Fredholm determinant formula for the KdV soliton gas

Manuela Girotti

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1 Pure-N-soliton solution setup

1.1 N-soliton case

Consider the case with N KdV solitons (reflectionless case; see [GT09]). For simplicity we set $t = 0$, but all the calculations can be carried out in the case $t > 0$. Find a 2-dimensional row vector $\boldsymbol{m}$ such that

1. $\boldsymbol{m}(\lambda)$ is meromorphic in $\mathbb{C}$, with simple poles at $\{\mathrm{i}\lambda_j\}_{j=1}^N$ in $\mathrm{i}\mathbb{R}_+$ ($\lambda_j > 0$), and at the corresponding conjugate points $\{-\mathrm{i}\lambda_j\}_{j=1}^N$ in $\mathrm{i}\mathbb{R}_-$;
2. $\boldsymbol{m}$ satisfies the residue conditions
$$\operatorname*{res}_{z=\mathrm{i}\lambda_j} \boldsymbol{m}(z) = \lim_{z\to\mathrm{i}\lambda_j} \boldsymbol{m}(z)\begin{bmatrix} 0 & 0 \\ \mathrm{i}c_j e^{2\mathrm{i}zx} & 0\end{bmatrix}, \quad \operatorname*{res}_{z=-\mathrm{i}\lambda_j} \boldsymbol{m}(z) = \lim_{z\to-\mathrm{i}\lambda_j} \boldsymbol{m}(z)\begin{bmatrix} 0 & -\mathrm{i}c_j e^{-2\mathrm{i}zx} \\ 0 & 0\end{bmatrix}, \tag{1}$$
where the norming constants $c_j \in \mathbb{R}_+$;
3. $\boldsymbol{m}(z) = \begin{bmatrix} 1 & 1\end{bmatrix} + \mathcal{O}\left(\frac{1}{z}\right)$ as $\lambda \to \infty$,
4. $\boldsymbol{m}$ satisfies the symmetry
$$\boldsymbol{m}(-z) = \boldsymbol{m}(z)\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}.$$

The ansatz for the solution is the following

$$\boldsymbol{m}(z) = \left[1 + \sum_{j=1}^N \frac{\mathrm{i}\alpha_j}{z-\mathrm{i}\lambda_j},\ 1 - \sum_{j=1}^N \frac{\mathrm{i}\alpha_j}{z+\mathrm{i}\lambda_j}\right], \tag{2}$$

with $\alpha_j = \alpha_j(x,t)$.

The N-soliton potential $u(x)$ is determined as

$$u(x) = 2\partial_x\left(\lim_{z\to\infty}\frac{z}{\mathrm{i}}\left(\boldsymbol{m}_1(z) - 1\right)\right) = 2\ \partial_x\left(\sum_{k=1}^N \alpha_k\right). \tag{3}$$

Plugging the ansatz into the residue conditions gives the equations

$$\alpha_k = \left(1 - \sum_{j=1}^{N} \frac{\alpha_j}{\lambda_k + \lambda_j}\right) c_k e^{-2\lambda_k x} \quad \Leftrightarrow \quad \frac{\alpha_k e^{\lambda_k x}}{\sqrt{c_k}} = \left(1 - \sum_{j=1}^{N} \frac{\sqrt{c_j} e^{-\lambda_j x}}{\lambda_k + \lambda_j} \frac{\alpha_j e^{\lambda_j x}}{\sqrt{c_j}}\right) \sqrt{c_k} e^{-\lambda_k x} \tag{4}$$

$\forall\ k = 1, \dots N$, i.e.

$$(\boldsymbol{I}_N + \boldsymbol{K})\, \tilde{\boldsymbol{\alpha}} = \tilde{\boldsymbol{c}}, \qquad \tilde{\boldsymbol{\alpha}} = (\boldsymbol{I}_N + \boldsymbol{K})^{-1} \tilde{\boldsymbol{c}} \tag{5}$$

with $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{c}} \in \mathbb{R}^N$ and

$$(\tilde{\boldsymbol{\alpha}})_k = \frac{\alpha_k}{\sqrt{c_k}} e^{\lambda_k x}\ , \quad (\tilde{\boldsymbol{c}})_k = \sqrt{c_k} e^{-\lambda_k x}\ , \quad (\boldsymbol{K})_{k\ell} = \frac{\sqrt{c_k}\sqrt{c_j} e^{-x(\lambda_k + \lambda_\ell)}}{\lambda_k + \lambda_\ell}\ . \tag{6}$$

Proposition 1. *The matrix $\boldsymbol{I}_N + \boldsymbol{K}$ is invertible.*

Proof. We follow the same argument as in [**?**, Proposition B1].

The matrix $\boldsymbol{K}$ is symmetric and (even better) it is positive definite, as it can be seen as an inner-product type of matrix:

$$\boldsymbol{K}_{kj} = \int_x^\infty \sqrt{c_k}\sqrt{c_k} e^{-s(\lambda_k + \lambda_j)} \mathrm{d}s = \left\langle \sqrt{c_k} e^{-s\lambda_k}, \sqrt{c_j} e^{-s\lambda_j} \right\rangle \tag{7}$$

and the functions $f_k(s) = \sqrt{c_k} e^{-s\lambda_k}$ are linearly independent in $L^2(x, +\infty)$ ($\forall\ x$), since $\lambda_k \neq \lambda_j$. $\square$

Notice now that

$$\alpha_k = \sum_{\ell=1}^{N} (\boldsymbol{I}_N + \boldsymbol{K})^{-1}_{k\ell} \sqrt{c_\ell} e^{-x(\lambda_j + \lambda_\ell)}, \tag{8}$$

therefore

$$\sum_{k=1}^{N} \alpha_k = -\operatorname{Tr}\left((\boldsymbol{I}_N + \boldsymbol{K})^{-1} \frac{\partial}{\partial x} \boldsymbol{K} \right) = -\frac{\partial}{\partial x} \ln\det\left(\boldsymbol{I}_N + \boldsymbol{K}\right) \tag{9}$$

where we used the fact that $\frac{\partial}{\partial x}\boldsymbol{K}_{k\ell} = -\sqrt{c_k} e^{-x(\lambda_k + \lambda_\ell)}$ and the formula $\frac{\partial}{\partial x} \ln\det \boldsymbol{A} = \operatorname{Tr}\left(\boldsymbol{A}^{-1} \frac{\partial}{\partial x} \boldsymbol{A}\right)$ for a generic matrix-valued function $\boldsymbol{A}(x)$.

In conclusion, our KdV N-soliton solution looks like

$$u(x) = -2\frac{\partial^2}{\partial x^2} \ln\det\left(\boldsymbol{I}_N + \boldsymbol{K}\right) \tag{10}$$

1.2 Fredholm determinant formula for the N-soliton solution of KdV

Theorem 1 (Matrix trick). *The finite $N \times N$ determinant (10) can be recast into the following Fredholm determinant:*

$$\det\left(\boldsymbol{I}_N + \boldsymbol{K}\right) = \det\left(\mathrm{Id}_{L^2(x,+\infty)} + \mathcal{K}_x\right) \tag{11}$$

where $\mathcal{K}_x : L^2(x,+\infty) \to L^2(x,+\infty)$ is an integrable, finite rank operator

$$\mathcal{K}_x[f](y) = \int_x^\infty K(s,y) f(s) \mathrm{d}s \tag{12}$$

with Hankel kernel

$$K(u,v) = F(u+v), \qquad F(s) := \sum_{j=1}^N c_j e^{-\lambda_j s}. \tag{13}$$

where $\{-\lambda_j^2\}_{j=1}^N$ are the the eigenvalues of the Schrödinger operator and $\{\gamma_j^2\}$ are the norming constants.

Proof. The matrix $\boldsymbol{K}$ can be written as the composition of the following two operators:

$$\boldsymbol{K} = \mathcal{A} \circ \mathcal{B} \tag{14}$$

with $\mathcal{A} : L^2(x,+\infty) \to \mathbb{C}^N$ and $\mathcal{B} : \mathbb{C}^N \to L^2(x,+\infty)$ are

$$(\mathcal{A}[f])_j = \int_x^\infty \gamma_j e^{-\lambda_j s} f(s) \mathrm{d}s \tag{15}$$

$$\mathcal{B}[\boldsymbol{v}] = \langle \mathbf{exp}(s), \boldsymbol{v} \rangle, \qquad (\mathbf{exp}(s))_k = \gamma_k e^{-\lambda_k s} \tag{16}$$

for all $j,k = 1,\dots,N$.

Therefore,

$$\det\left[\boldsymbol{I}_N + \mathcal{A} \circ \mathcal{B}\right] = \det\left[\mathrm{Id}_{L^2(x,+\infty)} + \mathcal{B} \circ \mathcal{A}\right] \tag{17}$$

and it is easy to see that $\mathcal{B} \circ \mathcal{A} = \mathcal{K}_x$. □

Remark 1. *It is easy to check that $\mathcal{A}$ and $\mathcal{B}$ are Hilbert-Schmidt, therefore $\mathcal{K}_x$ is trace class.*

2 Taking the limit $N \to +\infty$ in the Fredholm Determinant

We are now interested in the limit as $N \to +\infty$. We replace

$$c_j \mapsto \frac{c_j}{N},$$

as norming constant.

We also introduce some additional assumptions as usual:

1. The poles $\left\{\mathrm{i}\lambda_j^{(N)}\right\}_{j=1}^N$ are sampled from a density function $\varrho(\lambda)$ so that $\int_{\mathrm{i}\eta_1}^{\mathrm{i}\lambda_j} \varrho(\eta) d\eta = j/N$, for $j = 1,\dots,N$.

2. The coefficients $\{c_j\}_{j=1}^N$ are strictly positive and are assumed to be discretizations of a given function:

$$c_j = \frac{(\eta_2 - \eta_1) r_1(\mathrm{i}\lambda_j)}{2\pi} \qquad j = 1,\dots,N\ . \tag{18}$$

   where $r_1(z)$;

- is an analytic function for z near the intervals $(\mathrm{i}\eta_1, \mathrm{i}\eta_2)$ and $(-\mathrm{i}\eta_2, -\mathrm{i}\eta_1)$,
- satisfies the symmetry $r_1(\bar{z}) = r_1(z)$,
- is assumed to be a real valued positive and non-vanishing function of z for $z \in [\mathrm{i}\eta_1, \mathrm{i}\eta_2]$.

This way we have the following result.

Proposition 2. *For any open set K_+ containing the interval $[i\eta_1, i\eta_2]$, the following limit holds uniformly for all $z \in \mathbb{C}\backslash K_+$ (in particular, it holds for $x \in \mathbb{R}$):*

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{j=1}^{N} c_j e^{\mathrm{i}(\mathrm{i}\lambda_j)z} = \int_{\mathrm{i}\eta_1}^{\mathrm{i}\eta_2} r_1(\zeta) e^{\mathrm{i}\zeta z} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \tag{19}$$

Proof. Using (18), we have

$$\frac{1}{N}\sum_{j=1}^{N} c_j e^{-\lambda_j z} = \frac{1}{N}\sum_{j=1}^{N} e^{-\lambda_j z} \frac{(\eta_2-\eta_1) r_1(\mathrm{i}\lambda_j)}{2\pi} = \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^{N} r_1(\mathrm{i}\lambda_j) e^{\mathrm{i}(\mathrm{i}\lambda_j)x} \underbrace{\frac{\mathrm{i}(\eta_2-\eta_1)}{N}}_{\Delta\zeta} \tag{20}$$

The convergence follows from the convergence of the Riemann sum to the Riemann–Stieltjes integral. □

Therefore, in the limit we have an integral operator $\mathcal{K}_x^{(\infty)}$ with Hankel kernel

$$K_x^{(\infty)}(u,v) := F^{(\infty)}(u+v) \ , \qquad F^{(\infty)}(s) := \int_{[\mathrm{i}\eta_1,\mathrm{i}\eta_2]} r_1(\zeta) e^{\mathrm{i}\zeta s} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}. \tag{21}$$

Note again that $K_x^{(\infty)}$ is the composition of two Hilbert-Schmidt operators

$$\mathcal{K}_x^{(\infty)} = \mathcal{B}^{(\infty)} \circ \mathcal{A}^{(\infty)} \tag{22}$$

where

$$\mathcal{A}^{(\infty)} : L^2(x,+\infty) \to L^2(\mathrm{i}\eta_1,\mathrm{i}\eta_2), \qquad \mathcal{B}^{(\infty)} : L^2(\mathrm{i}\eta_1,\mathrm{i}\eta_2) \to L^2(x,+\infty) \tag{23}$$

$$\mathcal{A}^{(\infty)}[f](z) = \sqrt{r_1(z)} \int_x^{\infty} e^{\mathrm{i}zs} f(s)\mathrm{d}s, \qquad \mathcal{B}^{(\infty)}[g](y) = \int_{[\mathrm{i}\eta_1,\mathrm{i}\eta_2]} \sqrt{r_1(z)} g(z) e^{\mathrm{i}zy} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \tag{24}$$

Theorem 2. *The following convergence result holds:*

$$\det\left(\mathrm{Id}_{L^2(x,+\infty)} - \mathcal{K}_x\right) \to \det\left(\mathrm{Id}_{L^2(x,+\infty)} - \mathcal{K}_x^{(\infty)}\right), \qquad \textit{as } N \to \infty.$$

Proof. Convergence of the respective Fredholm determinants follows from the standard argument that if we have convergence in trace-class norm, then the corresponding Fredholm determinant converges as well (see [**?**]). □

Finally, we have

$$\det\left(\mathrm{Id}_{L^2(x,+\infty)} + \mathcal{B}^{(\infty)} \circ \mathcal{A}^{(\infty)}\right) = \det\left(\mathrm{Id}_{L^2(\mathrm{i}\eta_1,\mathrm{i}\eta_2)} + \mathcal{A}^{(\infty)} \circ \mathcal{B}^{(\infty)}\right) \ . \tag{25}$$

In conclusion, by setting $\mathcal{H}^{(\infty)} := \mathcal{A}^{(\infty)} \circ \mathcal{B}^{(\infty)}$, we define the limit function

$$u(x) = -2\frac{\partial^2}{\partial x^2} \ln\det\left(\mathrm{Id}_{L^2(\mathrm{i}\eta_1,\mathrm{i}\eta_2)} + \mathcal{H}^{(\infty)}\right) \tag{26}$$

where the integral operator $\mathcal{H}^{(\infty)}$ has kernel

$$H(z,\zeta) := \frac{\sqrt{r_1(z)}e^{\mathrm{i}zx}\sqrt{r_1(\zeta)}e^{\mathrm{i}\zeta x}}{2\pi(z+\zeta)}. \tag{27}$$

3 Recover the soliton gas RHP

What is left to show is that the function defined in (26) is indeed a solution to KdV and it carries an associated Riemann-Hilbert problem (which will be amenable to asymptotic analysis).

We first notice that the kernel (27) is of the form

$$K(z,\zeta) = \frac{E_1(z)E_2(\zeta)}{z+\zeta}, \qquad z,\zeta \in \gamma_+ \tag{28}$$

where γ_+ is an oriented in the upper half plane (in our case, the segment $[\mathrm{i}\eta_1, \mathrm{i}\eta_2]$, oriented upwards).

We also recall the following variational formula: given an operator $\mathcal{K} : L^2(\gamma_+) \to L^2(\gamma_+)$ depending on some parameter(s) s, then

$$\partial_s \ln\det(\mathrm{Id} + \mathcal{K}) = \mathrm{Tr}\left((\mathrm{Id} + \mathcal{R}) \circ \partial_s \mathcal{K}\right) \tag{29}$$

where $\mathcal{R} = -\mathcal{K} \circ (\mathrm{Id} + \mathcal{K})^{-1}$ is the resolvent operator.

It turns out (see [BC12]) that the resolvent operator of an integrable kernel of the form (28) can be conveniently expressed in terms of the solution to the following Riemann-Hilbert problem:

Riemann–Hilbert Problem 1. Find a meromorphic matrix-valued function $\mathbf{\Gamma} : \mathbb{C} \setminus \{\gamma_+ \cup \gamma_-\} \to \mathbb{R}^{2\times 2}$ (where $\gamma_- = \overline{\gamma_+}$) such that

$$\mathbf{\Gamma}_+(z) = \mathbf{\Gamma}_-(z)\boldsymbol{J}(z) \qquad z \in \gamma_+ \cup \gamma_- \tag{30}$$

$$\mathbf{\Gamma}(z) = \begin{bmatrix} 1 & 1 \\ -\mathrm{i}z & \mathrm{i}z \end{bmatrix}\left[\boldsymbol{I} + \frac{\mathbf{\Gamma}_1}{z} + \frac{\mathbf{\Gamma}_2}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right] \qquad z \to \infty \tag{31}$$

$$\mathbf{\Gamma}_1 = \begin{bmatrix} a_1 & 0 \\ 0 & -a_1 \end{bmatrix} \tag{32}$$

$$\mathbf{\Gamma}(-z) = \mathbf{\Gamma}(z)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{33}$$

with jump matrix

$$\boldsymbol{J}(z) = \begin{bmatrix} 1 & -2\pi\mathrm{i}R(z)\chi_{\gamma_+} \\ -2\pi\mathrm{i}R(-z)\chi_{\gamma_-} & 1 \end{bmatrix} \tag{34}$$

with $R(z) = E_1(z)E_2(z)$.

Skipping some (many) details, it follows that the right hand side of the variational formula (29) can be explicitly computed via the solution to the Riemann-Hilbert problem above:

$$\partial_s \ln \det\left(\mathrm{Id}_{L^2(\gamma_+)} + \mathcal{K}\right) = \frac{1}{2}\int_{\gamma_+\cup\gamma_-} \mathrm{Tr}\left(\boldsymbol{\Gamma}_-^{-1}\boldsymbol{\Gamma}_-' \partial_s \boldsymbol{J}\boldsymbol{J}^{-1}\right)\frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{35}$$

via the Bertola–Malgrange τ-function formula, where the $'$ notation indicates the derivative with respect to the variable z.

In our case, the Riemann-Hilbert problem above is precisely the (matrix) Riemann-Hilbert problem of the KdV equation with jump matrix

$$\boldsymbol{J}(z) = \begin{bmatrix} 1 & -\mathrm{i}r_1(z)e^{2\mathrm{i}zx}\chi_{[\mathrm{i}\eta_1,\mathrm{i}\eta_2]} \\ -\mathrm{i}r_1(z)e^{-2\mathrm{i}zx}\chi_{[-\mathrm{i}\eta_2,-\mathrm{i}\eta_1]} & 1 \end{bmatrix} \tag{36}$$

To be more precise, the triangularity of the jump needs to be "adjusted". To this end, we define $\hat{\boldsymbol{X}} = \boldsymbol{\sigma}_1\boldsymbol{\Gamma}(z)$. Then, considering the second row of $\hat{\boldsymbol{X}}$ ($\boldsymbol{X} = \begin{bmatrix}0 & 1\end{bmatrix}\hat{\boldsymbol{X}}$), we recover the (vector) KdV Riemann-Hilbert problem (see [GGMJ21]):

$$\boldsymbol{X}_+(z) = \boldsymbol{X}_-(z)\begin{cases}\begin{bmatrix} 1 & 0 \\ -\mathrm{i}r_1(z)e^{2izx} & 1\end{bmatrix} & z \in (\mathrm{i}\eta_1, \mathrm{i}\eta_2) \\ \begin{bmatrix} 1 & \mathrm{i}r_1(z)e^{-2\mathrm{i}zx} \\ 0 & 1\end{bmatrix} & z \in (-\mathrm{i}\eta_2, -\mathrm{i}\eta_1)\end{cases} \tag{37}$$

$$\boldsymbol{X}(z) = \begin{bmatrix}1 & 1\end{bmatrix} + \mathcal{O}\left(\frac{1}{z}\right) \qquad z\to\infty \tag{38}$$

$$\boldsymbol{X}(-z) = \boldsymbol{X}(z)\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \tag{39}$$

Existence of the solution of such a Riemann-Hilbert problem is addressed in [GGMJ21] (which implies that the operator $\mathrm{Id} + \mathcal{H}^{(\infty)}$ is invertible).

As a final remark, we can compute the integral in (35) as follows. Notice that the jump matrix can be written as

$$\boldsymbol{J}(z) = e^{\boldsymbol{T}(z)}\boldsymbol{J}_0(z)e^{-\boldsymbol{T}(z)} \tag{40}$$

where $\boldsymbol{J}_0$ is a matrix depending only on the variable z, and $\boldsymbol{T}(z) = \mathrm{i}zx\boldsymbol{\sigma}_3$ is a diagonal matrix, depending on the (deformation) parameter x and satisfying $\boldsymbol{T}(0) = \boldsymbol{0}$ and $\boldsymbol{T}(-z) = \boldsymbol{\sigma}_1\boldsymbol{T}(z)\boldsymbol{\sigma}_1$, then (see again [BC12])

$$\begin{aligned}\partial_x \ln\det\left(\mathrm{Id}_{L^2(\mathrm{i}\eta_1,\mathrm{i}\eta_2)} + \mathcal{H}^{(\infty)}\right) &= -\frac{1}{2}\operatorname*{res}_{z=\infty}\mathrm{Tr}\left(\boldsymbol{\Gamma}^{-1}(z)\boldsymbol{\Gamma}'(z)\partial_x\boldsymbol{T}(z)\right)\mathrm{d}z \\ &= \frac{1}{2}\operatorname*{res}_{z=\infty}\mathrm{i}z\,\mathrm{Tr}\left(\boldsymbol{\Gamma}^{-1}(z)\boldsymbol{\Gamma}'(z)\boldsymbol{\sigma}_3\right) \\ &= \mathrm{i}\left(-\boldsymbol{\Gamma}_{1;11} + \cancel{\boldsymbol{\Gamma}_{1;12}} - \cancel{\boldsymbol{\Gamma}_{1;21}} + \boldsymbol{\Gamma}_{1;22}\right) = -\mathrm{i}a_1\end{aligned} \tag{41}$$

References

[BC12] M. Bertola and M. Cafasso. Fredholm determinants and pole-free solutions to the noncommutative Painlevé II equation. *Comm. Math. Phys.*, 309:793–833, 2012.

[GGMJ21] M. Girotti, T. Grava, K. T-R McLaughlin, and R. Jenkins. Rigorous asymptotics of a KdV soliton gas. *Comm. Math. Phys.*, 384:733–784, 2021.

[GT09] K. Grunert and G. Teschl. Long-time asymptotics for the Korteweg-de vries equation via nonlinear steepest descent. *Math. Phys. Anal. Geom.*, 12(3), 2009.