Gauge transform for the Korteweg-de Vries equation and low regularity well-posedness

Andreia Chapouto

Monash University

Joint work with Simão Correia (IST-U. Lisbon) & João Pedro Ramos (IMPA)

Dispersive integrable equations: pathfinders in Hamiltonian PDEs

22-26 June 2026

The Korteweg-de Vries equation

(KdV) $$\begin{cases} \partial_t u + \partial_x^3 u = \partial_x(u^2) \\ u\big|_{t=0} = u_0 \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}$$

GOAL: Study low regularity local well-posedness of KdV

▷ Existence, uniqueness, and stability

$$u_0 \in X \quad \overset{?}{\mapsto} \quad u \in C([0,T]; X)$$

Q: What is the "largest" space X for which LWP holds?

▷ Complete integrability

▷ **(Multilinear) harmonic analysis**

Q: What is the "largest" space X for which LWP holds?

$$X = H^s(\mathbb{R}) \qquad \|f\|_{H^s} = \|\langle\xi\rangle^s \widehat{f}(\xi)\|_{L^2_\xi}$$

$$X = \mathcal{F}L^{\sigma,p}(\mathbb{R}) \qquad \|f\|_{\mathcal{F}L^{\sigma,p}} = \|\langle\xi\rangle^\sigma \widehat{f}(\xi)\|_{L^p_\xi}$$

Scaling heuristics:

- (KdV) is **invariant** under the scaling

$$u \text{ sol. on } [0,T] \quad \Longleftrightarrow \quad u_\lambda(t,x) = \lambda^2 u(\lambda^3 t, \lambda x) \text{ sol. on } [0, \lambda^{-3}T]$$

Effects of scaling on norms:

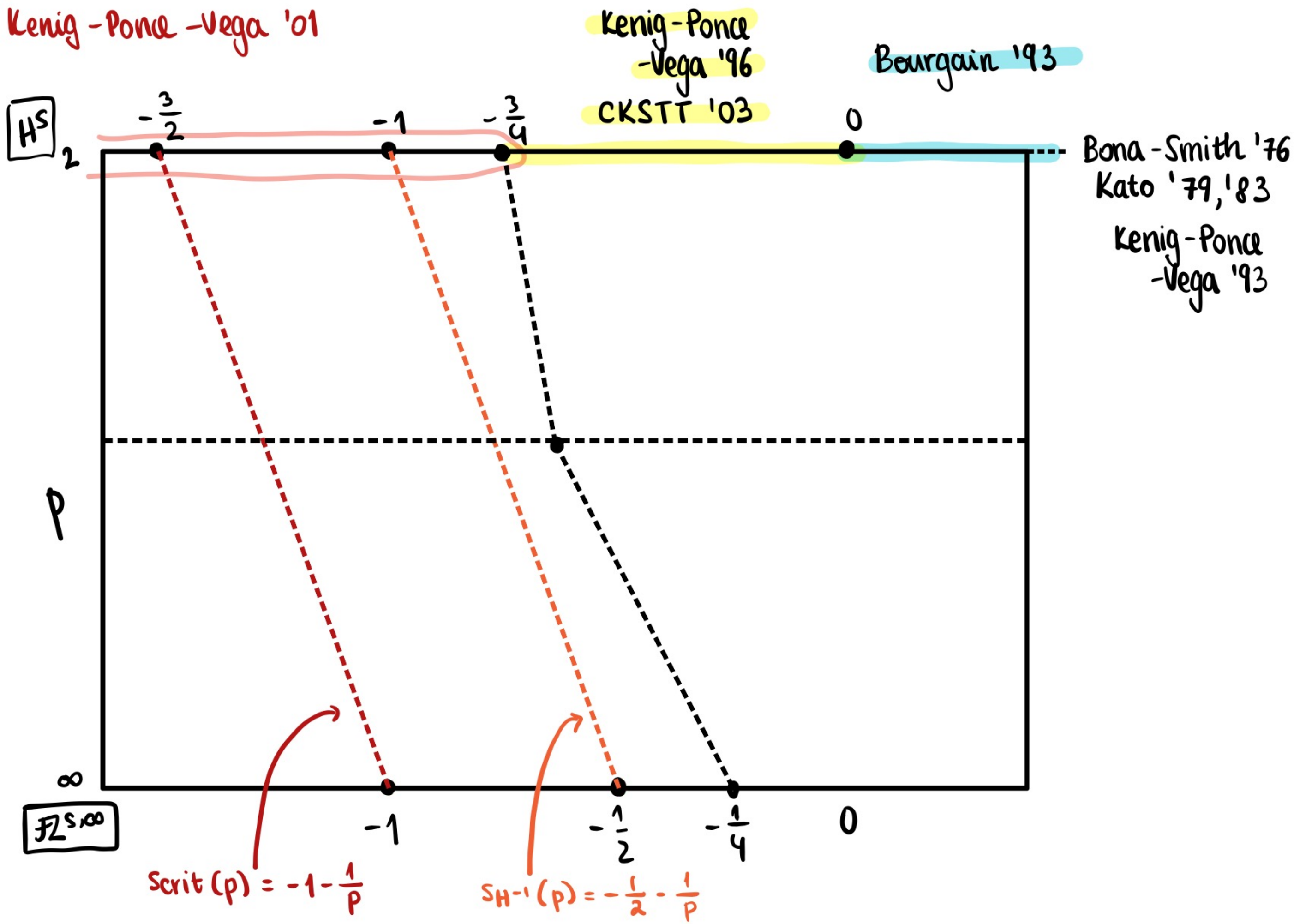
$$\|u_\lambda\|_{\dot{H}^s} = \lambda^{\frac{3}{2}+s}\|u\|_{\dot{H}^s} \quad \text{and} \quad \|u_\lambda\|_{\dot{\mathcal{F}L}^{\sigma,p}} = \lambda^{1+\frac{1}{p}+\sigma}\|u\|_{\dot{\mathcal{F}L}^{\sigma,p}}$$

IDEA: $\lambda \to 0$ makes X-norms small (and times longer) if

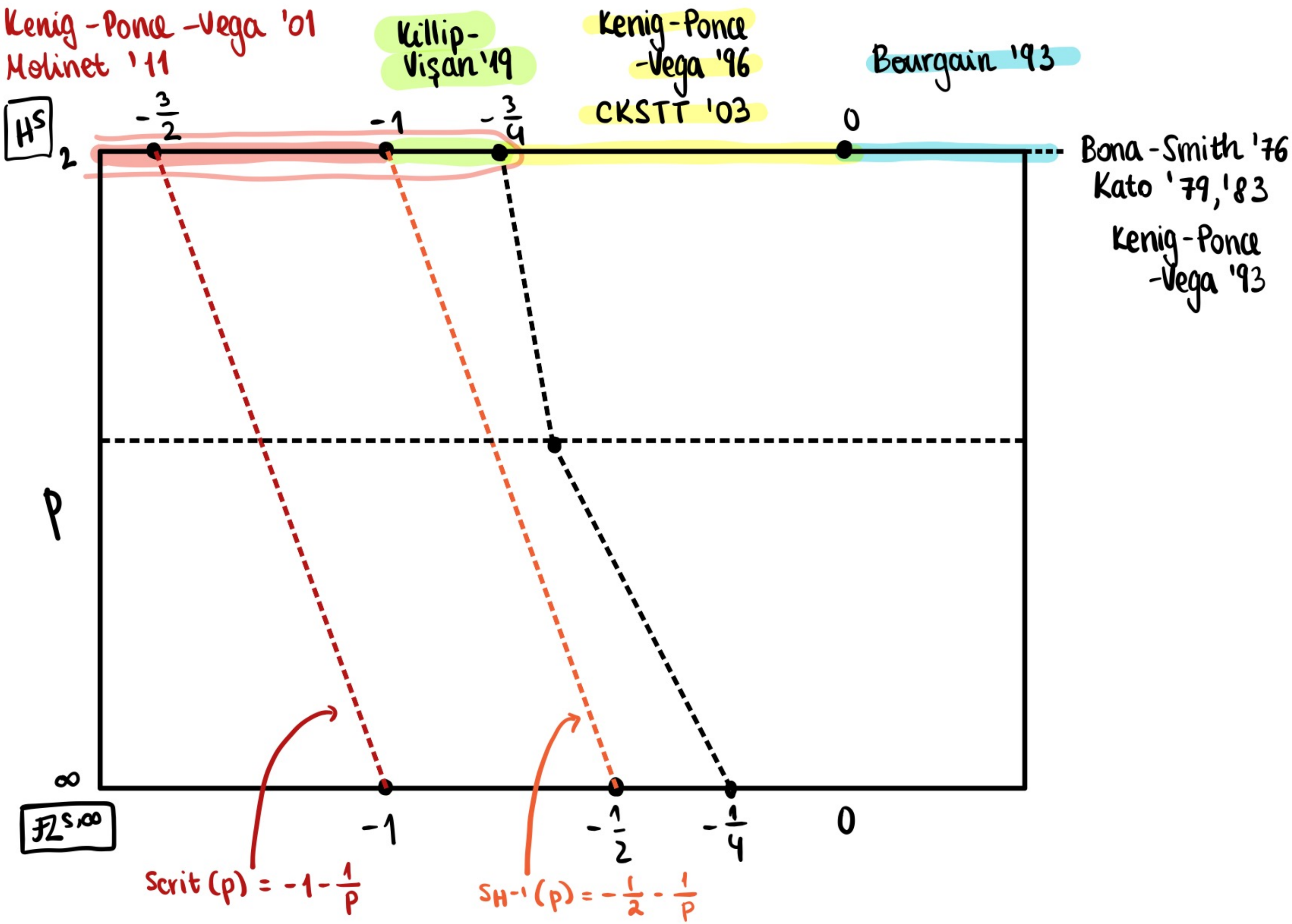
$$s > -\frac{3}{2} \qquad \text{and} \qquad \sigma > -1 - \frac{1}{p} =: s_{\text{crit}}(p)$$

Expect: LWP of (KdV) for $s > s_{\text{crit}}(2)$ and $\sigma > s_{\text{crit}}(p)$ ⚠

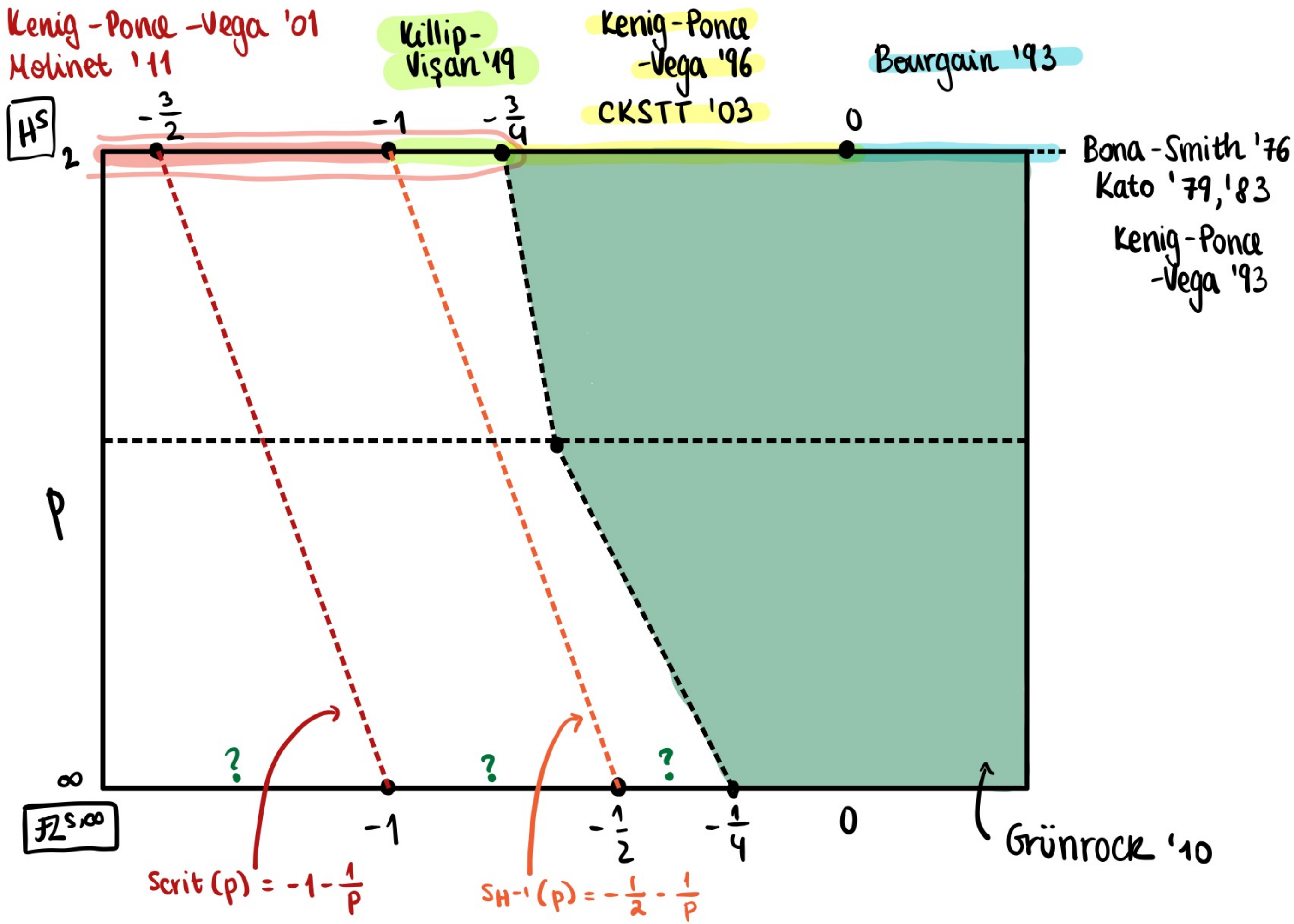
What is known?



What is known?



What is known?



Theorem (C.-Correia-Ramos '26)

Let $2 \leq p \leq \infty$.
(i) KdV is locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{R})$ for

$$s > \max(\underbrace{-\tfrac{1}{2} - \tfrac{1}{p}}_{s_{H^{-1}}(p)}, -1 + \tfrac{1}{2p})$$

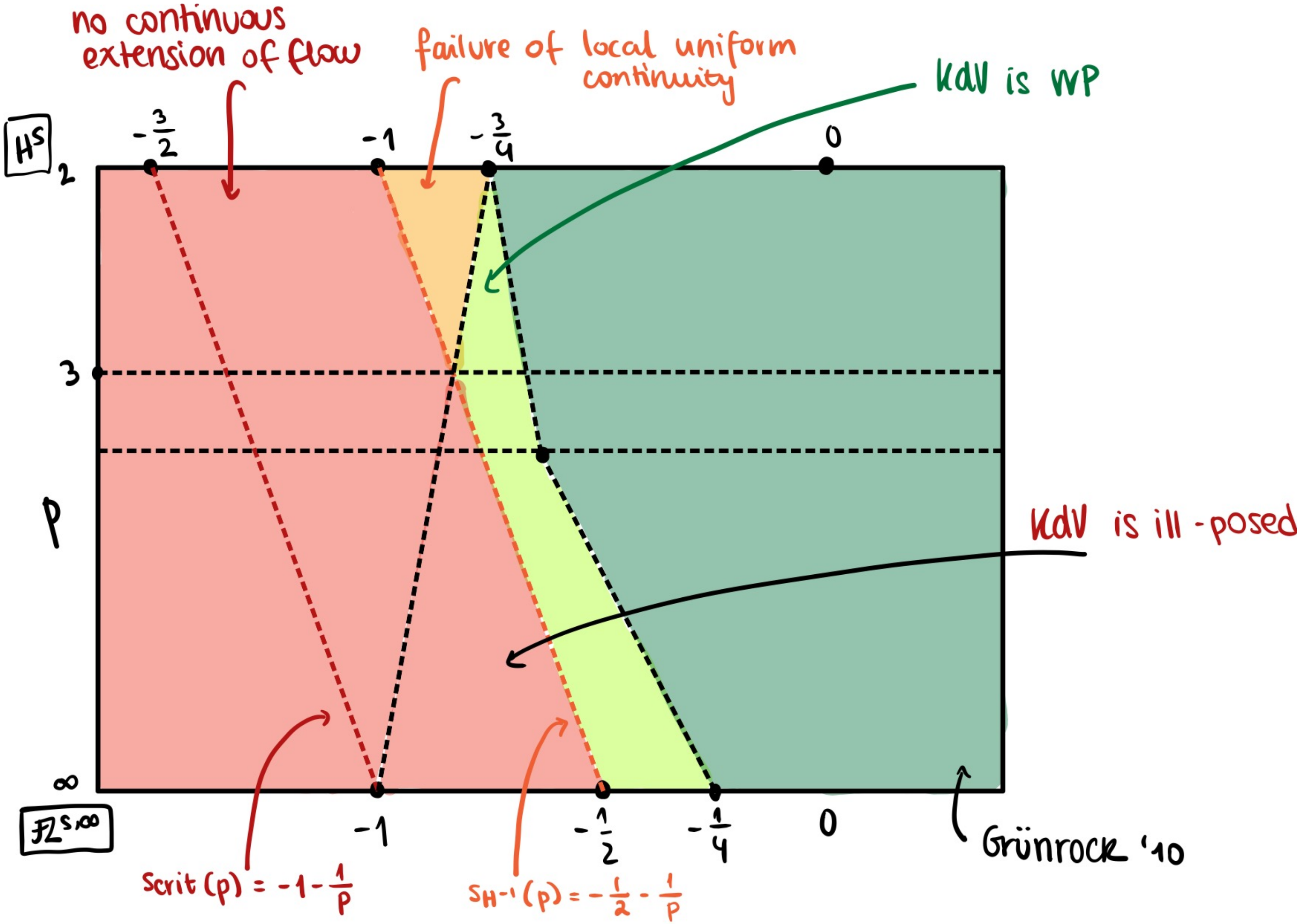
(ii) If $s < -\frac{1}{2} - \frac{1}{p}$, there is no continuous extension of the KdV flow from $\mathcal{F}L^{s,p}(\mathbb{R})$ to the space of distributions
(iii) If $s < -1 + \frac{1}{2p}$, the flow cannot be locally uniformly continuous

- Improves result by Grünrock '10

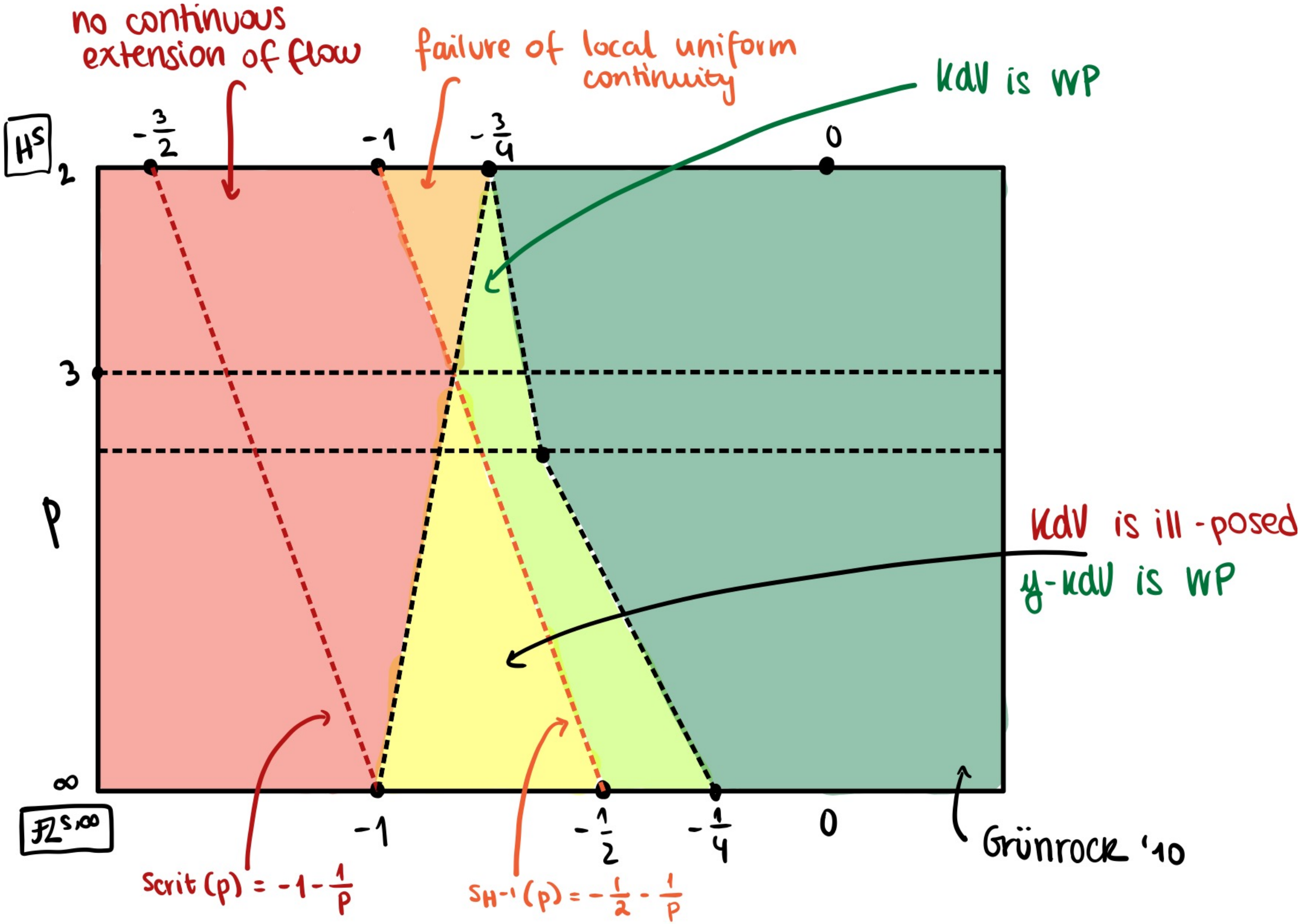
$$s > \max\left(-\frac{1}{2} - \frac{1}{2p}, -\frac{1}{4} - \frac{11}{8p}\right)$$

- **Optimal**:
  - ▷ (ii) & $p \geq 3 \rightsquigarrow$ if one requires a continuous flow
  - ▷ (iii) & $2 \leq p \leq 3 \rightsquigarrow$ wrt method

How does this fit in the literature?



How does this fit in the literature?



- To study (KdV) we combine:

∞-normal form reductions $+$ gauged KdV equation $+$ algebraic cancellations

$$(\mathcal{G}\text{-KdV}) \qquad \partial_t z + \partial_x^3 z = \sum_{j=2}^{\infty} N_j[z]$$

Theorem (C.-Correia-Ramos '26)

($\mathcal{G}$-KdV) is (small-data) locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{R})$ for

$$2 \leq p \leq \infty \quad \text{and} \quad s > \max(\underbrace{\cancel{-\tfrac{1}{2} - \tfrac{1}{p}}}_{s_{H^{-1}}(p)}, -1 + \tfrac{1}{2p})$$

- To study (KdV) we combine:

∞-normal form reductions $+$ gauged KdV equation $+$ algebraic cancellations

$$(\mathcal{G}\text{-KdV}) \qquad \partial_t z + \partial_x^3 z = \sum_{j=2}^{\infty} N_j[z]$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{(small)}\ u_0 \in \mathcal{F}L^{s,p}(\mathbb{R}) \arrow[r, "\mathcal{G}"] \arrow[d, dashed, "\boxed{\text{KdV}}"'] & z_0 := \mathcal{G}[u_0] \arrow[d, "\boxed{\mathcal{G}\text{-KdV}}"] \\
u := \mathcal{G}^{-1}[z] & z \arrow[l, "\mathcal{G}^{-1}"]
\end{tikzcd}$$

$s > -\frac{1}{2} - \frac{1}{p}$

$s > -1 + \frac{1}{2p}$

$s > \max\left(-\frac{1}{2} - \frac{1}{p}, -1 + \frac{1}{2p}\right)$

Theorem (C.-Correia-Ramos '26)

($\mathcal{G}$-KdV) is (small-data) locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{R})$ for

$$2 \le p \le \infty \quad \text{and} \quad s > \max\Big(\underbrace{\cancel{-\tfrac{1}{2} - \tfrac{1}{p}}}_{s_{H^{-1}}(p)}, -1 + \tfrac{1}{2p}\Big)$$

Gauge transform and other models

Miura transforms

$$\boxed{\mathcal{G}^{-1}[f] := f + \mathcal{F}_x^{-1}\left(\frac{1}{3}\int_{\xi=\xi_1+\xi_2} \mathbb{1}_{A^c}(\xi_1,\xi_2)\frac{\widehat{f}(\xi_1)\widehat{f}(\xi_2)}{(i\xi_1)(i\xi_2)}d\xi_1\right)}$$

Miura transforms

$$\boxed{\mathcal{G}^{-1}[f] := f + \mathcal{F}_x^{-1}\left(\frac{1}{3}\int_{\xi=\xi_1+\xi_2} \cancel{\mathbb{1}_{A^c}(\xi_1,\xi_2)}\,\frac{\widehat{f}(\xi_1)\widehat{f}(\xi_2)}{(i\xi_1)(i\xi_2)}\,d\xi_1\right)}$$

▷ Set $A = \emptyset$:

Miura transforms

$$\boxed{\mathcal{G}^{-1}[f] := f + \mathcal{F}_x^{-1}\Bigg(\frac{1}{3}\int_{\xi=\xi_1+\xi_2} \cancel{\mathbb{1}_{A^c}(\xi_1,\xi_2)}\, \frac{\widehat{f}(\xi_1)\widehat{f}(\xi_2)}{(i\xi_1)(i\xi_2)}\,d\xi_1\Bigg)}$$

▷ Set $A=\emptyset$:

$$\begin{aligned}\mathcal{G}^{-1}[f] = f + \tfrac{1}{3}[\partial_x^{-1}f]^2 &= \partial_x[\partial_x^{-1}f] + \tfrac{1}{3}[\partial_x^{-1}f]^2 \\ &= \mathbf{M}[\partial_x^{-1}f]\end{aligned}$$

▷ Miura transform for KdV: [Miura '68]

$$\mathbf{M}[z] := \partial_x z + \tfrac{1}{3}z^2$$

- z solves mKdV: $\partial_t z + \partial_x^3 z = \frac{2}{9}\partial_x(z^3) \implies \mathbf{M}[z]$ solves KdV [Colliander-Keel-Staffilani-Takaoka-Tao '03, Kappeler-Perry-Shubin-Topalov '05, Buckmaster-Koch '15]
- **M is not** invertible !

Miura transforms

$$\boxed{\mathcal{G}^{-1}[f] := f + \mathcal{F}_x^{-1}\left(\frac{1}{3}\int_{\xi=\xi_1+\xi_2} \mathbb{1}_{A^c}(\xi_1,\xi_2)\frac{\widehat{f}(\xi_1)\widehat{f}(\xi_2)}{(i\xi_1)(i\xi_2)}d\xi_1\right)}$$

▷ <u>Our gauge transform $\mathcal{G}^{-1}$:</u>

- **is** invertible ! [for suitable A]
- ($\mathcal{G}$-KdV) is an equation, not a system
- ($\mathcal{G}$-KdV) $\overset{\mathcal{G}}{\Longleftrightarrow}$ (KdV) for smooth solutions
- Can define $\mathcal{G}$ for non-integrable models and in higher dimensions

Example 1: dispersion generalized Benjamin-Ono equations

(dgBO) $$\partial_t u + |\partial_x|^\alpha \partial_x u = \partial_x(u^2)$$

- $\mathcal{F}_x(|\partial_x|^\alpha f)(\xi) = |\xi|^\alpha \widehat{f}(\xi)$
- $\alpha = 2$: KdV
- $\alpha = 1$: BO
- $1 < \alpha < 2$: not integrable

Example 1: dispersion generalized Benjamin-Ono equations

(dgBO) $$\partial_t u + |\partial_x|^\alpha \partial_x u = \partial_x(u^2)$$

- $\mathcal{F}_x(|\partial_x|^\alpha f)(\xi) = |\xi|^\alpha \widehat{f}(\xi)$
- $\alpha = 2$: KdV
- $\alpha = 1$: BO
- $1 < \alpha < 2$: not integrable

Example 2: Novikov-Veselov equation

(NV) $$\partial_t u + \tfrac{1}{4}(\partial_x^3 - 3\partial_x\partial_y^2)u = 3\partial_x\big(u\tfrac{\partial_x^2-\partial_y^2}{\Delta}u\big) + 3\partial_y\big(u\tfrac{2\partial_x\partial_y}{\Delta}u\big) + \tfrac{1}{4}E\tfrac{\partial_x^2-\partial_y^2}{\Delta}\partial_x u$$

- Miura-type transform: $\mathbf{M}[z] = \partial_x z - i\partial_y z + |z|^2$ [Bogdanov '87]
- Perry '14, Music-Perry '18, Kazeykina-Muñoz '16,'18, Angelopoulos '16, Adams-Grünrock '25, Nachman-Perry-Tataru '25

Example 1: dispersion generalized Benjamin-Ono equations

(dgBO) $$\partial_t u + |\partial_x|^\alpha \partial_x u = \partial_x(u^2)$$

- $\mathcal{F}_x(|\partial_x|^\alpha f)(\xi) = |\xi|^\alpha \widehat{f}(\xi)$
- $\alpha = 2$: KdV
- $\alpha = 1$: BO
- $1 < \alpha < 2$: not integrable

Example 2: Novikov-Veselov equation

(NV) $$\partial_t u + \tfrac{1}{4}(\partial_x^3 - 3\partial_x\partial_y^2)u = 3\partial_x\big(u\tfrac{\partial_x^2-\partial_y^2}{\Delta}u\big) + 3\partial_y\big(u\tfrac{2\partial_x\partial_y}{\Delta}u\big) + \tfrac{1}{4}E\tfrac{\partial_x^2-\partial_y^2}{\Delta}\partial_x u$$

- Miura-type transform: $\mathbf{M}[z] = \partial_x z - i\partial_y z + |z|^2$ [Bogdanov '87]
- Perry '14, Music-Perry '18, Kazeykina-Muñoz '16,'18, Angelopoulos '16, Adams-Grünrock '25, Nachman-Perry-Tataru '25

Theorem (C.-Correia-Ramos '26+)

For any $E \in \mathbb{R}$, (NV) is locally well-posed in $H^s(\mathbb{R}^2)$ for $\mathbf{s > -1}$

Back to KdV

LWP in $\mathcal{F}L^{s,\infty}(\mathbb{R})$ for $s > -\frac{1}{4}$: [Grünrock '10, CCR '26]

Duhamel formulation:

$$u(t) = e^{-t\partial_x^3} u_0 + \int_0^t e^{-(t-t')\partial_x^3} \partial_x (u^2(t'))dt'$$

Problem 1: Need to "beat" the derivative ∂_x in nonlinearity

Problem 2: dispersion $\not\Longrightarrow$ linear smoothing $\|e^{-t\partial_x^3} u_0\|_{\mathcal{F}L^{s,p}} = \|u_0\|_{\mathcal{F}L^{s,p}}$

IDEA: there may be "smoothing" from nonlinear interactions

Interaction representation: $v(t) = e^{t\partial_x^3} u(t)$ solves

$$\widehat{v}(t,\xi) = \widehat{u}_0(\xi) \pm \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi} (i\xi)\, \widehat{v}(t',\xi_1)\widehat{v}(t',\xi_2)\ dt'$$

Phase function: $\Phi = -\xi^3 + \xi_1^3 + \xi_2^3 \overset{\xi=\xi_1+\xi_2}{=} -3\xi\xi_1\xi_2$

▷ Capture oscillations via spaces adapted to linear dispersion [Bourgain '93]
[Klainerman-Machedon '93]

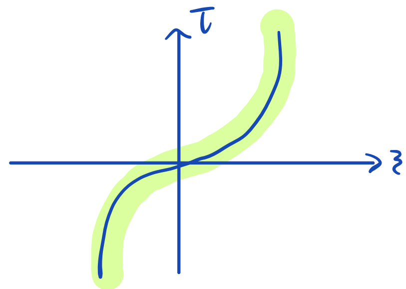
Fourier restriction norm method adapted to $\mathcal{F}L^{s,p}$ [G. '04, G.-Herr '08]

$$\|u\|_{X^{s,b}_{p,q}} = \|\langle\xi\rangle^s \langle\tau - \xi^3\rangle^b \widehat{u}(\tau,\xi)\|_{L^p_\xi L^q_\tau}$$

→ Airy equation :

$$0 = (\partial_t + \partial_x^3) u$$

$$\overset{\mathcal{F}_{t,x}}{\underset{(\tau,\xi)}{\Longrightarrow}} \quad 0 = i(\tau - \xi^3)\hat{u}(\tau,\xi)$$



Strategy: LWP follows from bilinear estimate in $X^{s,\frac{1}{p}+}_{p,p} \hookrightarrow C(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{R}))$

- Time derivatives $\rightsquigarrow$ "spatial derivatives"

$$\max\left(|\tau - \xi^3|, |\tau_1 - \xi_1^3|, |\tau_2 - \xi_2^3|\right) \gtrsim |\Phi(\xi,\xi_1,\xi_2)|$$

- Further gain when **non-stationary** [Kenig-Ponce-Vega '96]

$$|\partial_{\xi_1}\Phi| = 3|\xi(\xi_1 - \xi_2)| \gg 1 \quad \rightsquigarrow \quad \xi_1 \mapsto \Phi(\xi,\xi_1,\xi-\xi_1)$$

Problem: **stationary** region $|\xi_1| \sim |\xi_2| \gg \big||\xi_1| - |\xi_2|\big| \rightsquigarrow s > -\frac{1}{4}$

GOAL: go beyond $s > -\frac{1}{4}$

Solution: use more information on u

$\rightsquigarrow$ (Poincaré-Dulac) normal form reductions

General setup:

1. Separate nonlinear part into good and bad parts
2. "Eliminate" the bad part (integration-by-parts in *time*)
   $\Longrightarrow$ introduces higher order terms
3. Repeat (or terminate at a finite step)

[Shatah, Nikolenko, Babin-Ilyin-Titi, Kwon, Oh, Guo, Erdogan-Tzirakis, Kishimoto Forlano, Correia, Tzvetkov, Sun, de Suzzoni, Bambusi, Grébert, Sosoe, Y. Wang, ...]

Applications:

- small data global well-posedness and scattering: Shatah '85
- almost global existence: S. '85, Bambusi-Delort-Grébert-Szeftel '07
- construction of solutions: Babin-Ilyin-Titi '11, Guo-Kwon-Oh '13
- unconditional uniqueness: Kwon-Oh '12, Kishimoto '19, Oh-Sosoe-Wang '25
- nonlinear smoothing/growth of Sobolev norms/Talbot effect: Bourgain '04, Colliander-K.-Oh '12, Erdogan-Tzirakis '13, Bam.-Gr. '21, C.-Killip-Vişan '24
- energy estimates/quasi-invariance: CKSTT '03,'08, Oh-S.-Tzv.'18, Sun-Tzv.'25

① Infinite normal form reductions

Go back to Duhamel for **interaction representation** $v(t) = e^{t\partial_x^3} u(t)$

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi) = \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi} (i\xi) \widehat{v}_1 \widehat{v}_2 \, d\vec{\xi} dt' =: \int_0^t \mathcal{N}(t', v) dt'$$

where $\Phi = -\xi^3 + \xi_1^3 + \xi_2^3 \overset{\xi=\xi_1+\xi_2}{=} -3\xi\xi_1\xi_2$

good interactions (non-stat.) $\rightarrow$	$(\xi_1, \xi_2) \in A$
bad interactions (stat.) $\rightarrow$	$(\xi_1, \xi_2) \in A^c : \ |\xi_1|, |\xi_2| \geq 1$

- Split nonlinear term $\mathcal{N}(t, v) = ($good part$) + ($bad part$)$

(good part) $\rightarrow$ good estimate ✓

(bad part) $\rightarrow$ **cannot** estimate ✗ $\rightsquigarrow$ normal form reduction

Step 1: (= integration-by-parts *in time*)

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

$$= \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}(i\xi)(\qquad\qquad)\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

Step 1: (= integration-by-parts *in time*)

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

$$= \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}(i\xi)(\mathbb{1}_A + \mathbb{1}_{A^c})\widehat{v}_1\widehat{v}_2 d\vec{\xi} dt'$$

Step 1: (= integration-by-parts *in time*)

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

$$= \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}(i\xi)(\mathbb{1}_A + \mathbb{1}_{A^c})\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

Step 1: (= integration-by-parts *in time*)

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

$$= \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}(i\xi)(\mathbb{1}_A + \mathbb{1}_{A^c})\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

$$= (\dots) + \int_0^t \int_{\xi=\xi_1+\xi_2} \frac{d}{dt'}\Big(\frac{e^{it'\Phi}}{i\Phi}\Big)(i\xi)\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

Step 1: (= integration-by-parts *in time*)

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

$$= \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}(i\xi)(\mathbb{1}_A + \mathbb{1}_{A^c})\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

$$= (\dots) + \int_0^t \int_{\xi=\xi_1+\xi_2} \frac{d}{dt'}\Big(\frac{e^{it'\Phi}}{i\Phi}\Big)(i\xi)\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

$$= (\dots) + \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}\,\Big|_{t'=0}^{t'=t} - \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\partial_t(\widehat{v}_1\widehat{v}_2)d\vec{\xi}dt'$$

Step 1: (= integration-by-parts *in time*)

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

$$= \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}(i\xi)(\mathbb{1}_A + \mathbb{1}_{A^c})\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

$$= (\dots) + \int_0^t \int_{\xi=\xi_1+\xi_2} \frac{d}{dt'}\Big(\frac{e^{it'\Phi}}{i\Phi}\Big)(i\xi)\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

$$= (\dots) + \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}\,\Big|_{t'=0}^{t'=t} - \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\partial_t(\widehat{v}_1\widehat{v}_2)d\vec{\xi}dt'$$

Step 1: (= integration-by-parts *in time*)

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

$$= \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}(i\xi)(\mathbb{1}_A + \mathbb{1}_{A^c})\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

$$= (\dots) + \int_0^t \int_{\xi=\xi_1+\xi_2} \frac{d}{dt'}\Big(\frac{e^{it'\Phi}}{i\Phi}\Big)(i\xi)\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

$$= (\dots) + \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}\,\Big|_{t'=0}^{t'=t} - \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\partial_t(\widehat{v}_1\widehat{v}_2)d\vec{\xi}dt'$$

$$= (\dots) - \int_0^t \int_{\substack{\xi=\xi_1+\xi_2\\ \xi_1=\xi_{11}+\xi_{12}}} e^{it'[\Phi+\Phi_1]}\frac{(i\xi)(i\xi_1)}{i\Phi}\mathbb{1}_{A^c}\widehat{v}_{11}\widehat{v}_{12}\widehat{v}_2\, d\vec{\xi}dt'$$

$$- \int_0^t \int_{\substack{\xi=\xi_1+\xi_2\\ \xi_2=\xi_{21}+\xi_{22}}} e^{it'[\Phi+\Phi_2]}\frac{(i\xi)(i\xi_2)}{i\Phi}\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_{21}\widehat{v}_{22}\, d\vec{\xi}dt'$$

Step 1: (= integration-by-parts *in time*)

- $\rightsquigarrow$ repeat normal form reduction
- $\rightsquigarrow$ **need** to keep track of ∂_t
- $\rightsquigarrow$ **binary trees**

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

$$= \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}(i\xi)(\mathbb{1}_A + \mathbb{1}_{A^c})\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

$$= (\dots) + \int_0^t \int_{\xi=\xi_1+\xi_2} \frac{d}{dt'}\Big(\frac{e^{it'\Phi}}{i\Phi}\Big)(i\xi)\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

$$= (\dots) + \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}\,\Big|_{t'=0}^{t'=t} - \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\partial_t(\widehat{v}_1\widehat{v}_2)d\vec{\xi}dt'$$

$$= (\dots) - \int_0^t \int_{\substack{\xi=\xi_1+\xi_2\\ \xi_1=\xi_{11}+\xi_{12}}} e^{it'[\Phi+\Phi_1]}\frac{(i\xi)(i\xi_1)}{i\Phi}\mathbb{1}_{A^c}\widehat{v}_{11}\widehat{v}_{12}\widehat{v}_2\, d\vec{\xi}dt'$$

$$- \int_0^t \int_{\substack{\xi=\xi_1+\xi_2\\ \xi_2=\xi_{21}+\xi_{22}}} e^{it'[\Phi+\Phi_2]}\frac{(i\xi)(i\xi_2)}{i\Phi}\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_{21}\widehat{v}_{22}\, d\vec{\xi}dt'$$

⚠ Last terms still have bad regions A_1^c and A_2^c, resp.

Step 1: (= integration-by-parts *in time*)

- $\rightsquigarrow$ repeat normal form reduction
- $\rightsquigarrow$ **need** to keep track of ∂_t
- $\rightsquigarrow$ **binary trees**

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

N()

$$= \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}(i\xi)(\mathbb{1}_A + \mathbb{1}_{A^c})\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

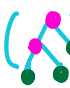
$$= (\dots) + \int_0^t \int_{\xi=\xi_1+\xi_2} \frac{d}{dt'}\Big(\frac{e^{it'\Phi}}{i\Phi}\Big)(i\xi)\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}dt'$$

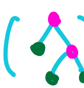
B()

$$= (\dots) + \boxed{\int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\widehat{v}_1\widehat{v}_2 d\vec{\xi}\Big|_{t'=0}^{t'=t}} - \int_0^t \int_{\xi=\xi_1+\xi_2} e^{it'\Phi}\frac{i\xi}{i\Phi}\mathbb{1}_{A^c}\partial_t(\widehat{v}_1\widehat{v}_2)d\vec{\xi}dt'$$

$$= (\dots) - \int_0^t \int_{\substack{\xi=\xi_1+\xi_2\\ \xi_1=\xi_{11}+\xi_{12}}} e^{it'[\Phi+\Phi_1]}\frac{(i\xi)(i\xi_1)}{i\Phi}\mathbb{1}_{A^c}\boxed{\widehat{v}_{11}\widehat{v}_{12}\widehat{v}_2}d\vec{\xi}dt'$$

$$- \int_0^t \int_{\substack{\xi=\xi_1+\xi_2\\ \xi_2=\xi_{21}+\xi_{22}}} e^{it'[\Phi+\Phi_2]}\frac{(i\xi)(i\xi_2)}{i\Phi}\mathbb{1}_{A^c}\boxed{\widehat{v}_1\widehat{v}_{21}\widehat{v}_{22}}d\vec{\xi}dt'$$

R()

R()

⚠ Last terms still have bad regions A_1^c and A_2^c, resp.

(Continuing the process ...)

$\widehat{v}(t,\xi) - \widehat{u_0}(\xi)$

$$\overset{1\times\mathrm{NFR}}{=} \mathrm{B}(\wedge, t', v)\Big|_{t'=0}^{t'=t} + \int_0^t \mathrm{N}(\wedge, t', v)\,dt' + \int_0^t \mathrm{R}({}_{\wedge}\!\wedge, t', v) + \mathrm{R}(\wedge_{\wedge}, t', v)\,dt'$$

(Continuing the process ...)

$\widehat{v}(t,\xi) - \widehat{u_0}(\xi)$

$$\overset{1\times\mathrm{NFR}}{=} \mathrm{B}(\wedge, t', v)\Big|_{t'=0}^{t'=t} + \int_0^t \mathrm{N}(\wedge, t', v)dt' + \int_0^t \mathrm{R}(\ , t', v) + \mathrm{R}(\ , t', v)dt'$$

$$\overset{2\times\mathrm{NFR}}{=} \mathrm{B}(\wedge, t', v)\Big|_{t'=0}^{t'=t} + \mathrm{B}(\ , t', v)\Big|_{t'=0}^{t'=t} + \mathrm{B}(\ , t', v)\Big|_{t'=0}^{t'=t}$$

$$+ \int_0^t \big[\mathrm{N}(\wedge, t', v) + \mathrm{N}(\ , t', v) + \mathrm{N}(\ , t', v)\big]dt'$$

$$+ \int_0^t \big[\mathrm{R}(\ , t', v) + \mathrm{R}(\ , t', v) + \mathrm{R}(\ , t', v) + \mathrm{R}(\ , t', v) + \mathrm{R}(\ , t', v)\big]dt'$$

$$\overset{J\times\mathrm{NFR}}{=} \sum_{j=1}^{J-1} \sum_{\mathcal{T}\in\mathrm{T}_j} \mathrm{B}(\mathcal{T}, t', v)\Big|_{t'=0}^{t'=t}$$

$$+ \sum_{j=1}^{J} \sum_{\mathcal{T}\in\mathrm{T}_j} \int_0^t \mathrm{N}(\mathcal{T}, t', v)dt' + \sum_{\mathcal{T}\in\mathrm{T}_J} \int_0^t \mathrm{R}(\mathcal{T}, t', v)dt'$$

Step J:

$$\widehat{v}(t,\xi) - \widehat{u_0}(\xi)$$

$$= \sum_{j=1}^{J-1} \sum_{\mathcal{T}\in \mathrm{T}_j} \mathrm{B}(\mathcal{T}, t', v)\Big|_{t'=0}^{t'=t} + \sum_{j=1}^{J} \sum_{\mathcal{T}\in \mathrm{T}_j} \int_0^t \mathrm{N}(\mathcal{T}, t', v) dt' + \sum_{\mathcal{T}\in \mathrm{T}_J} \int_0^t \mathrm{R}(\mathcal{T}, t', v) dt'$$

where

$$\mathcal{I}(\mathcal{T}, \mathrm{m}, t, v) = \int_{\Gamma(\mathcal{T})} e^{it\Psi(\mathcal{T})} \mathrm{m}(\vec{\xi}) \prod_{\bullet\in\mathcal{T}^\infty} \widehat{v}_\bullet(t') d\vec{\xi}$$

$$\mathrm{B}(\mathcal{T}) = \mathcal{I}(\mathcal{T}, \mathrm{mul_B}(\mathcal{T}))$$
$$\mathrm{N}(\mathcal{T}) = \mathcal{I}(\mathcal{T}, i\,\mathrm{mul_N}(\mathcal{T}))$$
$$\mathrm{R}(\mathcal{T}) = \mathcal{I}(\mathcal{T}, i\Psi(\mathcal{T})\mathrm{mul_B}(\mathcal{T}))$$

$$\mathrm{mul_B}(\mathcal{T}) = (-1)^{j+1} \prod_{\bullet\in\mathcal{T}^0} \mathbb{1}_{A^c_\bullet} \frac{\xi_\bullet}{\Phi_\bullet}$$
$$\mathrm{mul_N}(\mathcal{T}) = (\text{big formula})$$

- Systematic approach, integration-by-parts on trees
- One boundary piece and one time integral term per tree [No memory]
- Complexity is encoded in the Fourier multipliers [“Local” restrictions]

Limiting equation ($J \to \infty$):

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi)$$

$$= \sum_{j=1}^{\infty} \sum_{\mathcal{T}\in \mathrm{T}_j} \mathrm{B}(\mathcal{T}, t', v)\Big|_{t'=0}^{t'=t} + \sum_{j=1}^{\infty} \sum_{\mathcal{T}\in \mathrm{T}_j} \int_0^t \mathrm{N}(\mathcal{T}, t', v) dt' + \cancel{\sum_{\mathcal{T}\in \mathrm{T}_J} \int_0^t \mathrm{R}(\mathcal{T}, t', v) dt'}$$

where

$$\mathcal{I}(\mathcal{T}, \mathrm{m}, t, v) = \int_{\Gamma(\mathcal{T})} e^{it\Psi(\mathcal{T})} \mathrm{m}(\vec{\xi}) \prod_{\bullet \in \mathcal{T}^\infty} \widehat{v}_\bullet(t') d\vec{\xi}$$

$$\mathrm{B}(\mathcal{T}) = \mathcal{I}(\mathcal{T}, \mathrm{mul_B}(\mathcal{T}))$$
$$\mathrm{N}(\mathcal{T}) = \mathcal{I}(\mathcal{T}, i\mathrm{mul_N}(\mathcal{T}))$$
$$\mathrm{R}(\mathcal{T}) = \mathcal{I}(\mathcal{T}, i\Psi(\mathcal{T})\mathrm{mul_B}(\mathcal{T}))$$

$$\mathrm{mul_B}(\mathcal{T}) = (-1)^{j+1} \prod_{\bullet\in\mathcal{T}^0} \mathbb{1}_{A^c_\bullet} \frac{\xi_\bullet}{\Phi_\bullet}$$
$$\mathrm{mul_N}(\mathcal{T}) = (\text{big formula})$$

NEED: $X^{s,b}$-analysis for N-terms [less IBP $\rightsquigarrow$ less explicit smoothing]

Problem: $\mathrm{B}(\mathcal{T}, t, v) \in \mathcal{F}L^{s,\infty}$ for $s > -\frac{1}{2}$ BUT not in $X^{s,b}$ [no time integral]

Idea: remove B $\rightsquigarrow$ $\boxed{\text{gauge transform } \mathcal{G}}$

② Gauge transform

GOAL: Remove boundary terms via change of variables

Idea: Start by removing the first boundary term

$$\widehat{v}(t,\xi) - \widehat{u}_0(\xi) = \underbrace{\mathrm{B}(\,\Lambda\,, t', v)}_{=:\mathrm{D}[w,w](t')}\Big|_{t'=0}^{t'=t} + \text{(other boundary terms)} + \text{(integral terms)}$$

$$\rightsquigarrow \boxed{\widehat{v}(t,\xi) = \widehat{w}(t,\xi) + \mathrm{D}[w,w](t)}$$

Equation for w:

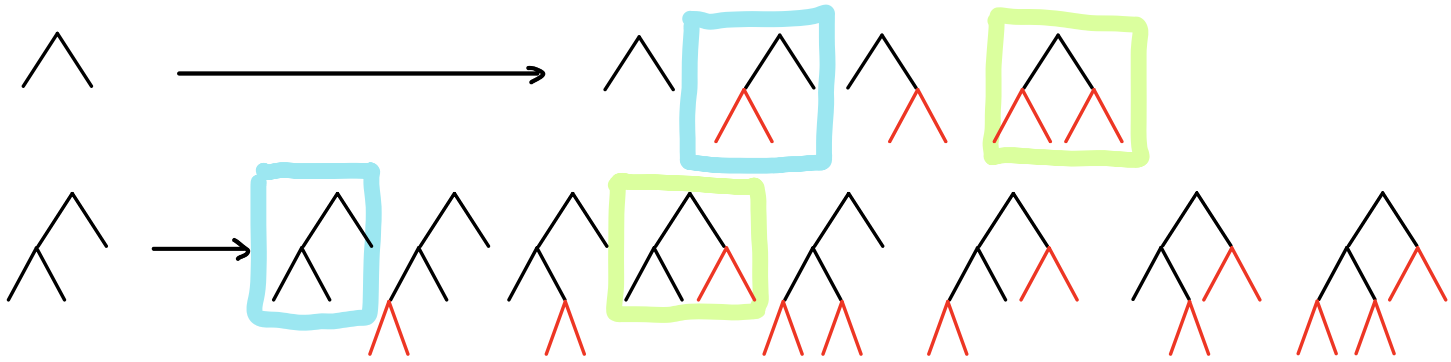
$$\widehat{w}(t,\xi) - \widehat{w}(0,\xi)$$
$$= \big[\mathrm{D}[v,v] - \mathrm{D}[w,w]\big]_{t'=0}^{t'=t} + \text{(other boundary terms)} + \text{(integral terms)}\big|_{w+\mathrm{D}[w,w]}$$

⚠ **Fundamental observation:** $\mathrm{D}[v,v]$ cancels

$+$ **ALL** other boundary terms cancel

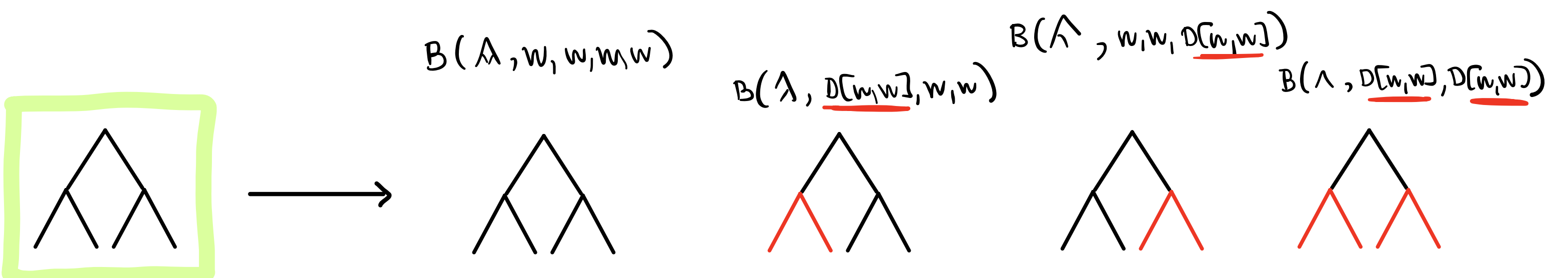
③ Exploit algebraic cancellations (Part I)

Original boundary terms $\qquad$ Gauged boundary terms

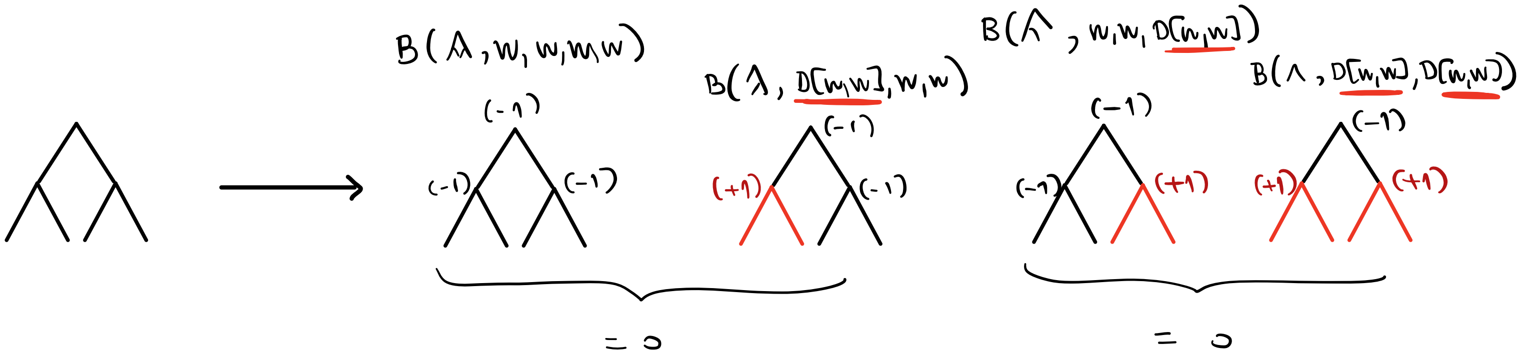


- For each tree, all possible black/red colourings appear in gauged equation

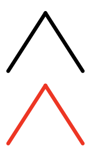
Example:



Example:



- The convolution structure is the same [same tree]
- The regions of integration are the same ⚠ [bad set $A^c_\bullet$ for each $\bullet \in \mathcal{T}^0$]
- Multiplier is the same (in absolute value) [$\frac{\xi_\bullet}{\Phi_\bullet}$ for each $\bullet \in \mathcal{T}^0$]
- The sign changes ($\sim$ integration-by-parts) !



Black cherry $\to -$

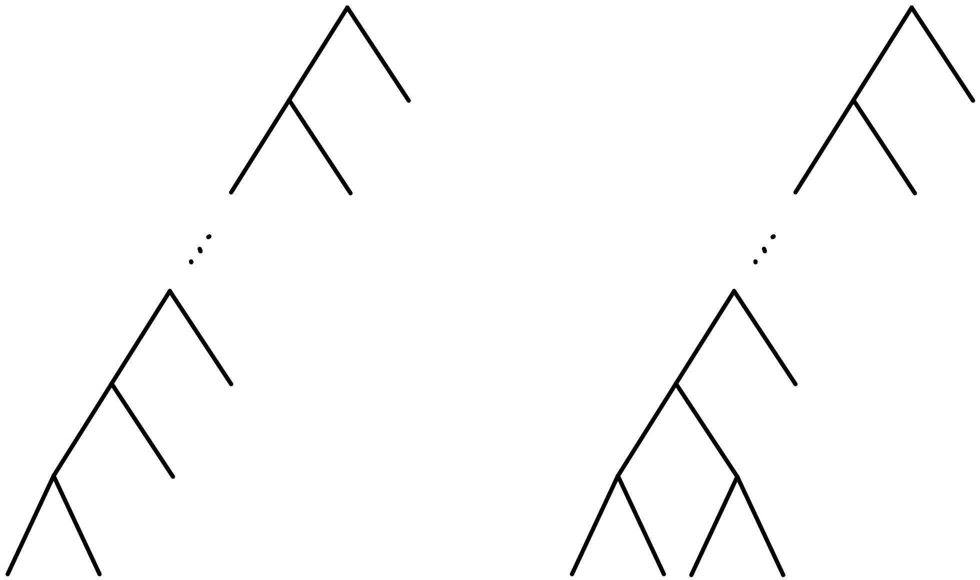
Red cherries $\to +$

$\Longrightarrow$ pairwise cancellation of gauged boundary terms

"Gauged" equation:

$$\widehat{w}(t,\xi) - \widehat{w}(0,\xi) = \cancel{\sum_{j=1}^{\infty} \sum_{\mathcal{T}\in \mathrm{T}_j} \mathrm{B}(\mathcal{T}, t', w)\Big|_{t'=0}^{t'=t}} + \sum_{j=1}^{\infty} \sum_{\mathcal{T}\in \mathrm{T}_j} \int_0^t \mathrm{N}(\mathcal{T}, t', w) dt'$$

⚠ There is also a cancellation for the integral terms: only see $\boxed{\text{"long trees" } \mathcal{T}}$



▷ Lastly, undo interaction representation $z(t) = e^{-t\partial_x^3} w(t)$

Summary:

$$\begin{tikzcd} \boxed{\text{KdV}} & \boxed{\text{NFE}} & \boxed{\mathcal{Y}\text{-KdV}} \\ u \arrow[r, "\infty\text{-NFR}"] & v \arrow[r, shift left, "\mathcal{Y}"] & z \arrow[l, shift left, "\mathcal{Y}^{-1}"] \end{tikzcd}$$

ill-posed for $s < s_{H^{-1}}(p)$

equivalent for smooth solutions

invertible for $s > s_{H^{-1}}(p)$

LWP for $s > \underbrace{-1 + \frac{1}{2p}}_{< s_{H^{-1}}(p),\ p > 3}$

1. KdV $\Leftrightarrow$ $\mathcal{Y}$-KdV for smooth solutions
2. BUT $\mathcal{Y}$-KdV is LWP below $s_{H^{-1}}(p)$ $\Rightarrow$ $\mathcal{Y}$-KdV is better equation at low regularity
3. Framework extends to other models (e.g., dg BO, NV)

Thank you for your attention!

Figure: Platanus orientalis planted by Buffon in the Jardin des Plantes in Paris in 1785. (2023, December 30).
Wikimedia Commons.
`https://commons.wikimedia.org/w/index.php?title=File:Buffplatree1.jpg&oldid=836804960`