An explicit formula for the Benjamin–Ono hierarchy with applications to traveling waves and zero–dispersion limits

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Joint work with Patrick Gérard

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Content of the talk

The Benjamin–Ono (BO) Equation

A nonlinear PDE that describes one-dimensional internal waves in deep water (introduced by Benjamin (1967) and Ono (1975))

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With periodic boundary conditions : Molinet (2008) in $L^2(\mathbb{T})$, Kappeler–Topalov–Gérard (2020) in $H^s(\mathbb{T}), s > -1/2$, with counterexample on $H^{-1/2}(\mathbb{T})$.

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Milestone : Explicit formula by Gérard (2022, mini-course in Hammamet), bypassing inverse spectral theory (but using the Lax pair structure).

Integrable Structure of BO and the Lax pair

The Hardy space is

$$L^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for } \xi < 0\}$$
$$= \{f \text{ holomorphic on } \mathbb{C}_+ : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < +\infty\}$$

The associated Riesz–Szegö projector is

$$\Pi : L^2(\mathbb{R}) \to L^2_+(\mathbb{R}), \quad \widehat{\Pi f}(\xi) = \mathbf{1}_{\xi>0}\hat{f}(\xi).$$

Given $b \in L^\infty(\mathbb{R})$, we define the Toeplitz operator on $L^2_+(\mathbb{R})$ of symbol b,

$$T_b : L^2_+ \to L^2_+, \quad f \mapsto T_b(f) := \Pi(bf).$$

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An important feature of BO is the fact that it admits a Lax pair,

$$
\partial_t L_{u(t)} = \left[B_{u(t)}, L_{u(t)}\right],
$$

where the Lax operator $L_u : H^1_+ = H^1 \cap L^2_+ \to L^2_+$ is given by $L_u(f) = \frac{1}{i}\frac{df}{dx} - T_u f$, and $B_u := i(T_{|D|u} - T_u^2)$. Notice that $B_u^* = -B_u$.

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The generating functional: $\mathcal{H}_\lambda(u) = \langle (L_u + \lambda)^{-1} \Pi u, \Pi u \rangle_{L^2}$. Its L^2-gradient is given by

$$\nabla \mathcal{H}_\lambda(u) = w_\lambda + \overline{w}_\lambda + |w_\lambda|^2, w_\lambda := (L_u + \lambda)^{-1} \Pi u$$

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$$\partial_t u = \partial_x (\nabla \mathcal{H}_\lambda(u)) = \partial_x \left(w_\lambda + \overline{w}_\lambda + |w_\lambda|^2 \right),$$

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Expanding $\mathcal{H}_\lambda(u)$ for spectral parameter λ,

$$\mathcal{H}_\lambda(u) = \sum_{n=0}^{\infty} \frac{(-1)^n E_n(u)}{\lambda^{n+1}} = \frac{E_0}{\lambda} - \frac{E_1}{\lambda^2} + \frac{E_2}{\lambda^3} - \dots, \text{ with } E_n(u) := \langle L_u^n \Pi u, \Pi u\rangle_{L^2}$$

$E_1(u) = \langle L_u \Pi u, \Pi u\rangle$: BO equation $\partial_t u = \partial_x(|D_x|u - u^2)$.
$E_2(u) = \langle L_u^2 \Pi u, \Pi u\rangle$: the third-order BO flow

$$\partial_t u + \partial_x^3 u + \frac{3}{2}\partial_x(u|D|u) + \frac{3}{2}|D|(u\partial_x u) - \partial_x(u^3) = 0.$$

The $(n+1)$-th Hamiltonian flow: $\partial_t u = \partial_x\big(\nabla E_n(u)\big)$.

The Lax pair structure extends to the entire BO hierarchy: one has (R. Sun, 2021)

$$\left[B_u^{[\lambda]}, L_u\right] = -\mathrm{i}\, T_{D(w_\lambda+\overline{w}_\lambda+|w_\lambda|^2)}, \quad \text{with } B_u^{[\lambda]} := \mathrm{i}\left(T_{w_\lambda} T_{\overline{w}_\lambda} + T_{\overline{w}_\lambda} + T_{w_\lambda}\right). \qquad (1)$$

Notice that $B_u^{[\lambda]}$ can be expressed by

$$B_u^{[\lambda]} = \frac{1}{\lambda} B_u^{(0)} - \frac{1}{\lambda^2} B_u^{(1)} + \frac{1}{\lambda^3} B_u^{(2)} - \frac{1}{\lambda^4} B_u^{(3)} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^{n+1}} B_u^{(n)}.$$

Each coefficient $B_u^{(n)}$ is the generator of the $(n+1)$-st flow in the BO hierarchy. On the other hand, we have $w_\lambda + \overline{w_\lambda} + |w_\lambda|^2 = \nabla \mathcal{H}_\lambda(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^{n+1}} \nabla E_n(u)$. Therefore,

$$-i\, T_{D(|w_\lambda|^2 + w_\lambda + \overline{w_\lambda})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^{n+1}} \left(-i\, T_{D(\nabla E_n(u))}\right).$$

Combining with the identity (1), and matching the coefficient of $\lambda^{-(n+1)}$,

$$[B_u^{(n)}, L_u] = -i\, T_{D(\nabla E_n(u))}.$$

Hence, the Lax pair for the $(n+1)$-th order flow

$$\partial_t L_{u(t)} = -T_{\partial_t u} = -i\, T_{D(\nabla E_n(u))} = [B_{u(t)}^{(n)}, L_{u(t)}].$$

Lax pair of the Benjamin–Ono Hierarchy

If $u \in C(\mathbb{R}, H^2_r(\mathbb{R}))$ solves the $(n+1)$-th flow of the BO hierarchy

$$\partial_t u = \partial_x\big(\nabla E_n(u)\big),$$

then

$$\partial_t L_{u(t)} = [B^{(n)}_{u(t)}, L_{u(t)}].$$

Corollary

Define the family of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ by

$$U'(t) = B^{(n)}_{u(t)} U(t), \qquad U(0) = Id.$$

Then,

$$L_{u(t)} = U(t) L_{u(0)} U(t)^*.$$

X^* is renormalized version of position operator $x \mapsto xf$ on L^2_+.

- The operator $X^* : \mathrm{Dom}(X^*) \subset L^2_+ \to L^2_+$ is defined as

$$\widehat{X^* f}(\xi) = i \frac{d}{d\xi}[\widehat{f}(\xi)]\mathbf{1}_{\xi>0},$$

$$\mathrm{Dom}(X^*) = \{f \in L^2_+(\mathbb{R}) : \hat{f}_{|]0,+\infty[} \in H^1(0,+\infty)\}.$$

- X and X^* are generators of canonical semigroups on L^2_+, i.e.

$$\{S(\eta)\}_{\eta\geq 0} = e^{i\eta X}(\text{right shifts}), \qquad \{S^*(\eta)\}_{\eta\geq 0} = e^{-i\eta X^*}(\text{left shifts})$$

For $f \in \mathrm{Dom}(X^*)$, we define

$$I_+(f) := \widehat{f}(0^+) = \lim_{\varepsilon\to 0^+} \langle f \mid \chi_\varepsilon \rangle_{L^2}, \qquad \chi_\varepsilon(x) := \frac{1}{1 - i\varepsilon x}.$$

The explicit formulas for the BO hierarchy

Theorem (J. He, P. Gérard, 2026)

Let $u \in C(\mathbb{R}, H^s_r(\mathbb{R}))$, $s > n + \frac{1}{2}$, be a solution to the $(n+1)$-th order flow of the BO hierarchy on $\mathbb{R}$ with $u(0) = u_0$. Then $u(t,x) = \Pi u(t,x) + \overline{\Pi u(t,x)}$, where

$$\Pi u(t,z) = \frac{1}{2i\pi} I_+ \left[\left(X^* - (n+1) t L_{u_0}^n - z\mathrm{Id} \right)^{-1} \Pi u_0 \right], \quad \mathrm{Im}(z) > 0.$$

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- In the proof, no explicit representation for operator B_u is needed.
- Based on the generating function, Killip, Laurens and Visan (2024) provided an analogous explicit formula for the full BO hierarchy. Our formula provides a direct representation for each individual flow in terms of powers of the Lax operator L_{u_0}.

Sketch of the proof

For every $z \in \mathbb{C}_+$, by the inverse Fourier transform and Plancherel Theorem, we have

$$\begin{aligned}
\Pi u(t,z) &= \frac{1}{2i\pi}\int_0^\infty e^{iz\xi}\,\widehat{\Pi u}(t,\xi)\,d\xi \\
&= \frac{1}{2i\pi}\int_0^\infty e^{iz\xi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} e^{-i\xi x}\frac{\Pi u(t,x)}{1+i\varepsilon x}\,dx \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2i\pi}\int_0^\infty e^{iz\xi}\left\langle e^{-i\xi X^*}\Pi u \mid \chi_\varepsilon\right\rangle d\xi \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2i\pi}\left\langle (X^* - z\mathrm{Id})^{-1}\,\Pi u \mid \chi_\varepsilon\right\rangle, \text{ by } \widehat{X^* f}(\xi) = i\partial_\xi[\hat{f}(\xi)]\mathbf{1}_{\xi>0}
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\end{aligned}$$

Since $U(t)$ is a unitary operator and it preserves the inner product, we infer that

$$\begin{aligned}
\Pi(u)(t,z) &= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\left\langle U^*(t)\left(X^* - z\mathrm{Id}\right)^{-1}\Pi(u_0) \mid U^*(t)\chi_\varepsilon\right\rangle \\
&= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\left\langle \left(U^*(t)X^*U(t) - z\mathrm{Id}\right)^{-1}U^*(t)\Pi(u_0) \mid U^*(t)\chi_\varepsilon\right\rangle.
\end{aligned}$$

Sketch of the proof

Compute the time derivative of the conjugated operator $U^*(t)X^*U(t)$

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\begin{aligned}
\frac{d}{dt}U^*(t)X^*U(t) &= (U^*(t))'X^*U(t) + U^*(t)X^*U'(t) \\
&= \left(-U^*(t)B^{(n)}_{u(t)}\right)X^*U(t) + U^*(t)X^*\left(B^{(n)}_{u(t)}U(t)\right) \\
&= U^*(t)[X^*, B^{(n)}_u]U(t)
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Proposition

For every $u \in H_r^s(\mathbb{R})$, $s > \frac{5}{2}$,

$$\left[X^*, B_u^{(2)}\right] = -3L_u^2 - i[X^*, L_u^3].$$

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Main steps:

- $[X^*, B_u^{(2)}]f = \frac{1}{2\pi}\Big(\langle f|L_u\Pi u\rangle \Pi u + \langle f|\Pi u\rangle L_u\Pi u - I_+(f)L_u^2\Pi u\Big)$.
- $[X^*, L_u^3]f = 3i\, L_u^3 f + \frac{i}{2\pi}\Big(\langle f|L_u\Pi u\rangle \Pi u + \langle f|\Pi u\rangle L_u\Pi u - I_+(f)L_u^2\Pi u\Big)$.

Proof of $[X^*, B_u^{(2)}]f = \frac{1}{2\pi}\Big(\langle f|L_u\Pi u\rangle \Pi u + \langle f|\Pi u\rangle L_u \Pi u - I_+(f) L_u^2 \Pi u\Big)$.

Split the commutator into three parts

$$\begin{aligned}[X^*, B_u^{[\lambda]}]f &= i\big[X^*, T_{w_\lambda}\big]f + i\big[X^*, T_{\overline{w_\lambda}}\big]f + i\big[X^*, T_{w_\lambda}\overline{T_{w_\lambda}}\big]f \\ &= -\frac{1}{2\pi}\Big(I_+(f)\, w_\lambda + \langle f|w_\lambda\rangle w_\lambda\Big), \text{ by } [X^*, T_u]f = \frac{i}{2\pi} I_+(f)\,\Pi u\end{aligned}$$

Proof of $[X^*, B_u^{(2)}]f = \frac{1}{2\pi}\Big(\langle f|L_u\Pi u\rangle \Pi u + \langle f|\Pi u\rangle L_u\Pi u - I_+(f)L_u^2\Pi u\Big)$.

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Expand w_λ in powers of λ^{-1},

$$w_\lambda = (L_u + \lambda)^{-1}\Pi u = \frac{1}{\lambda}\,\Pi u - \frac{1}{\lambda^2} L_u\,\Pi u + \frac{1}{\lambda^3} L_u^2\,\Pi u - \cdots$$

Recall

$$B_u^{[\lambda]} = \frac{1}{\lambda}B_u^{(0)} - \frac{1}{\lambda^2}B_u^{(1)} + \frac{1}{\lambda^3}B_u^{(2)} - \frac{1}{\lambda^4}B_u^{(3)} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda^{n+1}}B_u^{(n)}.$$

Matching the powers of λ on both sides of the commutator identity, we can extract the commutator $[X^*, B_u^{(2)}]$.

Sketch of the proof

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By using $L_{u(t)} = U(t)L_{u(0)}U(t)^*$,

$$U^*(t)X^*U(t) = -3tL_{u_0}^2 + e^{itL_{u_0}^3}X^*e^{-itL_{u_0}^3}.$$

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Recall the previous formula

$$\Pi(u)(t,z) = \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\left\langle\left(U^*(t)X^*U(t) - z\mathrm{Id}\right)^{-1}U^*(t)\Pi(u_0)\,\middle|\,U^*(t)\chi_\varepsilon\right\rangle.$$

Sketch of the proof

Proposition

For the operator $U(t)$ of the third equation of the Benjamin–Ono equation, we have

$$U^*(t)\chi_\varepsilon = e^{itL_{u_0}^3}\chi_\varepsilon + o(1), \qquad U^*(t)\Pi u = e^{itL_{u_0}^3}\Pi u_0.$$

Gathering all together,

$$\begin{aligned}\Pi(u)(t,z) &= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\left\langle\left(-3tL_{u_0}^2 + e^{itL_{u_0}^3}X^* e^{-itL_{u_0}^3} - z\,\mathrm{Id}\right)^{-1} e^{itL_{u_0}^3}\Pi(u_0)\,\middle|\, e^{itL_{u_0}^3}\chi_\varepsilon\right\rangle \\ &= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\left\langle\left(X^* - 3tL_{u_0}^2 - z\,\mathrm{Id}\right)^{-1}\Pi(u_0)\,\middle|\,\chi_\varepsilon\right\rangle.\end{aligned}$$

By the definition of I_+, we get

$$\forall z\in\mathbb{C}_+,\quad \Pi u(t,z) = \frac{1}{2i\pi} I_+\left[\left(X^* - 3tL_{u_0}^2 - z\mathrm{Id}\right)^{-1}\Pi u_0\right].$$

The zero dispersion limit

Application 1: The zero dispersion limit

Consider the BO with small dispersion $\varepsilon > 0$,

$$\partial_t u^\varepsilon + \partial_x((u^\varepsilon)^2) = \varepsilon \partial_x |D_x| u^\varepsilon, \quad u^\varepsilon(0,x) = u_0(x),$$

Question: what is the limit of $u^\varepsilon(t)$ as $\varepsilon \to 0$.

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Question: what is the limit of $u^\varepsilon(t)$ as $\varepsilon \to 0$.
If we neglect $\varepsilon \partial_x |D_x| u^\varepsilon$, then the initial datum evolve in time approximately according to Burgers' equation

$$\partial_t u + \partial_x(u^2) = 0.$$

But the solution of Burgers' equation forms shocks at some finite time.
Main question: What is the limit of $u^\varepsilon(t)$ after the time of shock formation?

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Main question: What is the limit of $u^\varepsilon(t)$ after the time of shock formation?
Starting in the analogous question for the KdV equation with Lax–Levermore (1983), Deift–Venakides–Zhou (1997), Claeys–Grava (2009), (using inverse scattering theory).
Results for Benjamin–Ono by Miller–Xu (2011), Miller–Wetzel (2016).
Results for BO hierarchy by Miller–Xu (2012).
More recently,

- On the circle, by Gassot (2021, 2022) for bell–shaped data, Mæhlen (2025).
- On the line, by Gérard (2024), Chen (2024), Blackstone, Gassot, Gérard, Miller(2025).

The zero dispersion limit

We use the explicit formula to revisit the zero dispersion limit for BO Hierarchy.
The small-dispersion $(n+1)$-st BO flow reads

$$\partial_t u^\varepsilon = \partial_x(\nabla E_n^\varepsilon(u^\varepsilon)), \quad E_n^\varepsilon(u^\varepsilon) := \langle (\varepsilon D - T_{u^\varepsilon})^n \Pi u^\varepsilon, \Pi u^\varepsilon \rangle_{L^2}.$$

Example: the third equation of the BO hierarchy with small dispersion $\varepsilon > 0$,

$$\partial_t u^\varepsilon + \varepsilon^2 \partial_x^3 u^\varepsilon + \frac{3}{2}\varepsilon \partial_x (u^\varepsilon |D| u^\varepsilon) + \frac{3}{2}\varepsilon |D|(u^\varepsilon \partial_x u^\varepsilon) - \partial_x (u^\varepsilon)^3 = 0, \quad u^\varepsilon(0) = u_0.$$

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After rescaling, using $L_u = D - T_u$,

$$\forall z \in \mathcal{C}_+, \quad \Pi u^\varepsilon(t,z) = \frac{1}{2i\pi} I_+ \left[\left(X^* - 3t(\varepsilon D - T_{u_0})^2 - z\mathrm{Id} \right)^{-1} \Pi u_0 \right]$$

The conservation law ensures that $\{u^\varepsilon\}_{\varepsilon > 0}$ is uniformly bounded in $L^2(\mathbb{R})$. By weak compactness, there exists $\varepsilon_j \to 0$ such that $u^{\varepsilon_j} \rightharpoonup u(t)$ in $L^2(\mathbb{R})$ with

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Question: can we give a more precise representation ?

Theorem (J. He, P. Gérard, 2026)

Let $n \in \mathbb{N}$ and $u_0 \in H^s_r(\mathbb{R})$ with $s > n + \frac{1}{2}$. For every $t \in \mathbb{R}$, the set $K_t(u_0)$ of critical values of the function $y \in \mathbb{R} \mapsto y - (-1)^n(n+1)tu_0^n(y)$ is a compact subset of measure 0. For every connected component Ω of $K_t(u_0)^c$, there exists a nonnegative integer ℓ such that, for every $x \in \Omega$, the equation $y - (-1)^n(n+1)tu_0^n(y) = x$ has $2\ell + 1$ simple real solutions $y_0(t,x) < y_1(t,x) < \cdots < y_{2\ell}(t,x)$, and the zero dispersion limit is given by

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Remark 1: The formula was proved by Miller–Xu (2012) for special initial data. We use a novel method developed by Gérard (2024), that differs from inverse scattering transforms. By transforming the formula into a linear system, this approach allows us to establish the signed sum without imposing any essential constraints on the initial data.

Remark 2: The alternating-sum formula was introduced by Brenier (1984) in his transport-collapse approximation of entropy solutions of the inviscid Burgers' equation. Our result extends the same structure from the classical BO equation to the entire BO hierarchy.

The case of rational data

$u_0(y)$ is a rational function with real coefficients of the form

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For the equation,

$$y - 3t\, u_0^2(y) = x \Rightarrow (y-x)(y^2+1)^2 - 3t = 0$$

there are three cases to consider for the number of real roots:

1. One real root y_0 and two pairs of complex conjugate roots $y_1 = \overline{y_2}$, $y_3 = \overline{y_4}$.
2. Three real roots y_0, y_1, y_2 and one pair of complex conjugate roots $y_3 = \overline{y_4}$.
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Note that, if x is slightly shifted into the upper half plane with $\mathrm{Im}(x) > 0$ small, then

$$\mathrm{Im}(y_0), \mathrm{Im}(y_2), \mathrm{Im}(y_4) > 0, \qquad \mathrm{Im}(y_1), \mathrm{Im}(y_3) < 0,$$

so the zeros of $y \mapsto y - 3t\, u_0(y)^2 - x$ in the upper half plane are y_0, y_2, y_4.

Notice that, for any suitable function f,

$$X^* f(y) = yf(y) + \frac{1}{2i\pi} I_+(f), \quad T_{u_0}^2 f = u_0(y)^2 f(y) + \frac{\mu\, u_0(y)}{y-i} + \frac{\nu}{y-i},$$

For $x \in \mathcal{C}_+$, with $\mathrm{Im} x > 0$ small, we calculate

$$f_{t,z} := (X^* - 3t\, T_{u_0}^2 - z\,\mathrm{Id})^{-1} \Pi u_0 \Rightarrow \Pi u(t,x) := \frac{1}{2i\pi} I_+[f_{t,x}] = \lambda(t,x)$$

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Therefore, the resolvent equation reads,

$$f_{t,z}(y)(y - z - 3tu_0^2(y)) = u_0 - \lambda(t,z) + 3t\mu \frac{u_0(y)}{y-i} + 3t\nu \frac{1}{y-i}.$$

where $\lambda(t,x), \mu(t,x), \nu(t,x)$ are chosen so that the solution is holomorphic in the upper half plane, namely so that the right hand side cancels at the zeroes y_0, y_2, y_4 in the upper half plane.

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$$\begin{pmatrix} 1 & \frac{u_0(y_0)}{y_0 - i} & \frac{1}{y_0 - i} \\ 1 & \frac{u_0(y_2)}{y_2 - i} & \frac{1}{y_2 - i} \\ 1 & \frac{u_0(y_4)}{y_4 - i} & \frac{1}{y_4 - i} \end{pmatrix} \begin{pmatrix} \lambda(t,x) \\ -3t\mu(t,x) \\ -3t\nu(t,x) \end{pmatrix} = \begin{pmatrix} u_0(y_0) \\ u_0(y_2) \\ u_0(y_4) \end{pmatrix}$$

Using Cramer's rule

$$\lambda(t,x) = u_0(y_0) + u_0(y_2) + u_0(y_4).$$

The case of rational data

Lemma

Set $p(y) := (y-x)(y^2+1)^2 - 1$, *and let* $y_0, \ldots, y_4$ *be the five roots of* $p(y)$. *Then,*

$$\sum_{j=0}^{4} u_0(y_j) = \sum_{j=0}^{4} \frac{1}{y_j^2+1} = 0.$$

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By $u(t,x) = \Pi u(t,x) + \overline{\Pi u(t,x)}$, we have

case 1 : One real root y_0

$$u(t,x) = 2u_0(y_0) + u_0(y_1) + u_0(y_2) + u_0(y_3) + u_0(y_4) = u_0(y_0).$$

case 2 : Three real root y_0, y_1, y_2

$$u(t,x) = 2(u_0(y_0) + u_0(y_2)) + u_0(y_3) + u_0(y_4)) = u_0(y_0) - u_0(y_1) + u_0(y_2).$$

case 3 : Five real root y_0, y_1, y_2, y_3, y_4

$$u(t,x) = 2(u_0(y_0) + u_0(y_2) + u_0(y_4)) = \sum_{j=0}^{4} (-1)^j u_0(y_j).$$

The case of rational data

For the $(n+1)$-th order flow of the BO hierarchy with general data

$$u_0(y) = \sum_{j=1}^{N} \frac{c_j}{y - p_j} + \frac{\bar{c}_j}{y - \bar{p}_j}, \qquad \mathrm{Im}\ p_j > 0.$$

we obtain a $(nN+1) \times (nN+1)$ linear system, the zero dispersion limit is given by

$$u(t,x) = \sum_{\substack{0 \le k \le 2nN \\ y_k \in \mathbb{R}}} (-1)^k\, u_0(y_k),$$

Classification of traveling waves

Application 2: Classification of traveling waves

A traveling wave is a solution of the form

$$u(t,x) = u_0(x - ct).$$

For the classical BO equation, every nontrivial traveling wave is a one-soliton of the form (by Amick and Toland, 1991)

$$u(t,x) = R_p(x - c_p t), \quad R_p(y) := \frac{2\mathrm{Im}p}{|y+p|^2}, \quad c_p := \frac{1}{\mathrm{Im}p}, \quad p \in \mathcal{C}_+.$$

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Let $n \in \mathbb{N}$ and $u_0 \in H^s_r(\mathbb{R})$ with $s > n + \frac{1}{2}$. If u is a traveling wave for the $(n+1)$st BO flow, then either $u_0 \equiv 0$, or there exists a unique $p \in \mathcal{C}_+$ such that

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Remark 1: In the periodic setting, Gassot (2021) proved that the third-order BO equation admits more traveling waves in $L^2_{r,0}(\mathbb{T})$ than the classical BO equation. By contrast, on the line, higher flows do *not* create new traveling-wave profiles and every nontrivial traveling wave remains a one-soliton.

Remark 2: For the KdV equation, inserting the ansatz $u(t,x) = Q_{KdV}(x - ct)$ reduces the equation to an ODE, whose solutions are exactly the well-known solitary waves

$$Q_{KdV}(x) = \frac{c}{2}\,\mathrm{sech}^2(\frac{\sqrt{c}}{2}(x - x_0)).$$

For the BO equation, the situation is delicate because the dispersion is nonlocal. Fortunately, by using the explicit formula for the BO hierarchy and the soliton limit, we can bypass this difficulty and the argument is both different from and simpler than the previously approaches of Amick and Toland.

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	KdV	**BO**
Traveling-wave	Phase-plane analysis, ODE techniques	nonlocal dispersion, Explicit formula, Lax pair, soliton limit
Zero-dispersion	Lax–Levermore theory, variational problem inverse scattering, bell-shaped initial data	Explicit formula, Lax pair , alternating sum, natural regularity for initial data

The existence of similar approach for zero dispersion limit problem of KdV is an interesting open problem.

Thanks for your attention!