From explicit formulas to Wave Kinetic theory for the Benjamin–Ono equation

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Dispersive integrable equations: pathfinders in Hamiltonian PDE
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Benjamin–Ono equation

Consider
$$\partial_t u(t,x) = \partial_x \left(|D|u - u^2 \right)(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{T},$$
with $D = \frac{1}{i}\frac{d}{dx}$, $\widehat{D}(k) = k$ and $u(t,x) \in \mathbb{R}$

GWP in $H^s(\mathbb{T})$, $s > -\frac{1}{2}$ *[Gérard–Kappeler–Topalov, '23]*

Different notions of integrability :

- Infinite number of conservation laws: the mass
$$M(u) = \int_0^{2\pi} u(x)\,dx,$$
the momentum
$$P(u) = \int_0^{2\pi} \frac{1}{2} u^2\,dx,$$
the energy
$$\mathcal{H}(u) = \int_0^{2\pi} \frac{1}{2} uDu - \frac{1}{3} u^3\,dx$$
- Lax pair : (L_u, B_u) *[Nakamura, '79], [Bock–Kruskal, '79]*
- Birkhoff coordinates *[Gérard–Kappeler, '21]*

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$$\begin{tikzcd} u_0 \arrow[r, "\Phi"] \arrow[d, "(BO)"'] & (\lambda_n), (\theta_n(0)) \arrow[d] \\ u(t) & (\lambda_n), (\theta_n(t)) \arrow[l, "\Phi^{-1}"] \end{tikzcd}$$

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$$\begin{tikzcd}
u_0 \arrow[r, "\Phi"] \arrow[d, "(BO)"'] \arrow[d, bend left, "\text{explicit formula}"] & (\lambda_n), (\theta_n(0)) \arrow[d] \\
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[Gérard, '23] : *Explicit formula*

$$\Pi u(t,z) = \left\langle \left(\mathrm{Id} - z e^{it} e^{2itL_{u_0}} S^* \right)^{-1} \Pi u_0, 1 \right\rangle, \quad \forall z \in \mathbb{C},\ |z| < 1$$

Work in the Hardy space $L^2_+ = \left\{ f \in L^2 : \widehat{f}(k) = 0,\ k < 0 \right\}$,

with the Szegő projector

$$\Pi : L^2 \to L^2_+$$

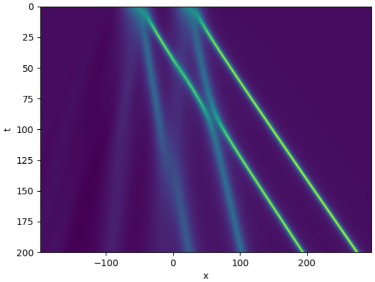
and the Lax and shift operators

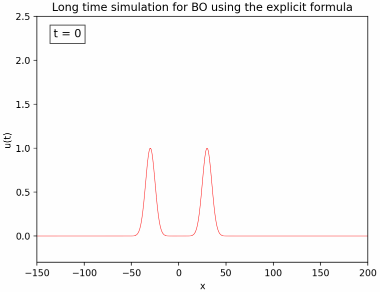
$$L_{u_0} f = Df - \Pi(u_0 f), \quad S^* f = \Pi(e^{-ix} f), \quad \forall f \in L^2_+$$

Applications of the explicit formulas for Benjamin–Ono (BO)

On the *theoretical* level:

- zero-dispersion limits for BO *[Maehlen, '25]*
- zero-dispersion limits on $\mathbb{R}$ *[Gérard, '24], [Chen, '25], [Blackstone–Gassot–Gérard–Miller, '24]* and classification of traveling wave solutions *[Gérard–He, '26]*
- the soliton resolution conjecture for BO *[Gassot–Gérard–Miller, '26]*





- large-torus limits for BO *[AB–Maehlen, '26+]* $\longrightarrow$ rigorous numerics on $\mathbb{R}$

On the *numerical* level:

- Long-time dynamics *[AB–Chen–Dolbeault, '25]*
- Low-regularity approximations *[AB–Laurens, '26]*

First step: reformulate Gérard's formula, in Fourier variables $\widehat{u}(t,k)$

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In Fourier space:

$$\begin{cases} \widehat{u}(t,k) = \langle (e^{it} e^{2itL_{u_0}} S^*)^k \Pi u_0 , 1 \rangle, & k \geq 0 \\ \widehat{u}(t,k) = \overline{\widehat{u}(t,-k)}, & k < 0 \end{cases}$$

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Work in the Hardy space $L^2_K = \Pi_K L^2_+ = \left\{ f \in L^2 : \widehat{f}(k) = 0,\ k < 0 \text{ or } k \geq K \right\}$,

with the Szegő projector

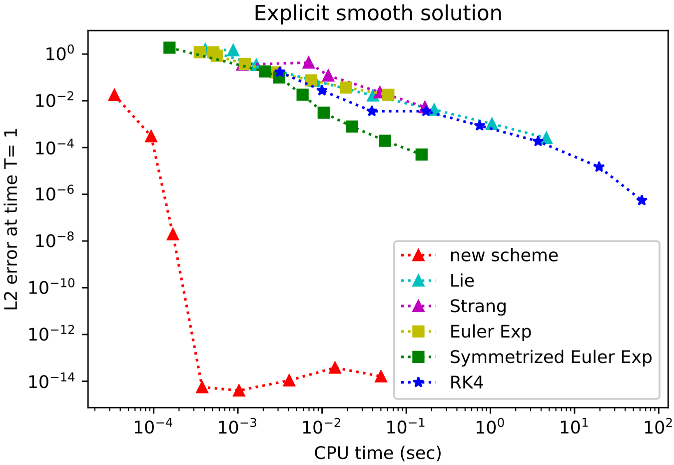
$$\Pi_K : L^2 \to L^2_K$$

and the Lax and shift operators

$$L_{u_0,K} f_K = Df_K - \Pi_K(u_0 f_K), \quad S^* f_K = \Pi(e^{-ix} f_K), \quad \forall f_K \in L^2_K$$

In Fourier space:

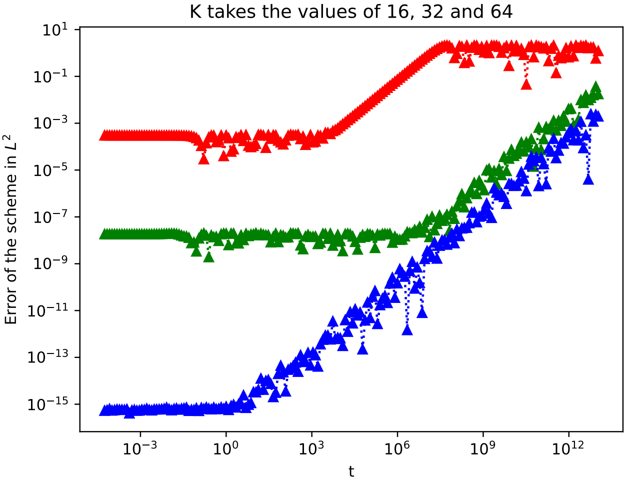
$$\begin{cases} \widehat{u_K}(t,k) = \langle (e^{it} e^{2itL_{u_0,K}} S^*)^k \Pi_K u_0, 1 \rangle, & 0 \leq k < K \\ \widehat{u_K}(t,k) = \overline{\widehat{u_K}(t,-k)}, & -K < k < 0 \end{cases}$$



Theorem (AB–Chen–Dolbeault, '25)

Let $s > 1$ and $u_0 \in H^s(\mathbb{T})$. There exists $C > 0$ depending on $||u_0||_{H^s}$ s.t.

$$||u(t) - u_K(t)||_{L^2} \leq C\,(1 + t)\,K^{-s+1}$$



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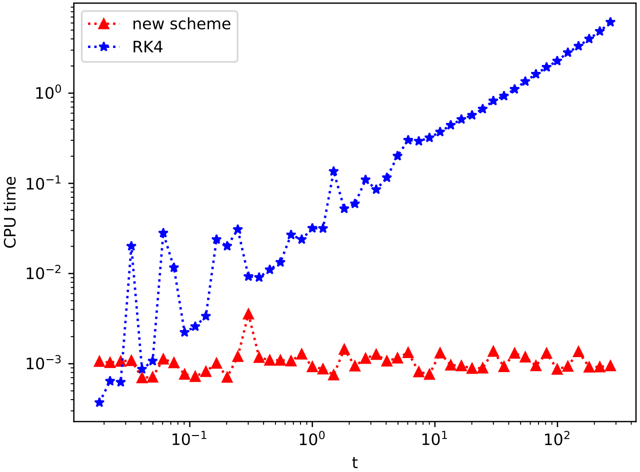

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Theorem (AB–Laurens, '26)

Given $u_0 \in L^2(\mathbb{T})$ *and* $T > 0$,

$$\sup_{t\in[-T,T]} ||u(t) - u_K(t)||_{L^2} \underset{K\to\infty}{\longrightarrow} 0$$

Tools: compactness and resolvent identities, inspired by: *[Killip–Laurens–Vişan, '24, '25]*

Better structure preserving properties:
Exactly preserve the *mass* $\int u$ and *momentum* $\int u^2$

$$||\Pi u(t)||_{L^2}^2 = ||\Pi u_0||_{L^2}^2 - \lim_{K\to\infty} ||(\mathrm{e}^{it}\mathrm{e}^{2itL_{u_0}} S^*)^K \Pi u_0||_{L^2}^2$$

Numerical artifact of the previous approximation u_K:

$$||(\mathrm{e}^{it}\mathrm{e}^{2itL_{u_0,K}} S^*)^K \Pi_K u_0||_{L^2} \neq 0$$

New proposed scheme : $\widehat{v}_K(t,k) = \langle v^k, 1\rangle$ with $v^0 = \Pi_K u_0$ and for $k \in \{1,\ldots,K-1\}$

$$v^k = \left(e^{it} e^{2iL_{u_0,K-k}} S^*\right) v^{k-1}, \qquad L_{u_0,j} = \Pi_j L_{u_0} \Pi_j$$

Preserves L^2-norm:

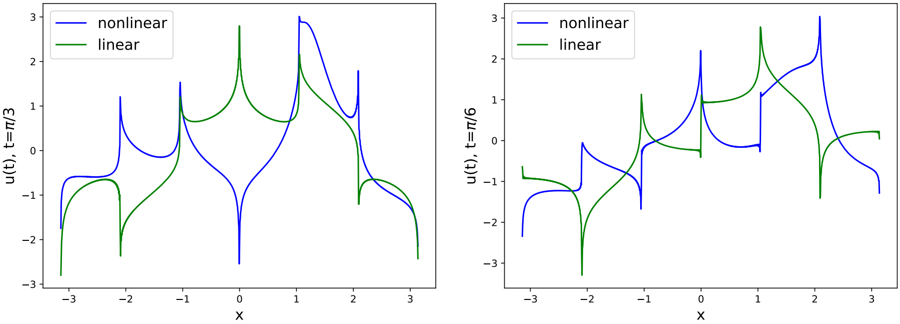
$$||\Pi v_K(t)||_{L^2} = ||\Pi_K u_0||_{L^2}, \quad t \geq 0, \quad K > 0$$

Theorem (AB–Laurens, '26)

Given $u_0 \in L^2(\mathbb{T})$ *and* $T > 0$,

$$\sup_{t\in[-T,T]} ||u(t) - u_K(t)||_{L^2} \underset{K\to\infty}{\longrightarrow} 0$$

Example: the Talbot Effet. BO evolution from $u_0(x) = \mathrm{sgn}(x)$.



Linear PDE: proven in *[Boulton–Olver–Pelloni–Smith, '21]*
Nonlinear PDE: First numerical evidence of the Talbot effect for (BO) supported by a rigorous convergence result

Bridging PDE integrability techniques and 1D Wave Kinetic Theory

- On the Wave Kinetic dynamics of the BO equation via an explicit formula *[AB – Hernández – Staffilani, '26+]*

Rescaled BO equation:

$$\partial_t u(t,x) = \partial_x \left(|D|u - \epsilon u^2\right)(t,x), \quad x \in \mathbb{T}_N = [-N\pi, N\pi],$$

with

$$u_0(x) = \sum_{k\in\mathbb{Z}} \eta_k \, \phi(k/N) \, \frac{e^{ikx/N}}{\sqrt{2\pi N}},$$

where η_k are i.i.d. complex Gaussians, $\eta_k = \overline{\eta_{-k}}$, $\mathbb{E}(\eta_k \overline{\eta_\ell}) = \delta_{k,\ell}$ and $\phi \in C_c^0(\mathbb{R})$

Goal: analyze the statistical dynamics of the solution in the kinetic regime, up to T_{kin}

Wave Kinetic Theory for nonlinear dispersive PDEs

Remarkable progress in the rigorous statistical description of wave interactions in dimensions $d \geq 2$, reaching up to the *kinetic timescale*

$$T_{\text{kin}} \sim \frac{1}{\epsilon^2},$$

threshold at which random nonlinear interactions accumulate.

Requires scaling law $\epsilon = N^{-\gamma}$, for $\gamma \in (0, 1]$.

Effective statistical description: the Wave Kinetic equation
[Peierls '29, Hasselmann '60, Zakharov '68]

Cubic NLS, in dimension $d \geq 2$:

- Up to $T_{\text{kin}}^{1/2}$ *[Buckmaster-Germain-Hani-Shatah, 19']*
- Up to $T_{\text{kin}}^{1-\delta}$, for some scaling laws *[Deng-Hani, Collot-Germain, '19]*
- Reached T_{kin}, *[Deng-Hani '21, '23]* and even arbitrarily long times *[Deng-Hani, '23]*

Other nonlinear PDEs in dimension $d \geq 2$:

- With stochastic forcing *[Staffilani-Tran, '21], [Grande-Hani, '26]*
- a Klein–Gordon system *[de Suzzoni-Stingo-Touati, '25]*

MMT (Majda–McLaughlin–Tabak) $d = 1$:

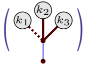
- Up to $T_{\text{kin}}^{5/8-}$ *[Vassilev, '25]*

Challenge in the 1D setting: counting problem is significantly more difficult

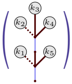
Key steps taken in the literature

Feynman diagram analysis: approximating $u(t)$ by *Duhamel expansions*

$$\widehat{u}(t,k) = \underbrace{e^{it\widehat{\mathcal{L}}(k)}\widehat{u}_0(k)}_{=\Psi(\;)} + \underbrace{e^{it\widehat{\mathcal{L}}(k)} \int_0^t e^{-is_1\widehat{\mathcal{L}}(k)}\widehat{p}_k(e^{is_1\widehat{\mathcal{L}}}\widehat{u}_0)ds_1}_{\Psi(\;)}$$

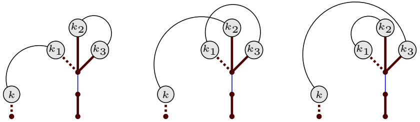


$$+ \underbrace{e^{it\widehat{\mathcal{L}}(k)} \int_0^t e^{-is_1\widehat{\mathcal{L}}(k)} \left[\widehat{p}'(e^{is_1\widehat{\mathcal{L}}}\widehat{u}_0) * e^{is_1\widehat{\mathcal{L}}} \int_0^{s_1} e^{-is_2\widehat{\mathcal{L}}}\widehat{p}(e^{is_2\widehat{\mathcal{L}}}\widehat{u}_0)ds_2\right]_k ds_1}_{\Psi(\;)} + \dots$$



Tree series expansion, for *randomized* u_0

$$\mathbb{E}(|\widehat{u}(t,k)|^2) - \mathbb{E}(|\widehat{u}_0(k)|^2) \simeq \sum_{T_1 \cdot T_2} \overline{\Psi}(T_1)\,\Psi(T_2)$$



Challenge in the 1D setting *[Vassilev, '25]* : divergence of some Feynman diagrams on time scales much shorter than the expected kinetic time

See also *[Deng–Ionescu–Pusateri, '25]* and *[Vassilev–Wu, '26]*

Exploiting integrability techniques

▷ Step 1. Let $w_0 = \epsilon u_0/\sqrt{N}$ and $t_N = t/N$.

$$\widehat{u}(t,k) = \frac{1}{\sqrt{2\pi N}} \left\langle \left(e^{it_N} e^{2it_N L_{w_0}} S^* \right)^k \Pi u_0, 1 \right\rangle$$

Instead of iterating Duhamel's formula for BO, we iterate along the diagonal flow $e^{it_N D}$:

$$\begin{aligned} e^{it_N L_{w_0}} = e^{it_N D} &+ \int_0^{t_N} e^{i(t_N - s)D}(-iT_{w_0})e^{isD}\, ds \\ &+ \int_0^{t_N}\int_0^{s} e^{i(t_N-s)D}(-iT_{w_0})e^{i(s-r)D}(-iT_{w_0})e^{irD}\, dr\, ds + \dots \end{aligned}$$

Obtain *Dyson series* for the operator appearing inside the explicit formula:

$$(e^{it_N L_{w_0}} S^*)^k - (e^{it_N D} S^*)^k = \sum_{j\geq 1} \mathcal{T}_j(k)$$

⇝ Recast the nonlinear dynamics into evolution of *linear operators* on L^2_+

Random Matrix Theory
Application to random Schrödinger equations: *[Black–Drogin–Hernández, '25]*

A first result on the wave kinetic theory for BO

Theorem (AB–Hernández–Staffilani)

Let $\epsilon = N^{-\gamma}$ with $\gamma \in [0,1]$. There exists $c > 0$ such that for all $c \leq t \leq \epsilon^{-2}(\log N)^{-4}$

$$\mathbb{E}||u(t) - u_{\text{lin}}(t)||_{L^2}^2 \lesssim t\epsilon^2(\log N)^4 \mathbb{E}||u_0||_{H^2}^2.$$

By applying the same elements of proof we obtain:

Given any $\ell \in \mathbb{R}$, and N large enough we have that for all $t \leq \epsilon^{-2}(\log N)^{-4}$

$$\left|\mathbb{E}|\widehat{u}(t,k)|^2 - \mathbb{E}|\widehat{u}(0,k)|^2\right| \lesssim t\epsilon^2 \log(N)^4$$

where $|k| \leq \ell N$.

Future work:

- Push these methods to *higher order*
- *Numerics* using explicit formula for exploring the statistical evolution over long times.
  Q: Do we have non-trivial dynamics, over which timescales, for which quantities ?
  [Flamarion–Pelinovsky, '23]: Numerically observe evolution of the *spectrum*, *skewness* and *kurtosis* for BO (but not in kinetic regime)

Thank you ! Merci !