The Hamiltonian formulation for the continuum Calogero–Moser model on the torus.

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joint work with Rowan Killip and Monica Vişan.

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UCLA

The continuum Calogero–Moser equation

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$$iq_t + \partial_x^2 q \mp 2iq\partial_x C_+(|q|^2) = 0, \tag{CCM}$$

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- Hardy space consists of L^2-traces of holomorphic functions on the upper-half plane (over $\mathbb{R}$) or the unit disc (over $\mathbb{T}$).
- The Hardy space is invariant by the Calogero–Moser flow, so a natural space to study the equation.

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$$H_{\mathrm{CSM}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\left(\frac{\pi}{L}\right)^2 \sum_{j\neq k} \frac{g^2}{\sin^2(\pi(x_j - x_k)/L)}.$$

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- Taking a thermodynamic limit of the associated Hamiltonian flow, i.e. $N, L \to \infty$ with density $\rho = N/L$ constant, yields the focusing CCM on the line.
- It is well-known that the finite dimensional models (H_{CSM}) are completely integrable (Calogero 1971, Moser 1975).

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- It follows that

$$\mathcal{E}_n(q) := \langle q, \mathcal{L}_q^n q\rangle$$

  is conserved by the flow for every $n \in \mathbb{N}_{\geq 0}$.
  E.g.

Mass:	$\mathcal{E}_0(q) = \|q\|_{L^2}^2$
Momentum:	$\mathcal{E}_1(q) = \langle q, -i\partial_x q \mp qC_+(\lvert q\rvert^2)\rangle$

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$$q(t,x) \mapsto \sqrt{\lambda} q(\lambda^2 t, \lambda x),$$

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See also [Chapouto-Forlano-Laurens '25, '26] in the non-chiral setting.

Hamiltonian structure

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such that

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Fun fact: $\Theta(q)^2 = 0$, so $(\Omega^\#(q))^{-1} = -(1 \pm \Theta(q))i$

$\Longrightarrow$ symplectic flows are easy to compute.

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The flow of $H(q) = \frac{1}{2}\mathcal{E}_2(q)$ with respect to the restriction is (CCM).

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"almost" generates (CCM) *with respect to the symplectic form*

$$\omega_q(f,g) = \operatorname{Re}\langle f, iC_+\big(g \mp 2iq\widetilde{\partial}^{-1}\operatorname{Re}(\overline{q}g)\big)\rangle \tag{1}$$

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"Almost": the Hamiltonian equation generated above is

$$i\partial_t q + q_{xx} \mp 2iq\partial_x C_+(|q|^2) \pm \pi^{-1}\mathcal{E}_0(q) q_x = 0,$$

which is (CCM) in a moving frame:

$$q(t,x) \mapsto q(t, x \pm \pi^{-1}\mathcal{E}_0 t).$$

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"Almost": the Hamiltonian equation generated above is

$$i\partial_t q + q_{xx} \mp 2iq\partial_x C_+(|q|^2) \pm \pi^{-1}\mathcal{E}_0(q)q_x = 0,$$

which is (CCM) in a moving frame:

$$q(t,x) \mapsto q(t, x \pm \pi^{-1}\mathcal{E}_0 t).$$

We found no Hamiltonian structure for the original (CCM) on $\mathbb{T}$.

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Not-so-good news: $\Omega(q)g = iC_+(q \mp 2iq\widetilde{\partial}^{-1}\,\mathrm{Re}(\bar{q}g))$ is not explicitly invertible. Fortunately, we are still able to compute the symplectic flows along the whole hierarchy.

The failure of non-degeneracy

It is easy to see that the focusing restricted form

$$\omega_q(f,g) = \operatorname{Re}\langle f, iC_+\big(g - 2iq\widetilde{\partial}^{-1}\operatorname{Re}(\overline{q}g)\big)\rangle$$

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Similarly, degeneracy in $L^2_+(\mathbb{R})$ occurs at the ground state

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Equipped with the Hamiltonian structure, we can give a new proof of global well-posedness for (CCM) in the Hardy space.

Theorem (Badreddine '23 on $\mathbb{T}$, Killip–Laurens–Vișan '25 on $\mathbb{R}$; alt. proof Killip, M., Vișan '26)

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Step 1: Commuting flows

Observe that

$$\beta(\kappa, q) = \langle q, (\mathcal{L}_q + \kappa)^{-1} q \rangle = \kappa^{-1}\mathcal{E}_0 - \kappa^{-2}\mathcal{E}_1 + \kappa^{-3}\mathcal{E}_2 + \dots$$

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Fix $\kappa \geq 1$. The flow

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Resolvent estimates

The simple well-posedness argument for the H_κ-flow is possible due to new uniform bounds for the resolvent of the Lax operator $\mathcal{L}_q = -i\partial \mp qC_+\bar{q}$ on bounded sets:

Lemma (Equivalence of Sobolev spaces.)

Let $M \le M_*$. Then for all $-1 \le s \le 1$ it holds

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The proof uses a connection to Carleman's proof of the isoperimetric inequality.

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is. S and T have the same spectrum, so we need to show

$$d(\sigma(S(q)),1) \ge C_M, \qquad (q\in B_M).$$

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Setting $f = q \cdot g \implies$

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$$\sigma(S(q)) \subset \{ |z| \leq M/2\pi \}$$

is bounded away from 1!

Equicontinuity

The previous estimates suffices to prove global well-posedness of the H_κ flows. We now turn to the equicontinuity of orbits.

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We can view the measures above as the spectral measures of the operator $\mathcal{L}_0$ on L^2_+. This is a definition which generalizes nicely to an arbitrary self-adjoint operator $\mathcal{L}$.

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Let $\mathcal{L}$ be a self-adjoint operator on L^2_+. A bounded set $\mathcal{Q} \subset L^2_+$ is called $\mathcal{L}$–equicontinuous if the family of spectral measures

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Equivalently

$$\lim_{\kappa\to\infty} \sup_{q\in\mathcal{Q}} \Big\langle q, \frac{(\mathcal{L}+1)^2}{(\mathcal{L}+1)^2+\kappa^2} q \Big\rangle = 0.$$

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Lemma

Let $M < M_*$, $u \in B_M$, $\mathcal{L}_u$ its associated Lax operator. A bounded set $\mathcal{Q} \subset B_M$ is $\mathcal{L}_u$–equicontinuous if and only if it is $\mathcal{L}_0$–equicontinuous.

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Key fact: The function $\varphi(X) = \frac{X}{X+\kappa^2}$ is operator monotone:

$$\langle q, \varphi(A) q\rangle \leq \langle q, \varphi(B) q\rangle$$

for all positive, self-adjoint operators $A \leq B$.

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The quantity $(*)$ characterises equicontinuity. And it is conserved by the flow! Thus equicontinuous data sets lead to equicontinuous orbits: for $\mathcal{Q} \subset B_M$ equicontinuous,

$$\mathcal{Q}_* := \{ e^{t \nabla_\omega H_\kappa} q_0 : \, q_0 \in \mathcal{Q}, \, t \in \mathbb{R}, \, \kappa \geq 1 \}$$

is also equicontinuous (and bounded).

Step 3: Limit $\kappa \to \infty$.

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- We are able to establish ($\dagger$) without any gauge transform, local smoothing estimates, ...this is thanks to the powerful chiral assumption.

Thank you!