Soliton gasses

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The scattering method [Gardner, Greene, Kruskal, Miura, '67]

Physical world

$$\begin{tikzcd}[column sep=huge, row sep=huge]
q(x,0) \arrow[r, dashed, "q_t - 6qq_x + q_{xxx} = 0"] \arrow[d, "\substack{\text{Direct Scattering:}\\ \text{Lax pair } \mathcal{L}, \mathcal{B}}"'] & q(x,t) \\
\mathcal{S}(0) \arrow[r, "\text{linear evolution}"] & \mathcal{S}(t) \arrow[u, "\substack{\text{Inverse Scattering:}\\ \text{Riemann-Hilbert}\\ \text{problem}}"']
\end{tikzcd}$$

Hyperuranion

The scattering data are the L^2-eigenvalues $\{\kappa_j\}$, the norming constants $\{c_j\}$, and the reflection coefficient $\rho(z)$.

What is a soliton?

- Solitons = localized travelling wave solutions, fundamental solutions to the KdV equation.
- Solitons = discrete eigenvalues of the operator $\mathcal{L}$:

$$\begin{aligned} \{\text{velocity, amplitude}\} \quad &\Leftrightarrow \quad \kappa_j \\ \{\text{position}\} \quad &\Leftrightarrow \quad c_j \end{aligned}$$

  They arise in the long-time behaviour!
  The reflection coefficient $\rho(z)$ corresponds to a radiative part that propagates to the left and decays at rate t^{-1}.

- Soliton interaction is elastic: they "survive" collisions [Zabusky, Kruskal, '65]

$$q(x,t) = \sum_{j=1}^{N} q_{\mathrm{sol,j}}(x+\delta_j^{\pm}, t) + o(1) \qquad \text{as } t \to \pm\infty$$

The Riemann–Hilbert problem for KdV

Find a 1×2 vector $\boldsymbol{m}$ such that:

1. $\boldsymbol{m}(z; x, t)$ is meromorphic in $\mathbb{C} \setminus \left\{ \mathbb{R} \cup \{\pm \mathrm{i}\kappa_j\}_{j=1}^{N} \right\}$
2. $\boldsymbol{m}$ has simple poles at $\{\pm \mathrm{i}\kappa_j\}$:

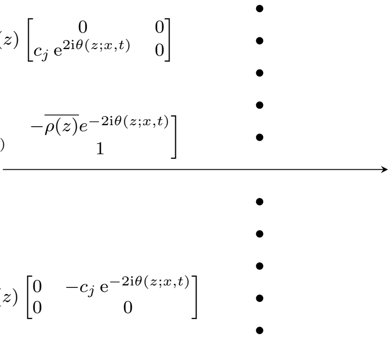
$$\operatorname{res}_{z=\mathrm{i}\kappa_j} \boldsymbol{m}(z) = \lim_{z \to \mathrm{i}\kappa_j} \boldsymbol{m}(z) \begin{bmatrix} 0 & 0 \\ c_j e^{2\mathrm{i}\,\theta(z;x,t)} & 0 \end{bmatrix},$$

$$\operatorname{res}_{z=-\mathrm{i}\kappa_j} \boldsymbol{m}(z) = \lim_{z \to -\mathrm{i}\kappa_j} \boldsymbol{m}(z) \begin{bmatrix} 0 & -c_j e^{-2\mathrm{i}\,\theta(z;x,t)} \\ 0 & 0 \end{bmatrix},$$

   where $\theta(z, x, t) = zx + 4tz^3$ and $c_j \in \mathrm{i}\mathbb{R}_+$:
3. $\boldsymbol{m}$ has jump condition on the real axis:

$$\boldsymbol{m}_+(z) = \boldsymbol{m}_-(z) \begin{bmatrix} 1 - |\rho(z)|^2 & -\overline{\rho(z)} e^{-2\mathrm{i}\theta(z;x,t)} \\ \rho(z) e^{2\mathrm{i}\,\theta(z;x,t)} & 1 \end{bmatrix}, \quad z \in \mathbb{R},$$

4. $\boldsymbol{m}(z) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\dfrac{1}{z}\right)$ as $z \to \infty$.



$$\operatorname*{res}_{z=\mathrm{i}\kappa_j} \boldsymbol{m} = \lim_{z\to \mathrm{i}\kappa_j} \boldsymbol{m}(z) \begin{bmatrix} 0 & 0 \\ c_j\,\mathrm{e}^{2\mathrm{i}\theta(z;x,t)} & 0 \end{bmatrix}$$

$$\boldsymbol{m}_+(z) = \boldsymbol{m}_-(z) \begin{bmatrix} 1-|\rho(z)|^2 & -\overline{\rho(z)}e^{-2\mathrm{i}\theta(z;x,t)} \\ \rho(z)e^{2\mathrm{i}\,\theta(z;x,t)} & 1 \end{bmatrix}$$

$$\operatorname*{res}_{z=-\mathrm{i}\kappa_j} \boldsymbol{m} = \lim_{z\to -\mathrm{i}\kappa_j} \boldsymbol{m}(z) \begin{bmatrix} 0 & -c_j\,\mathrm{e}^{-2\mathrm{i}\theta(z;x,t)} \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\theta = 4tz^3 + xz}$$

Pure solitonic solution $\rho \equiv 0$

Ansatz:

$$\boldsymbol{m}(z;x,t) = \left[1 + \sum_{j=1}^{N} \frac{\mathrm{i}\alpha_j(x,t)}{z-\mathrm{i}\kappa_j}\,, \quad 1 - \sum_{j=1}^{N} \frac{\mathrm{i}\alpha_j(x,t)}{z+\mathrm{i}\kappa_j}\right]$$

Then,

$$\begin{aligned} q_N(x,t) &= 2\mathrm{i}\,\frac{\partial}{\partial x} \lim_{z\to\infty} z\left(\boldsymbol{m}_1(z;x,t) - 1\right) \\ &= 2 \lim_{z\to\infty} z^2 \left(\boldsymbol{m}_1(z;x,t)\boldsymbol{m}_2(z;x,t) - 1\right) \quad \text{[Egorova, Piorkowski, Teschl, '19]} \\ &= -2\frac{\partial^2}{\partial x^2} \log\det(\boldsymbol{I} + \boldsymbol{A}) \quad \text{[Kay, Moses, '56]} \end{aligned}$$

with $\boldsymbol{A} \in \mathbb{R}^{N\times N}$,

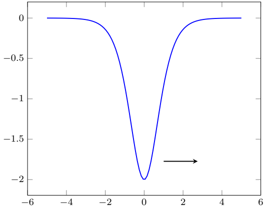
$$\boldsymbol{A}_{j\ell} = \frac{\sqrt{|c_j|}\sqrt{|c_\ell|}e^{-\mathrm{i}\left[\theta(\mathrm{i}\kappa_j;x,t)+\theta(\mathrm{i}\kappa_\ell;x,t)\right]}}{\kappa_j + \kappa_\ell}\,.$$

Example: $N = 1$

$$q_{\text{sol}}(x,t) = -2\kappa^2 \operatorname{sech}^2\left(2\kappa(x - 4\kappa^2 t) + x_0\right)$$

with initial position of the peak

$$x_0 = \frac{1}{2\kappa} \log \frac{2\kappa}{|c|} \ .$$



Example: $N = 2,\ \kappa_1 < \kappa_2$

$$q_2(x,t) = \frac{\left(4\kappa_1^2 - 4\kappa_2^2\right)\left[2\kappa_1 \cosh\left(2\kappa_2 (x - 4\kappa_2^2 t)\right) + 2\kappa_2 \cosh\left(2\kappa_1 (x - 4\kappa_1^2 t)\right)\right]}{16\kappa_1\kappa_2 + (2\kappa_1 - 2\kappa_2)^2 \cosh\left(2\kappa_1 (x - 4\kappa_1^2 t) + 2\kappa_2 (x - 4\kappa_2^2 t)\right) + (2\kappa_1 + 2\kappa_2)^2 \cosh\left(2\kappa_1 (x - 4\kappa_1^2 t) - 2\kappa_2 (x - 4\kappa_2^2 t)\right)}$$

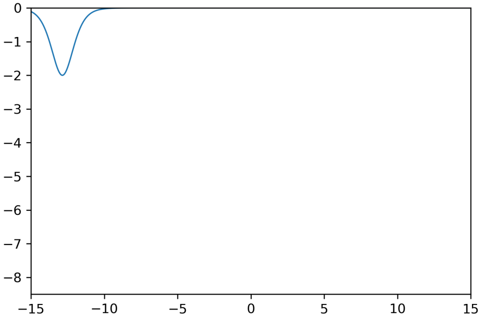


Figure: VIDEO

Example: $N=2,\ \kappa_1<\kappa_2$

$$q_2(x,t) \approx -12\kappa_1^2 \operatorname{sech}^2\left(\kappa_1(x-4\kappa_1^2 t)\right) - 12\kappa_2^2 \operatorname{sech}^2\left(\kappa_2(x-4\kappa_2^2 t)\right), \qquad t\to-\infty,$$

$$\begin{aligned} q_2(x,t) \approx {} & -12\kappa_1^2 \operatorname{sech}^2\left(\kappa_1(x-4\kappa_1^2 t) + \ln\left|\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right|\right) \\ & -12\kappa_2^2 \operatorname{sech}^2\left(\kappa_2(x-4\kappa_2^2 t) - \ln\left|\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right|\right), \quad t\to+\infty. \end{aligned}$$

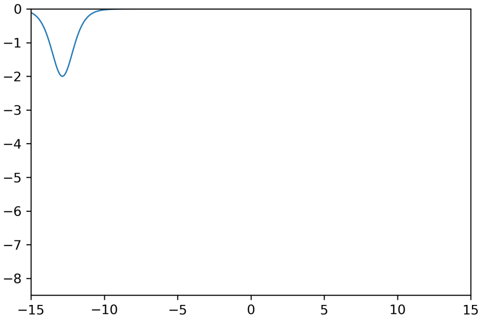


Figure: VIDEO

What is a soliton gas?

Definition (informal)

A **soliton gas/ensemble** is a -random- configuration of a large number of solitons.

The nonlinear wave field

$$q(x,t)$$

solving the PDE in this setting is called **integrable soliton turbulence**.

Recent interest revolves around the computation of statistical quantities describing the evolution of random configurations of a large number of solitons ("soliton ensemble").

Formulation of a soliton gas

1. **Continuum limit of solitons:** gas solutions belong to the closure of the set of multisoliton potentials
   [Marchenko '88–'91], [Zaitsev, Whitham, '83], [Boyd, '84], [Gesztesy, Karwowski, Zhao, '92]

2. **Kinetic theory:** soliton gas as a special large genus (thermodynamic) limit of a finite gap (N-phase nonlinear wave) solution to the PDE

   and a trial soliton velocity satisfies kinetic+continuity equations

   $$v(\kappa) = 4\kappa^2 + \frac{1}{\kappa}\int_0^\infty \ln\left|\frac{s+\kappa}{s-\kappa}\right| (v(\kappa) - v(s))\, f(s;x,t)\,\mathrm{d}s \ , \qquad f_t + (vf)_x = 0$$

   [Zakharov, '71], [El, '03], [El, Tovbis, '20], etc.

The Lax-Levermore zero-dispersion limit "soliton gas"

For each $\epsilon > 0$ there exists a unique global solution of

$$q_t + 6qq_x + \epsilon^2 q_{xxx} = 0 \qquad x \in \mathbb{R},\ t > 0.$$

with initial data $q_0(x)$, assumed to be ϵ-independent, positive, rapidly decreasing as $|x| \to \infty$, and having a single critical point (e.g. $q_0(x) = \operatorname{sech}^2(x)$).
The number of discrete eigenvalues (all simple) is

$$N(\epsilon) = \mathcal{O}(\epsilon^{-1})$$

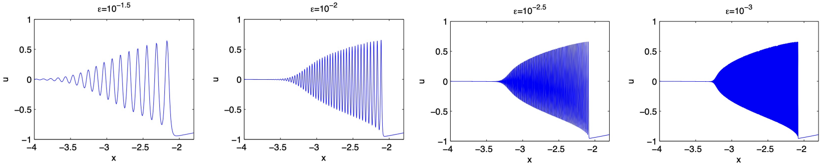
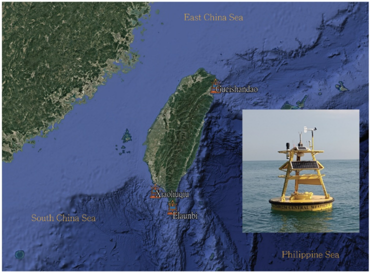


Figure: From [Grava, Klein, '07].

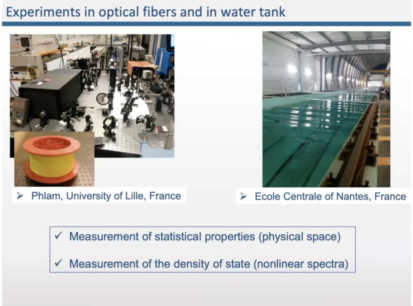
[Lax, Levermore, '83] proved that $q(x, t; \epsilon)$ has a weak limit $\bar{q}(x, t)$ as $\epsilon \to 0$.

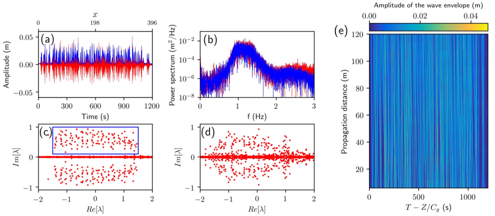
Does a soliton gas really exist?

- *real-life observation* in lagoons (Currituck Sound, NC [Costa *et al.*, '14] and Laguna Càhuil, Chile [Cienfuegos *et al.*, '25]) and deep ocean (Taiwan waters [Lee, Wahls, '25])



- *experimental observation* in water waves [Costa *et al.*, '14; Redor *et al.*, '19; Suret *et al.*, '20; ...] and in optics [Marcucci *et al.*, '19; ...]





- *numerical studies* [Gelash, Agafontsev, '18]. [Bonnemain, Congy, El, Roberti *et al.*, '18–'26], [Pelinovsky, Didenkulova *et al.*, '16–'24], etc.

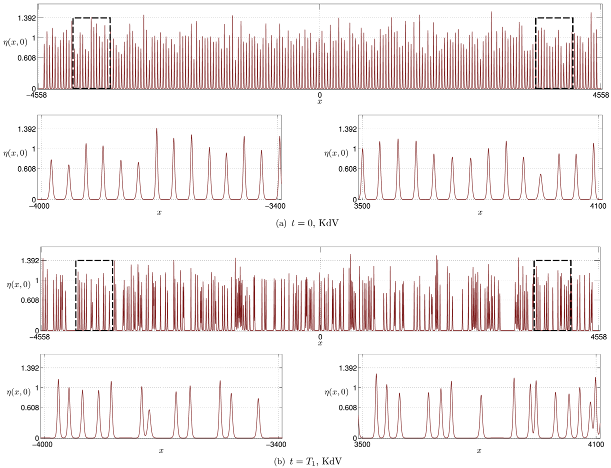


Figure: The initial condition (a) and the final state (b) of a random dilute KdV soliton gas simulated (number of solitons $N_s = 200$). From [Dutykh, Pelinovsky, '14].

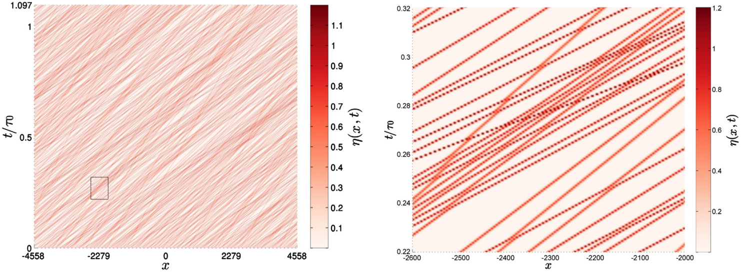


Figure: Space-time plot of a random KdV soliton gas. The time arrow is directed upwards (the bottom line being the initial state). The rectangular box shows the area zoomed in the image on the right. From [Dutykh, Pelinovsky, '14].

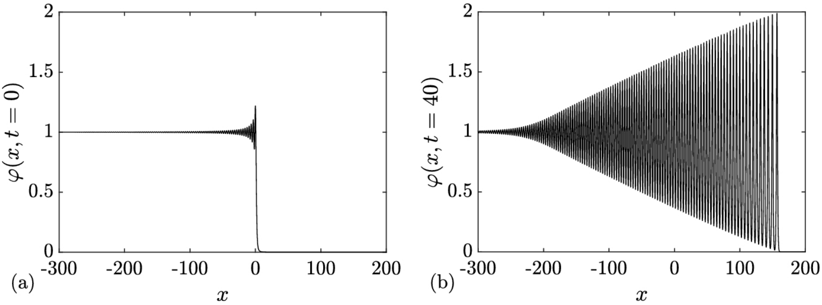


Figure: Wave field with step-like initial initial conditions: $\varphi = 1$ for $x \ll -1$ and $\varphi = 0$ for $x \gg 1$. The plots depict the variation of a condensate's realization at $t = 0$ and at $t = 40$. From [Congy, El, Roberti, Tovbis, '23].

Outline

1. *Study the behaviour of an N-soliton solution $q_N(x,t)$ in the limit as $N \to \infty$.*

   → Girotti, Grava, Jenkins, McLaughlin, *Rigorous Asymptotics of a KdV Soliton Gas*, Comm. Math. Phys. 384, 733–784 (2021).

2. *Study the dynamics of a big trial soliton travelling through the gas.*

   → Girotti, Grava, Jenkins, McLaughlin, Minakov, *Soliton versus the gas: Fredholm determinants, analysis, and the rapid oscillations behind the kinetic equation*, Comm. Pure Appl. Math., 10.1002/cpa.22106, 2023.

3. *Given random configurations of N solitons, study the behaviour of the random variable*

   $$q_N(x,t;\boldsymbol{\kappa}) - q(x,t)$$

   as $N \to \infty$.

   → Girotti, Grava, McLaughlin, Najnudel, *Law of Large Numbers and Central Limit Theorem for random sets of solitons of the focusing nonlinear Schrödinger equation*, to appear on Duke Math. J., 2026.

Upcoming efforts

- more *randomness*:
  - adding (independent) randomness to norming constants and reflection coefficient
  - evolution of density of random solitons (rogue waves?)
  - large deviation theory of rare events (rogue waves?)
- derivation of *kinetic equations* for random soliton ensembles and not-condensate gasses
- construct *new gasses* from different scaling limits as $N \to \infty$
  e.g. dilute gas, reflection coefficients *do not* scale like $\mathcal{O}(N^{-1})$, $(x, t) = \mathcal{O}((\log N)^{\alpha})$
- analyze *new random configurations*:
  - "bleeding" of solitons in random ensemble, rarefaction dense gas
  - tidal wave phenomenon at the edge of the quiescent region
  - soliton trapping and soliton tunneling
  - gas-gas iteraction