 = 1_{[\alpha,\infty)}(\lambda)e^{\alpha-\lambda}d\lambda$ with $\alpha = 1/4\kappa > 0$.