Global Well-Posedness for the mNV and NV Equations

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Dispersive Integral Equations:
Pathfinders in Hamiltonian PDE
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References

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2. Adrian Nachman, Peter Perry, Daniel Tataru: Large data global well-posedness for the modified Novikov-Veselov system (arXiV.2511.21564)
3. Peter Perry: The modified Novikov-Veselov equation and the inverse scattering transform (arXiV.2511.20579)

Why Listen to this Talk?

Historically the NV and the mNV equations have been considered from two essentially disjoint perspectives, (i) the completely integrable approach and (ii) the nonlinear dispersive PDE perspective. It is part of the goal of this paper to bring these two aspects together toward the study of the large data problems.

Preview

We'll prove large-data, global well-posedness for the Novikov-Veselov (NV) and modified Novikov-Veselov (mNV) equations, dispersive nonlinear equations in two space dimensions. These equations are completely integrable, and there are no known global well-posedness results for large data by PDE methods. We'll proceed as follows:

1. Prelude: Integrable PDE's in $2+1$ versus $1+1$
2. mNV, NV, and the Main Results
3. Function Spaces U^p, V^p, S^p
4. Global well-posedness: mNV
5. The Miura Map
6. Global well-posedness: NV

Prelude: The KdV Equation

The KdV equation (1+1)

$$q_t - 6qq_x + q_{xxx} = 0$$

has Lax pair

$$\begin{aligned} L(t) &= -\partial_x^2 + q(t,x) \\ A(t) &= -4\partial_x^3 + 3(q\partial_x + \partial_x q) \end{aligned}$$

The KdV equation is equivalent to

$$\dot{L} = i[L, A]$$

or

$$[L, A - \partial_t] = 0.$$

The operator L defines a *spectral problem*, and the operator A determines the evolution of scattering data.

Prelude: The Novikov-Veselov Equation

Let

$$\overline{\partial} = \frac{1}{2}\left(\partial_x + i\partial_y\right), \quad \partial = \frac{1}{2}\left(\partial_x - i\partial_y\right).$$

The NV equation (2+1) at zero energy

$$(\partial_t + \partial^3 + \overline{\partial}^3)v + \partial(u_1 v) + \overline{\partial}(u_2 v) = 0$$
$$\overline{\partial} u_1 = 3\partial v, \quad \partial u_2 = 3\overline{\partial} v$$

has a *Manakov triple* (Manakov, 1976)

$$L(t) = \partial\overline{\partial} + q(t,z)$$
$$A(t) = \partial^3 + \overline{\partial}^3 + u_1\partial + u_2\overline{\partial}$$
$$B(t) = \partial u_1 + \overline{\partial} u_2$$

The NV equation is equivalent to

$$[L, A + \partial_t] = BL$$

which becomes a Lax representation when restricted to zero energy solutions of $L(t)$.

Inverse Scattering: KdV Equation

KdV Direct Scattering Problem:

$$-\psi'' + q(x)\psi = k^2\psi, \quad k \in \mathbb{R}$$

KdV Direct Scattering Transform:

$$q(x) \mapsto \left(r(k), \{\lambda_j\}_{j=1}^N, \{c_j\}_{j=1}^N\right)$$

KdV Inverse Scattering Transform:

$$\left(r(k), \{\lambda_j\}_{j=1}^N, \{c_j\}_{j=1}^N\right) \mapsto q(x)$$

Time evolution of scattering data:

$$\begin{aligned} r(k,t) &= e^{8ik^3t} r(k,0) \\ \lambda_j(t) &= \lambda_j(0) \\ c_j(t) &= e^{8k_j^3 t} c_j(0) \end{aligned}$$

Inverse Scattering: NV Equation

NV Direct Scattering Problem:

$$\partial\overline{\partial}\psi + q(z)\psi = 0, \quad \psi(z,k) \underset{|z|\to\infty}{\sim} e^{ikz} \qquad k \in \mathbb{C} \setminus \mathcal{E}$$

NV Direct Scattering Transform:

$$\mathbf{t}^\sharp(k) = \frac{1}{4\pi\overline{k}} \int_{\mathbb{C}} e^{i\overline{k}\overline{z}} q(z)\psi(z,k)\, dz$$

NV Inverse Scattering Transform:

$$\overline{\partial}_k \mu = e_{-k}\mathbf{t}^\sharp(k)\overline{\mu}$$

$$q(z) = \frac{4i}{\pi}\int_{\mathbb{C}} e_{-k}(z)\mathbf{t}^\sharp(k)\overline{\mu(z,k)}\, dk, \quad e_k(z) = e^{i(kz+\overline{k}\overline{z})}$$

Time Evolution of Scattering Data:

$$\mathbf{t}^\sharp(k,t) = e^{i(k^3+\overline{k}^3)t}\mathbf{t}^\sharp(k,0)$$

KdV and NV

The mKdV Equation:

$$u_t + u_{xxx} = 6u^2 u_x$$

The KdV Equation:

$$q_t + q_{xxx} = 6qq_x$$

The *Miura Map*

$$\mathcal{M}(u) = u_x + u^2$$

maps solutions of mKdV to solutions of KdV

The potential $q = \mathcal{M}(u)$ has no bound states since

$$\begin{aligned} -\partial_x^2 + u_x + u^2 &= (\partial_x + u)(-\partial_x + u) \\ &= A^* A \end{aligned}$$

The mNV Equation:

$$u_t + (\partial^3 + \overline{\partial}^3)u = NL_{mNV}(u)$$

The NV Equation:

$$q_t + (\partial^3 + \overline{\partial}^3)q = NL_{NV}(q)$$

The map

$$\mathcal{M}(u) = 2\partial u + u^2$$

maps solutions of mNV to solutions of NV

The potential $q = \mathcal{M}(u)$ has no exceptional points.

Aside: The Nonlinearities

The nonlinearity in mNV is cubic:

$$\frac{4}{3}NL_{mNV} = u\overline{\partial}^{-1}\left(\partial(\overline{u}\partial u)\right) + (\partial u)\cdot\overline{\partial}^{-1}\left(\partial(|u|^2)\right) + u\partial^{-1}\left(\overline{\partial}(\overline{u}\overline{\partial}u)\right) + (\overline{\partial}u)\cdot\partial^{-1}\left(\overline{\partial}(|u|^2)\right)$$

while the nonlinearity in NV is quadratic:

$$\frac{4}{3}NL_{NV} = \partial\left(q\overline{\partial}^{-1}\partial q\right) + \overline{\partial}\left(\overline{q}\partial^{-1}\overline{\partial}\overline{q}\right).$$

These nonlinearities are nonlocal and involve the operators $B = \partial\overline{\partial}^{-1}$ and $B^* = \overline{\partial}\partial^{-1}$, which are bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$.

To interpret the PDE's with these nonlinearities, we need to work in a space-time setting and take advantage of dispersive norms.

mKdV and mNV

The mKdV equation has Lax operator

$$L = \begin{pmatrix} i\partial_x & -iu \\ iu & -i\partial x \end{pmatrix}$$

and belongs to the hierarchy of the defocussing NLS equation

$$iu_t + u_{xx} - |u|^2 u = 0$$

The mNV equation has Lax operator

$$L = \begin{pmatrix} \partial & -u \\ u & \overline{\partial} \end{pmatrix}$$

and belongs to the hierarchy of the Davey-Stewartson II (DS II) equation

$$\begin{cases} iq_t + (\partial^2 + \overline{\partial}^2)q = q(r + \overline{r}) \\ \overline{\partial} r + \partial(|q|^2) = 0 \end{cases}$$

The mNV and NV Equations

Who cares about this stupid equation?

Adrian Nachman

The modified Novikov-Veselov (mNV) equation

$$u_t + \left(\partial^3 + \overline{\partial}^3\right) u = NL_{mNV}(u)$$

and the Novikov-Veselov (NV) equation

$$q_t + (\partial^3 + \overline{\partial}^3)q = NL_{NV}(q)$$

are completely integrable, nonlinear dispersive equations in two space dimensions. The mNV equation is L^2-critical, and the NV equation is "$\dot{H}^{-1}$-critical."

The NV equations were introduced by Novikov and Veselov (1984) as flows associated to the two-dimensional Schrödinger operator at fixed energy. The mNV equation was proposed by Bogdanov (1987) as a natural analogue of the mKdV equation. Schottdorf (2013) proved global well-posedness of mNV for small initial data.

DS II, mNV, and NV

The mNV equation lies in the integrable hierarchy of the Davey-Stewartson II (DS II) equation, which is also L^2-critical. Nachman, Regev, and Tataru used inverse scattering methods to prove global well-posedness of DS II in $L^2(\mathbb{R}^2)$. In particular, they showed that the scattering map $\mathcal{S} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a global diffeomorphism.

Building on:

1. Schottdorf's proof of small data well-posedess of mNV, and
2. Nachman-Regev-Tataru's work on DS II,

we will prove large-data well-posedness of mNV.

The Miura map

$$\mathcal{M}(u) = 2\partial u + |u|^2$$

$$L^2(\mathbb{R}^2) \to \dot{H}^{-1}(\mathbb{R}^2) + L^1(\mathbb{R}^2)$$

maps solutions of mNV to solutions of NV but is *not* onto $\dot{H}^{-1} + L^1$.

Using the Miura map, we will prove global well-posedness of NV on a set of initial data for which is "soliton-free" in the sense that $-\Delta + q \geq 0$.

Overview - the mNV Equation

Building on the work of Nachman, Regev, and Tataru, we will prove:

Theorem 1

The mNV equation is globally well-posed for initial data $u_0 \in L^2(\mathbb{R}^2)$.

(i) Given $u_0 \in L^2$, there exists a unique solution $u \in U^2_{NV} \subset C(\mathbb{R}, L^2(\mathbb{R}))$

(Ii) The flow map $L^2 \ni u_0 \mapsto u \in U^2_{NV}$ is smooth uniformly on bounded subsets of $L^2(\mathbb{R}^2)$

(iii) For each $u_0 \in L^2$ there are $u_\pm \in L^2$ with

$$\lim_{t \to \pm\infty} \|u(t) - S_{NV}(t)u_\pm\|_{L^2} = 0$$

where $S_{NV}(t)$ is the linear evolution for mNV and $u \mapsto u_\pm$ are global diffeomorphisms of L^2

An essential part of our analysis is a nonlinear Gagliardo-Nirenberg inequality for the scattering operator which will allow us to bound a “control norm” and extend local solutions to global ones.

Overview: The Miura Map

Let

$$\mathcal{M}(u) = 2\partial u + |u|^2$$

If $u \in L^2(\mathbb{R}^2)$, $\partial u = \overline{\partial u}$, and $q = \mathcal{M}(u)$, then $q \in \dot{H}^{-1}(\mathbb{R}^2) + L^1(\mathbb{R}^2)$ is real-valued and defines the quadratic form of a Schrödinger operator H_q:

$$\langle H_q \varphi, \varphi \rangle = \int |\nabla \varphi|^2 - \langle 2u, \partial(|\varphi|^2) \rangle + \langle |u|^2 \varphi, \varphi \rangle$$

We will prove:

Theorem 2

1. The Miura map is injective from $L^2(\mathbb{R}^2)$ to $\dot{H}^{-1} + L^1$
2. The Miura map has closed range consists of $q \in \dot{H}^{-1} + L^2$ with $H_q \geq 0$
3. The inverse of the Miura map is continuous on $\mathcal{M}(L^2)$.

Our second result extends Agmon-Allegretto-Piepenbrink theory, which connects positive solutions of the Schrödinger equation at zero energy to operator positivity.

Overview - the NV Equation

Using our well-posedness result on mNV together with a careful study of the Miura-type map from solutions of mNV to solutions of NV, we prove:

Theorem 3

The NV equation is globally well-posed for initial data

$$q_0 \in \mathcal{M}(L^2(\mathbb{R}^2)) \subset \dot{H}^{-1}(\mathbb{R}^2) + L^1(\mathbb{R}^2)$$

in the sense that the data to solution map

$$L^2 \cap \mathcal{M}(L^2) \ni q_0 \mapsto q \in C(\mathbb{R}, L^2 \cap \mathcal{M}(L^2))$$

admits a continuous extension

$$\mathcal{M}(L^2) \ni q_0 \mapsto q \in C(\mathbb{R}, \mathcal{M}(L^2))$$

How to Choose Function Spaces

Let $S_{NV}(t)$ be the solution operator for the Cauchy problem

$$\begin{cases} v_t + \left(\partial^3 + \overline{\partial}^3\right) v = 0, \\ v|_{t=0} = v_0 \end{cases}$$

so $v(t) = S_{NV}(t)v_0$.

Let

$$\|v\|_{S^p} = \left\| |D|^{1/p} v \right\|_{L_t^p L_x^r}, \quad \frac{1}{p} + \frac{1}{r} = \frac{1}{2}, \quad 2 < p \leq \infty$$

If v solves the linear Cauchy problem, the Strichartz estimate

$$\|v\|_{S^p} \leq \|v_0\|_{L^2},$$

holds. This suggests that we study the mNV equation in the space S^p. Indeed the space S^p will play the role of a *control norm* in our analysis. We will prove a key continuation result with respect to this norm.

We will need finer function spaces to study well-posedness.

Function Spaces -V^p Spaces

To describe the spaces in which we will study the evolution equations, recall the U^p and V^p spaces.

If $\mathcal{B}$ is a Banach space and $I = [a,b] \subseteq \mathbb{R}$ is an interval,

- A function $u : I \to \mathcal{B}$ is *ruled* if u has left- and right-hand limits at each $t \in I$.
- A *partition* $\mathcal{P}$ of I is a set

$$a < t_1 < t_2 < \ldots < t_n = b$$

- A ruled function $u : [a,b] \to \mathcal{B}$ belongs to $V^p(I)$ if the p-variation

$$\|u\|_{V^p} = \sup_{\mathcal{P}} \left(\sum_{j=1}^{n-1} \left\| v(t_{j+1}) - v(t_j) \right\|_{\mathcal{B}}^p \right)^{1/p}$$

  is finite, where by convention $v(b) = 0$

The V^p space was introduced by Wiener (1924)

Function Spaces - U^p Spaces

If $\mathcal{B}$ is a Banach space and $I \subseteq \mathbb{R}$ is an interval, we define the atomic space $U^p(I)$ by first defining a p-atom to be a function of the form

$$a(t) = \sum_{j=1}^{n-1} \chi_{[t_j, j_{j+1}]}(t)\psi_i, \qquad \psi_i \in \mathcal{B}, \quad \sum_{j=1}^{n-1} \|\psi_i\|_{\mathcal{B}}^p \leq 1$$

Functions $u \in U^p(I)$ are sums of p-atoms

$$u(t) = \sum_{i=1}^{\infty} \lambda_i a_i(t) \quad \text{where } \lambda_i \in \mathbb{C}, \quad \sum_{i=1}^{\infty} |\lambda_i| < \infty$$

and have norm

$$\|u\|_{U^p} = \inf \left\{ \sum_{i=1}^{\infty} |\lambda_i| : u(t) = \sum_{i=1}^{\infty} \lambda_i a_i(t) \text{ for } p\text{-atoms } a_i \right\}$$

The continuous embedding $U^p \subset V^p$ holds.

Adapted Spaces

If $\mathcal{H}$ is a Hilbert space and $S(t)$ is a unitary evolution on $\mathcal{H}$, we define

$$V^p_S(I) = \{v : I \to \mathcal{H} : S(-t)v(t) \in V^p(I)\},$$
$$\|v\|_{V^p_S} = \|S(-\,\cdot\,)v(\,\cdot\,)\|_{V^p}$$

$$U^p_S(I) = \{u : I \to \mathcal{H} : S(-t)u(t) \in U^p(I)\},$$
$$\|u\|_{U^p_S} = \|S(-\,\cdot\,)u(\,\cdot\,)\|_{U^p}$$

We will use these spaces to study nonlinear equations with linear evolution $S(t)$ in $\mathcal{H} = L^2(\mathbb{R}^2)$.

It is important to note that functions in V^p_S have limits at $\pm\infty$, i.e., $\lim_{t\to\pm\infty} S(t)u(t)$ exists in $L^2(\mathbb{R}^2)$.

The adapted spaces were introduced by Tataru in unpublished work and applied to the study of dispersive equations by Koch-Tataru, Herr-Tataru-Tzvetkov, Hadac-Herr-Koch, Candy-Herr, and others.

Littlewood-Paley Decomposition

For $\mathcal{H} = L^2(\mathbb{R}^2)$ we denote by P_k the Littlewood-Paley operators localizing smoothly in dyadic annuli $2^{k-1} \leq |\xi| \leq 2^{k+1}$ in Fourier space, and we define $\dot{X}$ and $\dot{Y}$ to be the closures of $\mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$ in the respective norms

$$\|u\|_{\dot{X}}^2 = \sum_k \|P_k u\|_{U_S^p}^2, \qquad \|v\|_{\dot{Y}}^2 = \sum_k \|P_k v\|_{V_S^p}^2$$

Schottdorf's global well-posedness result for mNV with small initial data uses a fixed-point argument in $\dot{X}$ based on the Duhamel form of mNV:

$$u(t) = S_{NV}(t)u_0 + \int_0^t S_{NV}(t-s) N_{mNV}(u(s))\, ds$$

S^p Spaces

Motivated by Strichartz estimates for $S_{NV}(t)$, we define the space S^p of functions $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ with norm

$$\|u\|_{S^p} = \left\| |D|^{1/p} u \right\|_{L_t^p L_x^r}, \quad \frac{1}{p} + \frac{1}{r} = \frac{1}{2}, \quad 2 < p \leq \infty$$

We have the continuous inclusions

$$U^2_{NV} \subset V^2_{NV} \subset S^p \quad \Rightarrow \|u\|_{S^p} \lesssim \|u\|_{V^2_{NV}} \lesssim \|u\|_{U^2_{NV}}$$

We also define $\ell^2 S^p$ to be the space with norm

$$\|u\|^2_{\ell^2 S^p} = \sum_k \|P_k u\|^2_{S^p}$$

Finally, $\ell^\infty S^p$ is the space with norm

$$\|u\|_{\ell^\infty S^p} = \sup_k \|P_k u\|_{S^p} .$$

S^p Spaces

Let $I \subseteq \mathbb{R}$ be an interval. Then

$$\|u\|_{S^p(I)} = \left\| |D|^{1/p} u \right\|_{L_t^p(I;\, L_x^r(\mathbb{R}^2))}$$

By the Littlewood-Paley inequality, for $2 \leq p, r < \infty$,

$$\|u\|_{S^p(I)} \lesssim_{p,r} \|u\|_{\ell^2 S^p(I)}$$

where implied constants are independent of I (See, for example, Exercise A.14 in *Nonlinear Dispersive Equations* by Terence Tao)

Schottdorf's Theorem

Schottdorf (2013) showed that an equation closely related to the mNV equation is globally well-posed and scatters for small initial data in $L^2(\mathbb{R}^2)$.

We extend Schottdorf's result, adding a key continuation result, as follows:

Theorem 4

(a) Given $u_0 \in L^2(\mathbb{R}^2)$, there exists a time $T > 0$ and a unique local solution $u \in \dot{X}[0,T]$ depending smoothly on u_0

(b) The solutions may be continued so long as $\|u\|_{S^p}$ is finite. If $\|u_0\|_{L^2} = R$ and $2 < p < \infty$,

$$\|u\|_{\dot{X}} \leq R(1 + R\,\|u\|_{S^p})^{p/2}$$

The proof uses the Duhamel form of mNV.

Improved Local Well-Posedness

Writing

$$u(t) = S_{NV}(t)u_0 + NL(u), \qquad NL(u) = \int_0^t S_{NV}(t-s)N_{mNV}(u(s))\,ds$$

we find a solution in $\dot{X}[0,T]$ via:

(1) Proving trilinear estimates on $NL(u)$ of the form

$$\|NL(u)\|_{\dot{X}} \lesssim \|u\|_{\dot{X}} \|u\|_{\dot{X}} \|u\|_{\ell^\infty S^p}$$

(2) Setting $v(t) = u(t) - S_{NV}(t)u_0$ we solve

$$v(t) = \int_0^t S(t-s)N_{mNV}(v(s) + S(s)u_0)\,ds$$

by contraction mapping in $B_\delta(\dot{X}[I])$

Well-Posedness Near Infinity

For the scattering problem, we may write

$$u(t) = S(t)u_+ - \int_t^\infty S(t-s)N_{mNV}(u(s))\,ds$$

For $\|u_+\|_{L^2} = R$, we can choose T large so that on $I = [T, \infty]$

$$\|S(t)u_0\|_{S^p[I]} \leq \varepsilon.$$

We set

$$v(t) = u(t) - S_{NV}(t)u_+$$

and solve

$$v(t) = \int_t^\infty S(t-s)N_{mNV}(v(s) + S(s)u_0)\,ds$$

by contraction mapping on $B_\delta(\dot{X}[I])$

Results of Local Well-Posedness near 0 and ∞

The local well-posedness results for mNV has several important corollaries.

(1) If T is chosen so $\|S_{NV}(t)u_0\|_{\ell^\infty S^p} \ll \|u_0\|_{L^2}^{-1}$, then a solution $u(t)$ exists in $[0,T]$ with

$$\|u\|_{\dot{X}} \lesssim \|u_0\|_{L^2}, \quad \|u\|_{S^p[0,T]} \lesssim \|S(t)u_0\|_{S^p[0,T]}$$

(2) If u is an L^2 solution in $[0,T]$ so that $\|u\|_{\ell^\infty S^p[0,T]} \ll \|u_0\|_{L^2}^{-1}$, then

$$\|u\|_{\dot{X}} \lesssim \|u_0\|_{L^2}, \quad \|S_{NV}(t)u_0\|_{S^p[0,T]} \lesssim \|u\|_{S^p[0,T]}$$

(3) The scattering problem

$$\begin{cases} u_t + (\partial^3 + \overline{\partial}^3)u = N_{mNV}(u), \\ \lim_{t\to\infty} S_{NV}(-t)u(t) = u_+ \in L^2(\mathbb{R}^2) \end{cases}$$

is locally well-posed in $\dot{X}[T,\infty]$ for sufficiently large T, and u depends smoothly on u_+

It remains to show that the S^p norm of the solution is controlled in time

Nonlinear Gagliardo-Nirenberg Inequality

The Gagliardo-Nirenberg inequality on $\mathbb{R}^n$ (Brezis-Mironescu form) is as follows. Suppose that $1 \leq p, p_1, p_2 \leq +\infty, \quad s, s_1, s_2 \geq 0, \quad \theta \in (0,1), \quad s_1 \leq s_2$, and

$$s = \theta s_1 + (1-\theta) s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Then

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} \lesssim \|u\|^{\theta}_{W^{s_1,p_1}(\mathbb{R}^n)} \|u\|^{1-\theta}_{W^{s_2,p_2}(\mathbb{R}^n)}$$

The nonlinear Gagliardo-Nirenberg inequality for the scattering transform is

Theorem 5 (Fixed-Time NGN)

Let $2 < r < 4$, $\dfrac{1}{2} + \dfrac{1}{r_1} = \dfrac{2}{r}$ and $0 < s < \dfrac{1}{2}$. The scattering transform obeys the fixed-time bound

$$\|\mathcal{S}u\|_{\dot{B}^{s,r}_2} \underset{\|u\|_{L^2}}{\lesssim} \|\widehat{u}\|^{\frac{1}{2}}_{L^2} \|\widehat{u}\|^{\frac{1}{2}}_{\dot{B}^{2s,r_1}_2}$$

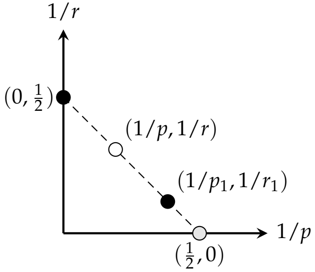
Nonlinear Gagliardo-Nirenberg Inequality

Assume

$$\frac{1}{p} + \frac{1}{r} = \frac{1}{p_1} + \frac{1}{r_1} = 1$$

and (p, r) lies between (p_1, r_1) and $(\infty, 2)$

Recall that $S^p = L^p_t(\dot{W}^{1/p,r})$



Theorem 6 (Space-Time NGN)

Let (p_1, r_1) and (p_2, r_2) be Strichartz pairs. The scattering transform obeys the fixed-time bound

$$\|\mathcal{S}u\|_{\ell^2 S^p} \underset{\|u\|_{L^2}}{\lesssim} \|\widehat{u}\|^{\frac{1}{2}}_{\ell^2 L^\infty L^2} \|\widehat{u}\|^{\frac{1}{2}}_{\ell^2 S^{p_1}}$$

The mNV Equation

We now consider the inverse scattering solution

$$u(t) = \mathcal{S}^{-1}\left(e^{-it((\diamond)^3+(\overline{\diamond})^3)}(\mathcal{S}u_0)\right)$$

As $\mathcal{F}\left(e^{-it((\diamond)^3+\overline{\diamond}^3)}(\mathcal{S}u_0)\right)$ solves the linear mNV flow we have

$$\left\|\mathcal{F}\left(e^{-it((\diamond)^3+\overline{\diamond}^3)}(\mathcal{S}u_0)\right)\right\|_{\ell^2 L_t^{p_1}\dot{W}^{\frac{1}{p_1},r_1}} \lesssim \|\mathcal{S}(u_0)\|_{L^2} = \|u_0\|_{L^2}$$

By the nonlinear GN inequality we get

$$\|u\|_{\ell^2 L_t^p \dot{W}^{1/p,r}} \underset{C(\|u_0\|_{L^2})}{\lesssim} \|u_0\|_{L^2}$$

We can now obtain L^2 solutions in two ways:

(a) We can extend the IST solution map from $u_0 \in \mathcal{S}(\mathbb{R}^2)$ to $u_0 \in L^2(\mathbb{R}^2)$ by density

(b) We can extend local solutions to global solutions using local well-posedness and uniform bounds on the control norm

We'll show that the two resulting solutions are identical

The mNV Equation

(a) We can extend the IST solution map from $u_0 \in \mathcal{S}(\mathbb{R}^2)$ to $u_0 \in L^2(\mathbb{R}^2)$

(b) We can extend local solutions to global solutions using local well-posedness and uniform bounds on the control norm

Let $u_{0,n} \underset{L^2(\mathbb{R}^2)}{\rightarrow} u_0$ with $u_{n,0} \in \mathcal{S}(\mathbb{R}^2)$, and let $u^n \rightarrow u$ be the corresponding solutions.

Let $T_{max} \in (0, \infty]$ be the maximal time of existence from part (b). For $t < T_{max}$ both solutions are unique limits of $\mathcal{S}(\mathbb{R}^2)$ solutions while, by (b),

$$u^n \rightarrow u \text{ in } U^2_{NV}([0,T]) \quad \Rightarrow \quad u^n \rightarrow u \text{ in } \ell^2 L^p([0,T], \dot{W}^{\frac{1}{p},r})$$

From the bound

$$\|u\|_{\ell^2 L^p([0,T],\dot{W}^{1/p,r})} \underset{C(\|u_0\|_{L^2})}{\lesssim} \|u_0\|_{L^2}$$

we pass to $T = T_{max}$ and get

$$\|u\|_{S^p[0,T]} \lesssim \|u\|_{\ell^2 L^p([0,T_{max}],\dot{W}^{1/p,r})} \underset{C(\|u_0\|_{L^2})}{\lesssim} \|u_0\|_{L^2}$$

If $T_{max} < \infty$ this is a contradiction–the solution can be extended beyond T_{max}.

The mNV Equation

We have established global existence for $u_0 \in L^2(\mathbb{R}^2)$ and identified the global solution with the solution by inverse scattering.

Scattering: Since $u \in \dot{X}$, the limits

$$u_\pm = \lim_{t \to \pm\infty} S_{NV}(-t)u(t)$$

exist in L^2. We can also show that

$$u_\pm = \mathcal{S}(u_0)$$

The Miura Map, One Dimension

Recall

$$\mathcal{M}_1(u) = u_x + |u|^2$$

and, for $q \in \dot{H}^{-1}_{\text{loc}}(\mathbb{R})$, let

$$L_q = -\frac{d^2}{dx^2} + q$$

Kappeler, Perry, Shubin, and Topalov proved:

Theorem 7 (Miura Map in One Dimension)

Let q be a real-valued distribution in $\dot{H}^{-1}(\mathbb{R})$. The following are equivalent:

(i) $q \in \operatorname{ran}(\mathcal{M}_1)$, i.e., $q = r' + r^2$ for some $r \in L^2_{\text{loc}}(\mathbb{R})$

(ii) The equation $L_q y = 0$ has a strictly positive solution $y \in H^1_{\text{loc}}(\mathbb{R})$

(iii) $L_q \geq 0$ as a self-adjoint operator

The range of $\mathcal{M}_1$ consists of potentials with no negative-energy bound states

The Miura Map, Two Dimensions

Bogdanov (1987) showed that, in two dimensions, the Miura map

$$\mathcal{M}(u) = 2\partial u + |u|^2, \qquad u \in L^2(\mathbb{R}^2), \quad \partial u = \overline{\partial u}$$

takes solutions of the mNV equation to solutions of the NV equation. The range of $\mathcal{M}$ lies in $\dot{H}^{-1}(\mathbb{R}^2) + L^1(\mathbb{R}^2)$. Nachman, Perry, and Tataru proved:

Theorem 8 (Miura Map in Two Dimensions)

Let q be a real-valued distribution in $\dot{H}^{-1}(\mathbb{R}^2) + L^1(\mathbb{R}^2)$. The following are equivalent:

(i) $q \in \mathrm{ran}(\mathcal{M})$, i.e., $q = \partial u + |u|^2$ for some $u \in L^2(\mathbb{R}^2)$ with $\partial u = \overline{\partial u}$

(ii) The equation $L_q \psi = 0$ has a strictly positive solution ψ with $\ln \psi \in \dot{H}^1(\mathbb{R})$

(iii) $L_q \geq 0$ as a self-adjoint operator

One should think of the range of $\mathcal{M}$ as "soliton-free" potentials

Global Well-Posedness for NV

To our knowledge, it hasn't been proven so far that (1.1) possesses solutions for data in any "reasonable" space (where by "reasonable" here we mean a standard Sobolev space), although many results have been obtained in other directions (see [22] and references therein).

Angelos Angelopoulos, 2014

Perry (2014) showed that inverse scattering methods can be used:

1. To solve the mNV equation with initial data $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with $\partial u_0 = \overline{\partial u_0}$ and $\int u_0(z)\, dA(z) = 0$ via
$$u(t) = \mathcal{S}^{-1}\left(e^{it\omega_{NV}}\mathcal{S}(u_0)\right)$$
2. To solve the NV equation with initial data $q_0 = \mathcal{M}(u_0)$, we set
$$q(t) = \mathcal{M}\left[\mathcal{S}^{-1}\left(e^{it\omega_{NV}}\mathcal{S}(u_0)\right)\right]$$

Global Well-Posedness for NV

In the present work, we can solve the NV equation for a spectrally defined subset of $\dot{H}^{-1}(\mathbb{R}^2) + L^1(\mathbb{R}^2)$

Using the Miura map, we can solve the NV equation for potentials $q \in \operatorname{ran} \mathcal{M}$.

First, we establish that $\mathcal{M}$ has a continuous inverse on $\operatorname{ran} \mathcal{M}$.

Next, to solve the NV equation:

(i) Given $q_0 \in \operatorname{ran} \mathcal{M}$, we set $u_0 = \mathcal{M}^{-1} q_0$

(ii) As $\mathcal{M}^{-1}(q_0) \in L^2(\mathbb{R}^2)$ with $\partial u = \overline{\partial u}$, we can solve the mNV equation with initial data u_0, preserving the condition $\partial u = \overline{\partial u}$

It remains to show that $q = \mathcal{M}(u)$ solves NV

Global Well-Posedness for NV

The proposed solution map is

$$q_0 \mapsto u_0 := \mathcal{M}^{-1}(q_0) \mapsto \mathcal{M}(u(t))$$

Setting $q(t) = \mathcal{M}(u)(t)$:

(i) By the continuity of the Miura map, $q \in C(\mathbb{R}, \dot{H}^{-1}(\mathbb{R}^2) + L^1(\mathbb{R}^2))$ and the data to solution map is continuous

(ii) By showing that L^2 is dense in $\mathcal{M}(L^2)$, and using local well-posedness for NV in L^2, we can show that the map above is the continuous extension of the solution map on more regular data.