Explicit Formulae for Nonlocal Integrable PDEs and Applications

Lecture notes for a mini-course at the Institut Henri Poincaré

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Abstract

These lecture notes discuss explicit formulae for certain nonlocal integrable partial differential equations, with emphasis on their use in qualitative analysis. The guiding examples are

- Benjamin–Ono equation
- Half-Wave Maps equation
- Calogero–Moser derivative NLS
- Cubic Szegő equation

which all feature a Lax pair structure on Hardy spaces involving nonlocal projections, Toeplitz operators, and Hankel operators. Although these equations lack some of the standard dispersive features of local PDEs, their integrable structure provides powerful substitutes: Lax pairs on Hardy spaces, invariant subspaces, and – most notably – they all feature so-called *explicit formulae.*

The main theme of this mini-course is that explicit formulae should not be viewed merely as closed-form representations of solutions. Rather, they provide powerful tools for studying global well-posedness in scaling critical spaces, growth of Sobolev norms, zero-dispersion limits, finite-time blowup, as well as qualitative long-time behavior including soliton resolution and almost periodicity. After briefly recalling the explicit formulae for the equations above, we formulate a general and unifying *stability principle* and a canonical decomposition into stable and unstable subspaces. Our discussion is aimed at a PDE audience without any expert knowledge in operator theory on Hardy spaces. We then briefly outline several applications in the space-periodic setting (i.e. the equations are posed on $\mathbb{T}$), including global well-posedness, finite-time blowup, and almost periodicity in time. The second part of the mini-course, presented by Patrick Gérard, will explore the real line setting, including a discussion of the zero-dispersion limit and soliton resolution for Benjamin–Ono on the real line.

Contents

1 Introduction

The aim of this mini-course is to explain how explicit formulae arise in nonlocal integrable partial differential equations and how they can be used in qualitative analysis. The main focus is on understanding such explicit formulae in a conceptual way, yielding a set of new tools, perspectives, and results that have so far been beyond the scope of known techniques developed for PDEs and/or completely integrable systems.

Our guiding examples will be the following PDEs with one-dimensional spatial variable $x \in \mathbb{T}$ or $x \in \mathbb{R}$, i.e., in the space-periodic setting or on the real line. [Here $D = -i\partial_x$ and Π denotes the Cauchy–Szegő projection onto positive Fourier modes; see below.]

1. **Benjamin–Ono equation (BO):**
   $$\partial_t u = \partial_x(|D|u - u^2) \tag{BO}$$
   for real-valued $u = u(t,x) \in \mathbb{R}$.
2. **Calogero–Sutherland/Moser derivative (NLS):**
   $$i\partial_t u = -\partial_{xx} u \pm 2\Pi D(|u|^2)u \tag{CM/CS}$$
   for complex-valued $u = u(t,x) \in \mathbb{C}$ subject to a so-called chirality condition, i.e., the membership in the Hardy space L^2_+; see below. Here the choices of signs $+$ and $-$ correspond to the *defocusing* and *focusing* cases, respectively.[1]
3. **Half-wave maps equation (HWM):**
   $$\partial_t \mathbf{u} = \mathbf{u} \times |D|\mathbf{u} \tag{HWM}$$
   for maps $\mathbf{u} = \mathbf{u}(t,x) \in \mathbb{S}^2$ valued in the standard two-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.
4. **Cubic Szegő equation:**
   $$\partial_t u = \Pi(|u|^2 u) \tag{Szegő}$$
   for complex-valued $u = u(t,x) \in \mathbb{C}$ subject to a so-called chirality condition, i.e., the membership in the Hardy space L^2_+.

The overarching feature of these models is the presence of a *Lax pair structure* on the *Hardy spaces* given by
$$L^2_+(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \widehat{f}_n = 0 \text{ for } n \in \mathbb{Z}_{<0}\},$$
$$L^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \widehat{f}(\xi) = 0 \text{ for a.e. } \xi < 0\}.$$
depending on whether we consider the space-periodic or the real line setting. Moreover, the Lax operators for these models involve so-called Toeplitz and/or Hankel operators, which form a natural class of operators on these Hardy spaces. However, this course will not aim at studying spectral properties of such operators! Rather, we will focus on another common property that arises from a delicate interplay of the Lax pair structure with the canonical shift operators on L^2_+, i.e., the presence of so-called *explicit formulae* which can be derived for (BO), (CM/CS), (HWM), and (Szegő). Indeed, the use of explicit formulae (initiated by the seminal works [10, 8]) has generated a vast list of results, ranging from global well-posedness in critical spaces, analysis of zero-dispersion limits, soliton resolution, scattering, weak turbulence, classification of traveling solitary waves, almost periodicity and Poincaré recurrence, and finite-time blowup of solutions; see [9, 13, 14, 7, 2, 11, 1, 12, 5, 3, 15].

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[1] We adopt the convention that the name Calogero–Moser derivative NLS (CM) refers to the equation posed on $\mathbb{R}$, whereas Calogero–Sutherland derivative NLS (CS) refers to the space-periodic case on $\mathbb{T}$.

1.1 First glimpse at explicit formulae

Let us take a first (and informal) look at explicit formulae by looking at (CS) posed on $\mathbb{T}$ first derived in [1]. By standard techniques, we obtain local well-posedness for smooth initial (chiral) data $u_0 \in H^\infty_+(\mathbb{T}) = H^\infty(\mathbb{T}) \cap L^2_+(\mathbb{T})$. Let $u \in C(I; H^\infty_+)$ denote the corresponding solution with $u(0) = u_0$ with $I \subset \mathbb{R}$ denoting the maximal time interval of existence. By exploiting the Lax structure and further key commutator identities for the model, we deduce that $u = u(t, z$ when identified as a holomorphic map on $\mathbb{D}$ can be written as

$$\boxed{u(t,z) = \left\langle (\mathrm{Id} - z\mathrm{e}^{it}\mathrm{e}^{-2itL_{u_0}} S^*)^{-1} u_0, 1 \right\rangle \quad \text{for } (t,z) \in I \times \mathbb{D}} \tag{EF}$$

Here $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product, $1 \in L^2_+(\mathbb{T})$ is the constant function with value 1, and the remaining ingredients above are:

- **Lax operator at time $t = 0$:** The self-adjoint operator

  $$L_{u_0} = D \pm T_{u_0} T_{\bar{u}_0} \quad \text{acting on } L^2_+(\mathbb{T}),$$

  where the choice of sign ± correspond to the defocusing and focusing case, respectively. Here $D = -i\partial_x$ and $T_g(f) = \Pi(gf)$ denotes the Toeplitz operator on L^2_+ with symbol g and Π the Szegő projection onto $L^2_+(\mathbb{T})$.

- **Backward (or left) shift:** The operator $S^* : L^2_+ \to L^2_+$ is the backward (or left) shift with

  $$(S^* f)(z) = S^* \left(\sum_{n=0}^{\infty} \widehat{f}_n z^n \right) = \sum_{n=0}^{\infty} \widehat{f}_{n+1} z^n$$

  which amounts to shifting the Fourier coefficients of f by one step to the 'left'.

But here is an important **warning** to the reader: Although the right-hand side in (EF) makes sense for all $t \in \mathbb{R}$ by producing an element in L^2_+, we *cannot* simply deduce that smooth solutions, say, exists for all times! The deeper problem is that right-hand side may undergo a loss of L^2-mass at some finite time $T > 0$, which would signal the breakdown of strong solutions (and possible blowup of higher Sobolev norms). Ruling out this scenario, and obtaining global existence of strong solutions, requires a non-trivial analysis depending on L_{u_0} and its interactions with S^*. This mini-course will provide an introduction to the general understanding of explicit formulae, with some further concrete applications, e..g, the proof of almost periodicity in time using (EF) for the defocusing (CS).

Remark 1. For (BO), (HWM), and (Szegő) on $\mathbb{T}$, analogous explicit formulae have been derived and used; see, e.g., [8, 10, 11, 13].

Before we start with the main contents of this part of the mini-course, let us mention that explicit formulae on $\mathbb{R}$, involving the Hardy space $L^2_+(\mathbb{R})$, exist too. For instance, if we consider (CM) on $\mathbb{R}$, the corresponding explicit formula reads

$$\boxed{u(t,z) = \frac{1}{2\pi i} I_+ \left(X^* - 2tL_{u_0} - z\,\mathrm{Id})^{-1} u_0 \right) \quad \text{with } z \in \mathbb{C}_+} \tag{EF$_\mathbb{R}$}$$

Here $L_{u_0} = D \pm T_{u_0} T_{\bar{u}_0}$ acting on $L^2_+(\mathbb{R})$ and X^* denotes generator of the continuous backward (or left) shifts on $L^2_+(\mathbb{R})$, i.e.,

$$\widehat{(X^* f)}(\xi) = i \frac{d\widehat{f}}{d\xi},$$

which is an unbounded operator on the suitable domain $\mathrm{dom}(X^*) = \{f \in L^2_+ : \frac{d}{d\xi}\widehat{f} \in L^2\}$. Here, the (unbounded) functional $I_+ : \mathrm{dom}(X^*) \to \mathbb{C}$ is given by

$$I_+(g) = \widehat{g}(0^+) = \lim_{\xi \to 0^+} \widehat{g}(\xi).$$

However, the fact that X^* and I_+ are unbounded linear operators requires some additional technical efforts. Thus, we have chosen to first discuss the setting of explicit formulae on $\mathbb{T}$, which is analytically simpler as S^* is a bounded operator. A discussion of ($\mathrm{EF}_{\mathbb{R}}$) together with applications, i.e., soliton resolution and zero-dispersion limit of (BO) on $\mathbb{R}$ is presented in the second part of this mini-course held by P. Gérard.

2 Explicit formulae on $\mathbb{T}$: General approach

In this section, we will study explicit formulae from a general perspective by developing a unifying operator-theoretic approach. However, as the mini-course is aimed at a more PDE-based audience, we chose to follow a mostly self-contained discussion *without* any prerequisite knowledge on contractions on Hilbert spaces, Wold decomposition, and semigroup theory.

In order to keep things directly accessible while avoiding some technical issues (like unbounded operators and domain questions), we will henceforth focus on the space-periodic setting on the one-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \cong \partial\mathbb{D}$. [Further below, the reader will find some remarks on the real line case posed on $\mathbb{R}$.]

2.1 The Hardy space $L^2_+(\mathbb{T})$, shifts, and all that

We consider the L^2-based Hardy space

$$L^2_+(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \widehat{f}_n = 0 \text{ for } n \in \mathbb{Z}_{<0}\} \quad \text{with} \quad \widehat{f}_n = \frac{1}{2\pi}\int_{\mathbb{T}} f(x) \mathrm{e}^{-inx}\, dx,$$

of square-integrable functions $f : \mathbb{T} \to \mathbb{C}$ whose negative Fourier coefficients vanish. We recall that $L^2_+(\mathbb{T})$ is a Hilbert space equipped with the inner product and norm

$$\langle f | g \rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\overline{g(x)}\, dx, \quad \|f\| = \sqrt{\langle f | f \rangle}.$$

The corresponding orthogonal projection

$$\Pi : L^2(\mathbb{T}) \to L^2_+(\mathbb{T}), \quad f \mapsto \Pi\left(\sum_{n\in\mathbb{Z}} \widehat{f}_n e^{inx}\right) = \sum_{n \geq 0} \widehat{f}_n \mathrm{e}^{inx}$$

is the so-called *Szegő projection*. We shall often make use of the classical fact that elements $f \in L^2_+(\mathbb{T})$ can be identified (via radial boundary limits on $\mathbb{T} \cong \partial\mathbb{D}$) with holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ with power series $f(z) = \sum_{n\geq 0} c_n z^n$ such that $\sum_{n\geq 0} |c_n|^2 < \infty$; see [4] for a general background on Hardy spaces. Furthermore, for notational convenience, we will henceforth write L^2_+ to denote $L^2_+(\mathbb{T})$.

On L^2_+, there exist two canonical bounded operators $S : L^2_+ \to L^2_+$ and its adjoint $S^* : L^2_+ \to L^2_+$, whose importance in the analysis of L^2_+ can hardly be overrated. Indeed, we define the operator

$$S : L^2_+ \to L^2_+, \quad f \mapsto Sf = S\left(\sum_{n\geq 0} \widehat{f}_n e^{inx}\right) := \sum_{n\geq 1} \widehat{f}_{n-1}\mathrm{e}^{inx},$$

which is referred to as the *forward shift* (or *right shift*) on L^2_+. Clearly, the action of S on f amounts to shifting its Fourier coefficients by one to the right. Equivalently, when $f \in L^2_+$ is regarded as a holomorphic function on $\mathbb{D}$, the action of S is simply the multiplication by z, i.e., we have $(Sf)(z) = zf(z)$. The adjoint $S^* : L^2_+ \to L^2_+$ is easily found to be

$$S^* : L^2_+ \to L^2_+, \quad f \mapsto S^* f = S^* \left(\sum_{n \geq 0} \widehat{f}_n e^{inx} \right) := \sum_{n \geq 0} \widehat{f}_{n+1} \mathrm{e}^{inx},$$

and S^* is referred to as the *backward shift* (or *left shift*) on L^2_+. When f is regarded as a holomorphic function on $\mathbb{D}$, we readily check that $(S^* f)(z) = \frac{f(z)-f(0)}{z}$ holds.

A direct computation shows the fundamental identities

$$\boxed{S^* S = \mathrm{Id} \quad \text{and} \quad SS^* = \mathrm{Id} - \langle \cdot | 1 \rangle} \tag{2.1}$$

where $1 \in L^2_+$ denotes the constant function on $\mathbb{T}$ with value 1. From (2.1) we immediately deduce the following facts.

- S is an *isometry*, i.e., we have $\|Sf\| = \|f\|$ for $f \in L^2_+$.
- S^* is a *contraction*, i.e., we have $\|S^* f\| \leq \|f\|$ for $f \in L^2_+$.
- $\ker S^* = \mathbb{C}$, i.e., the kernel of S^* is the subspace of constant functions in L^2_+.
- $\|S^* f\| = \|f\|$ if and only if $f \perp 1$.
- S is injective (but not surjective) and S^* is surjective (but not injective).

The spectra of the bounded operators S and S^* can also be determined by straightforward arguments (see, e.g., [4]) and are found to be $\sigma(S) = \sigma(S^*) = \overline{\mathbb{D}}$, i.e., they coincide with the closed unit disk. However, it turns out that S has no point spectrum, i.e., $\sigma_p(S) = \emptyset$, whereas the point spectrum of S^* is given by $\sigma_p(S^*) = \mathbb{D}$ with simple eigenvalues $\overline{\lambda} \in \mathbb{D}$ and corresponding eigenfunctions

$$S^* c_\lambda = \overline{\lambda} c_\lambda \quad \text{with} \quad c_\lambda(z) = \frac{1}{1 - \overline{\lambda} z} = \sum_{n=0}^{\infty} \overline{\lambda}^n z^n \in L^2_+.$$

In fact, the eigenfunctions $\{c_\lambda\}_{\lambda \in \mathbb{D}}$ turn out to be the *reproducing kernels* (also known as *Szegő kernels*) in L^2_+, i.e., we obtain

$$f(\lambda) = \langle f | c_\lambda \rangle \quad \text{for } f \in L^2_+ \text{ and } \lambda \in \mathbb{D}.$$

Along the same lines, we can also deduce the fundamental identity

$$\boxed{f(z) = \left\langle (\mathrm{Id} - zS^*)^{-1} f | 1 \right\rangle \quad \text{for } f \in L^2_+ \text{ and } z \in \mathbb{D}} \tag{2.2}$$

which will be the initial step to derive the explicit formulae in our analysis below. Indeed, by the geometric series expansion in $z \in \mathbb{D}$, we see that the right-hand side in (2.2) is given by

$$\left\langle (\mathrm{Id} - zS^*)^{-1} f | 1 \right\rangle = \sum_{n \geq 0} \langle (S^*)^n f | 1 \rangle z^n = \sum_{n \geq 0} \langle f | S^n 1 \rangle z^n = \sum_{n \geq 0} \widehat{f}_n z^n,$$

which is just the Fourier series expansion of $f \in L^2_+$, using the elementary facts that $S^n 1 = z^n$ for $n \in \mathbb{Z}_{\geq 0}$ and hence $\widehat{f}_n = \langle f | z^n \rangle$ is the n-th Fourier coefficient of f. This proves that (2.2) holds. Let us also remark that (2.2) can be seen as an operator-theoretic phrasing of Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(w)}{w - z} \, dw \quad \text{for } z \in \mathbb{D}.$$

Remark 2. Further below, we will add some remarks on the natural generalization to *vector-valued Hardy spaces* and its shifts. That is, for a given complex Hilbert space E, we can study the Hardy space $L^2_+(\mathbb{T}; E)$ of holomorphic maps $f : \mathbb{D} \to E$ with $f(z) = \sum_{n \geq 0} c_n z^n$ and $c_n \in E$ such that $\sum_{n \geq 0} |c_n|^2_E < \infty$. In particular, the choice $E = \mathbb{C}^{d \times d}$ with $d \geq 2$ occurs in the analysis of the half-wave maps equation (HWM) in [13]. However, to keep the discussion focused, we will mostly consider the scalar case when $E = \mathbb{C}$ in what follows.

2.2 Functional properties of explicit formulae

We now develop a general framework for explicit formulae posed on $\mathbb{T}$. For our discussion, we make the following standing assumption:

$$\boxed{\mathcal{L} \text{ is a self-adjoint (possibly unbounded) operator on } L^2_+}$$

Let $\{\mathrm{e}^{-it\mathcal{L}}\}_{t\in\mathbb{R}}$ denote the corresponding strongly continuous one-parameter group of unitary maps, generated by the self-adjoint operator $\mathcal{L}$. For $t \in \mathbb{R}$, we introduce the following linear maps Σ_t on L^2_+ together with the adjoint Σ_t^* given by

$$\boxed{\Sigma_t = Se^{it\mathcal{L}} \quad \text{and} \quad \Sigma_t^* = e^{-it\mathcal{L}}S^*}$$

The unitarity of $\mathrm{e}^{-it\mathcal{L}}$ directly implies that the classical shift identities (2.1) translate into

$$\Sigma_t^*\Sigma_t = \mathrm{Id} \quad \text{and} \quad \Sigma_t\Sigma_t^* = \mathrm{Id} - \langle\cdot|1\rangle. \tag{2.3}$$

Thus, Σ_t is also an isometry with $\ker \Sigma_t^* = \mathbb{C}$ for any $t \in \mathbb{R}$. Or in the language of operator theory, we can say that the isometry Σ_t has deficiency index $\dim\ker\Sigma_t^* = 1$.

Next, we introduce the following linear map $\mathcal{U}(t) : L^2_+ \to L^2_+$ with $t \in \mathbb{R}$ defined as

$$\boxed{(\mathcal{U}(t)f)(z) = \Big\langle (\mathrm{Id} - z\mathrm{e}^{-it\mathcal{L}}S^*)^{-1}f, 1\Big\rangle \quad \text{for } f \in L^2_+ \text{ and } z \in \mathbb{D}} \tag{2.4}$$

The next lemma shows that $\mathcal{U}(t)$ is always a contraction and, moreover, we obtain a key identity which links the map $\mathcal{U}(t)$ to the strong-stability property of the discrete semigroup $\{(\Sigma_t^*)^N\}_{N\in\mathbb{N}}$.

Lemma 1 (Stability identity)**.** *For all $t \in \mathbb{R}$ and $f \in L^2_+$, we have*

$$\|\mathcal{U}(t)f\|^2 = \|f\|^2 - \lim_{N\to\infty} \|(\Sigma_t^*)^N f\|^2.$$

In particular, we have $\|\mathcal{U}(t)f\| \leq \|f\|$, i.e., the map $\mathcal{U}(t) : L^2_+ \to L^2_+$ is a contraction.

Remark 3. 1) Since Σ_t^* is a contraction, the non-negative sequence $\|(\Sigma_t^*)^N f\|$ is monotone decreasing. Hence the limit above always exists and we have $\lim_{N\to\infty}\|(\Sigma_t^*)^N f\| = \inf_{N\geq 0}\|(\Sigma_t^*)^N f\|$.

2) Let $T : H \to H$ be a contraction on a Hilbert space, i.e., $\|Tf\|_H \leq \|f\|_H$ for $f \in H$. Recall that the discrete semigroup $\{T^N\}_{N\in\mathbb{N}}$ is said to be *strongly stable* if $\|T^N f\|_H \to 0$ as $N \to \infty$ for any $f \in H$. The identity above shows that $\mathcal{U}(t) : L^2_+ \to L^2_+$ is an isometry if and only if the discrete semigroup generated by Σ_t^* is strongly stable.

3) In operator analysis, an isometry $\Sigma : H \to H$ on a Hilbert space H is called a *pure isometry* if $\|(\Sigma^*)^N f\| \to 0$ as $N \to \infty$ for every $f \in H$. Thus, we the strong stability property in 2) exactly means that $\Sigma_t = S\mathrm{e}^{it\mathcal{L}_{u_0}}$ is a pure isometry.

Proof. Let $t \in \mathbb{R}$ and $f \in L^2_+$ be given. Since $(\mathcal{U}(t)f)(z) = \sum_{n\geq 0}\langle(\Sigma_t^*)^n f|1\rangle z^n$ for $z \in \mathbb{D}$, it follows

$$\|\mathcal{U}(t)f\|^2 = \sum_{n=0}^{\infty} |\langle(\Sigma_t^*)^n f|1\rangle|^2.$$

Hence $\mathcal{U}(t)f \in L^2_+$ if and only if the series above is finite. Now, for any $g \in L^2_+$, we deduce from (2.3) that $\|\Sigma_t^* g\|^2 = \langle\Sigma_t\Sigma_t^* g|g\rangle = \|g\|^2 - |\langle g|1\rangle|^2$. Taking $g = (\Sigma_t^*)^N f$ with integer $N \geq 1$, we find

$$|\langle(\Sigma_t^*)^N f|1\rangle|^2 = \|(\Sigma_t^*)^N f\|^2 - \|(\Sigma_t^*)^{N+1} f\|^2.$$

Therefore, we obtain the telescopic sum

$$\sum_{n=0}^{N-1} |\langle (\Sigma_t^*)^n f | 1\rangle|^2 = \|f\|^2 - \|(\Sigma_t^*)^N f\|^2 \quad \text{for any } N \geq 1.$$

Hence, it follows

$$\|\mathcal{U}(t) f\|^2 = \sum_{n=0}^{\infty} |\langle (\Sigma_t^*)^N f | 1\rangle|^2 = \|f\|^2 - \lim_{N\to\infty} \|(\Sigma_t^*)^N f\|^2.$$

In particular, we see that $\|\mathcal{U}(t) f\| \leq \|f\|$. □

Lemma 2 (On unitarity of $\mathcal{U}(t)$)**.** *Let $t \in \mathbb{R}$ be given. The map $\mathcal{U}(t) : L^2_+ \to L^2_+$ is* ***unitary*** *if and only if one of the following equivalent conditions holds.*

- (i) $\ker \mathcal{U}(t) = \{0\}$.
- (ii) $\lim_{N\to\infty} \|(\Sigma_t^*)^N f\| = 0$ *for all* $f \in L^2_+$.

Proof. The equivalence of (i) and (ii) is a direct consequence of Lemma 1. Also, it is clear that (i) is necessary for $\mathcal{U}(t)$ to be unitary.

It remains to show that (i) implies that $\mathcal{U}(t)$ is a unitary map on L^2_+. We argue as follows and claim that

$$\|\mathcal{U}(t)^* f\| = \|f\| \quad \text{for all } f \in L^2_+ \tag{2.5}$$

i.e., the adjoint $\mathcal{U}(t)^*$ is always an isometry. Once this is shown, it easily follows that (i) implies that $\mathcal{U}(t)$ is unitary. Indeed, to prove (2.5), we first recall that $(\mathcal{U}(t)f)(z) = \sum_{n\geq 0} \langle (\Sigma_t^*)^n f | 1\rangle z^n$ for $z \in \mathbb{D}$. Let $e_n = z^n$ for $n \geq 0$ denote the standard Fourier orthonormal basis in L^2_+. Then

$$\langle \mathcal{U}(t) f, e_n\rangle = \langle f, \Sigma_t^n 1\rangle \quad \text{for all } n \geq 0.$$

By definition of the adjoint, we have $\langle \mathcal{U}(t) f, e_n\rangle = \langle f, \mathcal{U}(t)^* e_n\rangle$ and we deduce

$$\mathcal{U}(t)^* e_n = \Sigma_t^n 1 \quad \text{for all } n \geq 0.$$

To show that $\mathcal{U}(t)^*$ is an isometry, i.e. that (2.5) holds, it remains to prove that $(\Sigma_t^n 1)_{n\geq 0}$ forms an orthonormal system. Indeed,

$$\|\Sigma_t^n 1\| = \|1\| = 1$$

since $\Sigma_t = S \mathrm{e}^{it\mathcal{L}}$ is an isometry and, if $m < n$, then

$$\langle \Sigma_t^n 1, \Sigma_t^m 1\rangle = \langle \Sigma_t^{n-m} 1, 1\rangle = \langle \Sigma_t^{n-m-1} 1, \Sigma_t^* 1\rangle = 0,$$

using that $\Sigma_t^* \Sigma_t = \mathrm{Id}$ and $1 \in \ker \Sigma_t^*$. Hence, for every polynomial $p(z) = \sum_{n=0}^N a_n z^n$, we find $\mathcal{U}(t)^* p = \sum_{n=0}^N a_n \Sigma_t^n 1$ and therefore $\|\mathcal{U}(t)^* p\|^2 = \sum_{n=0}^N |a_n|^2 = \|p\|^2$. Since polynomials are dense in L^2_+, we conclude that (2.5) holds.

Suppose now that (i) holds, i.e., we have $\ker \mathcal{U}(t) = \{0\}$. Since $\mathcal{U}(t)^*$ has closed range (because it is an isometry), we deduce

$$L^2_+ = \ker \mathcal{U}(t) \oplus \operatorname{ran} \mathcal{U}(t)^* = \{0\} \oplus \operatorname{ran} \mathcal{U}(t)^* = \operatorname{ran} \mathcal{U}(t)^*.$$

Therefore, the isometry $\mathcal{U}(t)^*$ is surjective and hence it is a unitary map together with its adjoint $\mathcal{U}(t)$. □

Finally, we record the following general fact about the weak continuity of the map $t \mapsto \mathcal{U}(t) f$ for given $f \in L^2_+$.

Lemma 3 (Weak continuity)**.** *Let $f \in L^2_+$ be given. Then the map $t \mapsto \mathcal{U}(t)f$ is weakly continuous, i.e.,*

$$\mathcal{U}(t_n)f \rightharpoonup \mathcal{U}(t)f \text{ weakly in } L^2_+ \text{ whenever } t_n \to t.$$

Proof. Assume that $t_n \to t$ and let $f \in L^2_+$ be given. We need to show that

$$\mathcal{U}(t_n)f \rightharpoonup \mathcal{U}(t)f \text{ weakly in } L^2_+. \tag{2.6}$$

Since $\|\mathcal{U}(t_n)f\| \leq \|f\|$ for all n, the sequence $\mathcal{U}(t_n)f$ is bounded in L^2_+ and, by passing to a subsequence, we can assume that $\mathcal{U}(t_n)f \rightharpoonup g$ for some $g \in L^2_+$. We claim that

$$g(z) = (\mathcal{U}(t)f)(z) \quad \text{for } z \in \mathbb{D}. \tag{2.7}$$

Observe that by the strong continuity of the unitary group $\{\mathrm{e}^{-it\mathcal{L}}\}_{t\in\mathbb{R}}$, we can deduce

$$\Sigma^*_{t_n} = \mathrm{e}^{-it_n\mathcal{L}}S^* \to \Sigma^*_t = \mathrm{e}^{-it\mathcal{L}}S^* \quad \text{strongly as operators.}$$

Next, by holomorphicity, it suffices to prove (2.7) on some non-empty open subset in $\mathbb{D}$. Let us take $\mathbb{D}_{1/2} = \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$. Now, we observe

$$\begin{aligned}(\mathcal{U}(t_n)f)(z) - (\mathcal{U}(t)f)(z) &= \left\langle (\mathrm{Id} - z\Sigma^*_{t_n})^{-1}f - (\mathrm{Id} - z\Sigma^*_t)^{-1}f, 1\right\rangle \\ &= \left\langle (\mathrm{Id} - z\Sigma^*_{t_n})^{-1}(z(\Sigma^*_t - \Sigma^*_{t_n}))(\mathrm{Id} - z\Sigma^*_t)^{-1}f, 1\right\rangle.\end{aligned}$$

Since $\sigma(\Sigma^*_{t_n}) \subseteq \overline{\mathbb{D}}$, we see that $\|(\mathrm{Id} - z\Sigma^*_{t_n})^{-1}\| \leq \frac{1}{1/2} = 2$ for $z \in \mathbb{D}_{1/2}$. Hence, for any $z \in \mathbb{D}_{1/2}$, we find

$$|(\mathcal{U}(t_n)f)(z) - (\mathcal{U}(t)f)(z)| \leq \|(\Sigma^*_t - \Sigma^*_{t_n})(\mathrm{Id} - z\Sigma^*_t)^{-1}f\| \to 0 \quad \text{as } n \to \infty,$$

using that $\Sigma^*_{t_n} \to \Sigma^*_t$ strongly as operators. This proves that (2.7) holds. Since the limit is independent of the chosen subsequence, we obtain that (2.6) holds. ☐

With help of Lemmas 1–3 above, we obtain the following main result.

Theorem 1 (Stability principle for explicit formulae)**.** *The map $\mathcal{U}(t) : L^2_+ \to L^2_+$ is* ***unitary*** *for all $t \in \mathbb{R}$ if and only if one of the following equivalent properties holds.*

(*i*) ***Nondegeneracy:*** $\ker \mathcal{U}(t) = \{0\}$ *for all $t \in \mathbb{R}$.*

(*ii*) ***Strong stability:*** $\lim_{N\to\infty} \|(\mathrm{e}^{-it\mathcal{L}}S^*)^N f\| = 0$ *for all $f \in L^2_+$ and $t \in \mathbb{R}$.*

In this case, we have that, for every $f \in L^2_+$, the map $t \mapsto \mathcal{U}(t)f$ is ***strongly continuous****, i.e.,*

$$\mathcal{U}(t_n)f \to \mathcal{U}(t)f \text{ strongly in } L^2_+ \text{ whenever } t_n \to t.$$

Proof. The above theorem directly follows from Lemmas 1–3 above. Concerning the claim about strong continuity of $t \mapsto \mathcal{U}(t)f$ for given $f \in L^2_+$, we remark that this follows from weak continuity in Lemma 3 together with the fact that no loss of L^2-norm can occur, i.e., $\|f\| = \|\mathcal{U}(t_n)f\| = \|\mathcal{U}(t)f\|$, when $\mathcal{U}(t)$ is unitary for all $t \in \mathbb{R}$. ☐

We close this subsection by making the following remarks. First, we can regard Theorem 1 as some sort of *nonlinear extension of Stone's theorem* in the sense that $\{\mathcal{U}(t)\}_{t\in\mathbb{R}}$ is a strongly continuous family of unitaries if and only if the self-adjoint operator $\mathcal{L}$ satisfies the strong stability property stated in (ii) above, or equivalently, we have that (i) holds, i.e.

$$\ker \mathcal{U}(t) = \{0\} \quad \text{for all } t \in \mathbb{R}.$$

Thus, in order to study the global existence of strong solutions (or their breakdown in finite time), we need to study whether the above stability criterion can be proved or disproved for the equation under consideration depending on the initial datum.

- For (BO) and the defocusing (CS) on $\mathbb{T}$, the relevant self-adjoint operator $\mathcal{L}$ is given by

  $$\mathcal{L} = -2(D - T_{u_0}) - \mathrm{Id} \quad \text{for (BO)}, \quad \mathcal{L} = 2(D + T_{u_0} T_{\bar{u}_0}) + \mathrm{Id} \quad \text{for defocusing (CS)}$$

  with $D = -i\partial_x$ and the Toeplitz operator $T_g f = \Pi(gf)$, and u_0 is the initial datum. In these cases, the stability property holds true and we thus obtain that the explicit formula yields a strongly continuous solution map defined for all times $t \in \mathbb{R}$.

- For the focusing (CS) on $\mathbb{T}$, the relevant self-adjoint operator $\mathcal{L}$ becomes

  $$\mathcal{L} = 2(D - T_{u_0} T_{\bar{u}_0}) + \mathrm{Id} \quad \text{for focusing (CS)}$$

  Under the smallness assumption $\|u_0\|_{L^2} < 1$, the global well-posedness results in [1] imply the stability property. However, as recently shown in [3], the *strong stability* of $\mathrm{e}^{-it\mathcal{L}} S^*$ fails for certain (smooth) initial data $u_0 \in L^2_+$ with $\|u_0\|_{L^2} > 1$, leading to finite-time blowup of the corresponding solution! That is, there exists smooth initial data $u_0 \in L^2_+$ and some finite time $T > 0$ such that

  $$\ker \mathcal{U}(T) \neq \{0\}$$

  is nontrivial. Via the explicit formula, the inspection of the nontrivial kernel $\ker \mathcal{U}(T) \neq \{0\}$ allows us to discuss the *blowup dynamics* of the solution $u(t)$ as $t \to T$. More specifically, the explicit formula approach induces a canonical decomposition as

  $$u(t) = \mathcal{U}(t) u_{0,b} + \mathcal{U}(t) u_{0,*} = \text{“blowup part”} + \text{“regular part”} \quad \text{for } t \in [0, T)$$

  with $u_{0,b}$ being the orthogonal projection of the initial datum u_0 onto $\ker \mathcal{U}(T)$ and $u_{0,*}$ its orthogonal complement.

2.3 Decomposition into stable and unstable subspaces

From the previous discussion we deduce the orthogonal decomposition

$$L^2_+ = X_{\mathrm{u}}(t) \oplus X_{\mathrm{s}}(t) \quad \text{for every } t \in \mathbb{R}$$

where

$$X_{\mathrm{u}}(t) = \ker \mathcal{U}(t) \quad \text{and} \quad X_{\mathrm{s}}(t) = X_{\mathrm{u}}(t)^{\perp} = \operatorname{ran} \mathcal{U}(t)^*,$$

recalling that the adjoint $\mathcal{U}(t)^*$ is always an isometry (and hence its range is automatically closed). The subspace $X_{\mathrm{u}}(t)$ can be written as

$$\begin{aligned} X_{\mathrm{u}}(t) &= \left\{ f \in L^2_+ : \lim_{N \to \infty} \|(\Sigma_t^*)^N f\| = \|f\| \right\} \qquad \text{(by Lemma 1)} \\ &= \left\{ f \in L^2_+ : \|(\Sigma_t^*)^N f\| = \|f\| \text{ for all } N \in \mathbb{N} \right\}, \end{aligned}$$

where the last equality follows from the fact that the sequence $\|f\| \geq \|\Sigma_t^* f\| \geq \|(\Sigma_t^*)^2 f\| \geq \ldots \geq 0$ is monotone-decreasing, since Σ_t^* is a contraction. As for an explicit description of $X_{\mathrm{s}}(t) = X_{\mathrm{u}}(t)^{\perp}$, we claim that

$$X_{\mathrm{s}}(t) = \overline{\operatorname{span}} \, \{\Sigma_t^k 1 : k \in \mathbb{N}\} = \left\{ f \in L^2_+ : \lim_{N \to \infty} \|(\Sigma_t^*)^N f\| = 0 \right\}. \tag{2.8}$$

To see that the first equality holds, we simply recall from the proof of Lemma 2 that $X_{\mathrm{s}}(t) = \operatorname{ran} U(t)^*$ is the closed linear span of the orthonormal system $(\Sigma_t^k 1)_{k \in \mathbb{N}}$. The second equality is also elementary to see using that $\mathcal{U}(t)^*$ is an isometry and left as an exercise.

In view of the identifications above, we shall refer to $X_{\mathrm{s}}(t)$ as the *stable subspace* and $X_{\mathrm{u}}(t)$ as the *unstable subspace* for $\mathcal{U}(t)$, reflecting the presence and failure of the strong stability of the discrete semigroup $\{(\Sigma_t^*)^N\}_{N \in \mathbb{N}}$, respectively. We summarize our discussion as follows.

Lemma 4 (Decomposition for explicit formulae)**.** *For every $t \in \mathbb{R}$, we have the orthogonal decomposition*

$$L^2_+ = X_{\mathrm{u}}(t) \oplus X_{\mathrm{s}}(t)$$

together with the following properties:

- ***Stable dynamics:*** *For $f \in X_{\mathrm{s}}(t)$, we have $\mathcal{U}(t_n)f \to \mathcal{U}(t)f$ strongly in L^2_+ as $t_n \to t$.*
- ***Unstable dynamics:*** *For $f \in X_{\mathrm{u}}(t)$ with $f \neq 0$, we have that $\mathcal{U}(t_n)f \rightharpoonup \mathcal{U}(t)f$ weakly in L^2_+ but $\mathcal{U}(t_n)f \not\to \mathcal{U}(t)f$ strongly in L^2_+ as $t_n \to t$.*

Remark 4. 1) If we aim for proving good global dynamics (i.e. strong continuity of solutions in L^2_+ for all $t \in \mathbb{R}$), we need to show that $X_{\mathrm{u}}(t) = \{0\}$ is always trivial.

2) If we are interested in possible blowup, i.e. breakdown of strong continuity at some finite time $T > 0$, we need to look for non-trivial $X_{\mathrm{u}}(T) \neq \{0\}$. In fact, it can be shown that necessarily we then have that the projection of the initial datum u_0 onto $X_{\mathrm{u}}(T)$ is non-vanishing, leading to the "blowup part", which can be further analyzed using the explicit formula in the limit $t \nearrow T$.

2.4 Vector-valued Hardy spaces

Let us briefly remark that the preceding discussion directly carries over to the setting of the vector-valued Hardy space $L^2_+(\mathbb{T}; E)$ of functions with values in a given complex Hilbert space E. In particular, this setting arises for (HWM), where $E = \mathbb{C}^{d \times d}$ with $d \geq 2$ is considered. The explicit formulae in this E-valued setting then read

$$\Pi u(t,z) = \mathcal{M}\left((\mathrm{Id} - z\mathrm{e}^{-it\mathcal{L}_{u_0}} S^*)^{-1} \Pi u_0 \right), \tag{2.9}$$

where $\mathcal{M} : L^2_+(\mathbb{T}; E) \to E$ denotes the orthogonal projection onto the subspace $E \subset L^2_+(\mathbb{T}; E)$ of constant functions E-valued maps defined on $\mathbb{T}$. Note that, for $E = \mathbb{C}$, we have $\mathcal{M}(f) = \langle f, 1 \rangle$, which is in full agreement with our discussion above. We refer to [13] for a general discussion for the general stability principle on vector-valued Hardy spaces.

3 On the derivation of explicit formulae

In this section, we present a short proof of the derivation of explicit formulae in the integrable PDEs discussed above with a Lax pair on the Hardy space L^2_+. To simplify our discussion, we will henceforth consider the focusing (CS) on $\mathbb{T}$, i.e.,

$$i\partial_t u = -\partial_{xx} u - 2\Pi D(|u|^2)u \quad \text{for } (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{3.1}$$

By an iterative Kato-type scheme, we can deduce local well-posedness for initial data in the Hardy–Sobolev spaces $H^s_+ = L^2_+ \cap H^s$ with $s > \frac{3}{2}$. We fix $s > 3/2$ in what follows and let $u \in C(I; H^s_+)$ be the solution of (CS) with initial datum $u(0) = u_0 \in H^s_+$, where I denotes the maximal time interval of existence. Also, this amount of regularity of $u(t)$ allows us to justify the following calculations.

From [1] we recall that the following Lax equation holds

$$\frac{d}{dt} L_{u(t)} = [B_{u(t)}, L_{u(t)}] \tag{3.2}$$

where $[X,Y] = XY - YX$ denotes the commutator and

$$L_u = D - T_u T_{\bar{u}}, \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$$

Note that $L_u = L_u^*$ is self-adjoint on L^2_+ with domain $\operatorname{dom} L_u = H^1_+$, whereas $B_u = -B_u^*$ is bounded and skew-adjoint. To derive the explicit formula for (CS), it is convenient (but ultimately not necessary) to "re-gauge" the operators L_u and B_u by defining the new operators

$$\mathcal{L}_u = 2L_u + \mathrm{Id}, \quad \mathcal{B}_u = B_u - iL_u^2 = -iD^2 + 2iT_u D T_{\bar{u}}.$$

We readily check that $\mathcal{L}_u$ and $\mathcal{B}_u$ are again a Lax pair, i.e.,

$$\frac{d}{dt}\mathcal{L}_{u(t)} = [\mathcal{B}_{u(t)}, \mathcal{L}_{u(t)}]. \tag{3.3}$$

Note that $\mathcal{B}_u = -\mathcal{B}_u^*$ is still skew-adjoint, but now we obtain an unbounded operator[2] with $\operatorname{dom} \mathcal{B}_u = \operatorname{dom} L_u^2 = H^2_+$. However, with this new choice of $\mathcal{B}_u$, we can write (CS) as

$$\partial_t u(t) = \mathcal{B}_{u(t)} u(t) \quad \text{with } t \in I, \tag{3.4}$$

which enables us to find a rather slick derivation of the explicit formula below.

As a next preliminary step, let us study (3.4) as follows. Let $U : I \to \mathcal{B}(L^2_+)$ denote the solution of the operator-valued ODE:

$$\frac{d}{dt}U(t) = \mathcal{B}_{u(t)} U(t) \quad \text{for } t \in I, \quad U(0) = \mathrm{Id}\,. \tag{3.5}$$

By the skew-adjointness of $\mathcal{B}_{u(t)}$, the maps $\{U(t)\}_{t\in I}$ are unitary maps on L^2_+. Also, a quick calculation shows that

$$\mathcal{B}_{u(t)} 1 = 0, \tag{3.6}$$

i.e., the constant function 1 belongs to $\ker \mathcal{B}_{u(t)}$, which implies that

$$U(t)^* 1 = 1.$$

Indeed, this follows from $\frac{d}{dt}U(t)^*1 = -U(t)^*\mathcal{B}_{u(t)}1 = 0$ and $U(0)^* = \mathrm{Id}$.

The following result now establishes the link to the explicit formulae studied above.

Lemma 5. *For every $f \in L^2_+$, we have*

$$(U(t)f)(z) = \left\langle (\mathrm{Id} - z\mathrm{e}^{-it\mathcal{L}_{u_0}} S^*)^{-1} f, 1 \right\rangle \quad \textit{for } (t,z) \in I \times \mathbb{D},$$

with $\mathcal{L}_{u_0} = 2L_{u_0} + \mathrm{Id}$.

Remark 5. This shows that the family of unitary maps $U(t)$ for $t \in I$ can be identified with the explicit formula maps $\mathcal{U}(t) : L^2_+ \to L^2_+$ discussed above.

Proof. Let $f \in L^2_+$ and $t \in I$ be given. From the identity (2.2) we deduce

$$(U(t)f)(z) = \langle (\mathrm{Id} - zS^*)^{-1} U(t) f, 1 \rangle \quad \text{for } z \in \mathbb{D}.$$

By Lemma 6 shown below, we have the intertwining relation

$$S^* U(t) = U(t) \mathrm{e}^{-it\mathcal{L}_{u_0}} S^*.$$

Thus, by the geometric series expansion in $z \in \mathbb{D}$, we deduce the identity

$$(\mathrm{Id} - zS^*)^{-1} U(t) = U(t)(\mathrm{Id} - z\mathrm{e}^{-it\mathcal{L}_{u_0}} S^*)^{-1} \quad \text{for } z \in \mathbb{D}.$$

Therefore,

$$(U(t)f)(z) = \left\langle U(t)(\mathrm{Id} - z\mathrm{e}^{-it\mathcal{L}_{u_0}} S^*)^{-1} f, 1 \right\rangle = \left\langle (\mathrm{Id} - z\mathrm{e}^{-it\mathcal{L}_{u_0}} S^*)^{-1} f, 1 \right\rangle,$$

where the last equation follows from $U(t)^*1 = 1$ above. ☐

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[2] The fact that $\mathcal{B}_u$ is unbounded requires some technicalities, which we omit in our presentation here. Alternatively, we could use the bounded operator B_u, which however gives a less clean calculation below.

In view of (3.4) and Lemma 5 with $f = u_0$, we immediately obtain the following result.

Theorem 2 (Explicit formula for focusing (CS))**.** *Let $s > \frac{3}{2}$ and suppose that $u \in C(I; H^s_+)$ solves (3.1) with initial datum $u(0) = u_0 \in H^s_+$. Then it holds*

$$u(t,z) = \left\langle (\mathrm{Id} - z\mathrm{e}^{-it\mathcal{L}_{u_0}} S^*)^{-1} u_0, 1 \right\rangle \quad \text{for } (t,z) \in I \times \mathbb{D}$$

with $\mathcal{L}_{u_0} = 2L_{u_0} + \mathrm{Id}$.

To complete the proof of Lemma 5, we are left with showing the intertwining relation used above.

Lemma 6. *We have $S^* U(t) = U(t) \mathrm{e}^{-it\mathcal{L}_{u_0}} S^*$ for $t \in I$.*

Proof. The proof is based on commutator calculations for S^*, D, and Toeplitz operators T_f, whose details we skip here. The main outcome is that

$$[S^*, \mathcal{B}_u] = -i\mathcal{L}_u S^*. \tag{3.7}$$

Next, by using this identity together with $\frac{d}{dt}U(t) = \mathcal{B}_{u(t)} U(t)$ and $\frac{d}{dt}U(t)^* = -U(t)\mathcal{B}_{u(t)}$, we obtain

$$\frac{d}{dt} U(t)^* S^* U(t) = U(t)^* [S^*, \mathcal{B}_{u(t)}] U(t) = U(t)^* (-i\mathcal{L}_{u(t)}) U(t).$$

Now, from the Lax equation (3.3) we deduce that $\mathcal{L}_{u(t)} = U(t)\mathcal{L}_{u_0} U(t)^*$. Since $U(t)^* U(t) = \mathrm{Id}$, we conclude

$$\frac{d}{dt} U(t)^* S^* U(t) = -i\mathcal{L}_{u_0} U(t)^* S^* U(t).$$

Hence by integration in t and using that $U(0) = \mathrm{Id}$, this yields

$$U(t)^* S^* U(t) = \mathrm{e}^{-it\mathcal{L}_{u_0}} S^*.$$

Applying $U(t)$ on both sides and using that $U(t)U(t)^* = \mathrm{Id}$, we finish the proof. ☐

4 Applications

In this final section, we discuss some applications of the approach via explicit formulae. A non-exhaustive list of results in the space-periodic setting on $\mathbb{T}$ is as follows.

- **Global well-posedness in scaling-critical spaces:** For the defocusing (CS) and (HWM) on $\mathbb{T}$, the approach via explicit formula provides a unique continuous flow map on the scaling-critical spaces $L^2_+(\mathbb{T})$ and $H^{1/2}(\mathbb{T})$, respectively. See [1, 13]. Also, for the focusing (CS) on $\mathbb{T}$, we obtain GWP in $L^2_+(\mathbb{T})$ with initial data $\|u_0\|_{L^2} < 1$.
- **Zero-dispersion limit**: Using the explicit formula for (BO), we can study the zero-dispersion limit leading to an explicit-type formula

  $$\Pi ZD[u_0](t,z) = \left\langle (\mathrm{Id} - z\mathrm{e}^{-2iT_{u_0}} S^*)^{-1} \Pi u_0, 1 \right\rangle$$

  The loss of L^2-mass of the right-hand side for some finite $T > 0$ corresponds to the (first) shock formation of the inviscid Burgers equation with real initial datum $u_0 : \mathbb{T} \to \mathbb{R}$. See [6, 15].
- **Classification of traveling solitary waves and finite-gap potentials:**

- **Finite-time blowup:** For the focusing (CS) on $\mathbb{T}$, the explicit formula is used in [3] to explicitly construct finite-time blowup solutions with smooth initial data $u_0 \in H^\infty_+(\mathbb{T})$ in the full range $1 < \|u_0\|^2_{L^2} < 2$. The blowup dynamics can be calculated completely and the blowup rate is given by

$$\|u(t)\|_{H^s(\mathbb{T})} \sim (T-t)^{-2s} \quad \text{as } t \nearrow T \text{ for any } s > 0.$$

- **Almost periodicity:** Using the explicit formula and that the spectrum of the Lax operator $\mathcal{L}_{u_0}$ is at most countable, we can derive almost periodic solutions for globally well-posed models. As a by-product, we obtain Poincaré recurrence of solutions.

Let us also remark that in the real line setting (with the corresponding explicit formula on $L^2_+(\mathbb{R})$, see second part of this mini-course), one can prove soliton resolution and scattering results.

4.1 Almost periodicity

We wish to discuss one of the previous results, namely how to prove almost periodicity via the explicit formula, in more detail here. The arguments originated in the recent work [13] on (HWM) on $\mathbb{T}$. However, we shall discuss its adaptation to defocusing (CS) on $\mathbb{T}$, i.e.

$$i\partial_t u = -\partial_{xx} u + 2D\Pi(|u|^2)u \quad \text{with } (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{4.1}$$

Global well-posedness in H^s_+ for any $s \geq 0$ was proven in [1].

To simplify our discussion, we consider smooth initial data $u_0 \in H^\infty_+(\mathbb{T})$ and the corresponding solution $u \in C(\mathbb{R}; H^\infty_+)$ is known to satisfy the a priori bounds

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} \leq C(u_0, s) \quad \text{for any } s \geq 0 \tag{4.2}$$

due to the hierarchy of conserved quantities. We have the explicit formula

$$u(t,z) = \left\langle (\mathrm{Id} - z\mathrm{e}^{-it\mathcal{L}_{u_0}} S^*)^{-1} u_0, 1 \right\rangle \quad \text{for } (t,z) \in \mathbb{R} \times \mathbb{D} \tag{4.3}$$

with the Lax operator $\mathcal{L}_{u_0} = 2L_{u_0} + \mathrm{Id}$, where $L_{u_0} = D - T_{u_0} T_{\bar{u}_0}$.

Theorem 3 (Almost periodicity for defocusing (CS), smooth case)**.** *Let $u \in C(\mathbb{R}; H^\infty_+)$ be a solution of the defocusing (CS). Then, for every $s \geq 0$, the map $t \mapsto u(t)$ is almost periodic from $\mathbb{R}$ to $H^s_+(\mathbb{T})$.*

Remark 6. 1) Let X be a Banach space and suppose that $f \in C_b(\mathbb{R}; X)$, i.e., $f : \mathbb{R} \to X$ is a bounded and continuous function. Recall that f is said to be *almost periodic* (in the sense of Bohr) if for every $\epsilon > 0$, there exists some $L > 0$ such that every interval $I \subset \mathbb{R}$ of length L contains some $\tau \in I$ with

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\|_X \leq \epsilon.$$

2) From the almost periodicity of $u \in C(\mathbb{R}; H^s_+)$ we easily conclude that the orbit $\{u(t) : t \in \mathbb{R}\}$ is relatively compact in $H^s_+(\mathbb{T})$. Also, another direct consequence is *Poincaré recurrence* in the sense that for every $\epsilon > 0$ and $T > 0$, there exists some time $t_* \geq T$ such that

$$\|u(t_*) - u_0\|_{H^s} \leq \epsilon.$$

That is, the solution $u(t)$ always returns at some sufficiently large time to its initial datum u_0 up to an error of at most ϵ.

As a first step towards the proof of Theorem 3, we recall *Bochner's criterion* which characterizes almost periodic functions $f \in C_b(\mathbb{R}; X)$ by the following compactness property.

Lemma 7 (Bochner's criterion)**.** *Let X be a Banach space. A function $f \in C_b(\mathbb{R}; X)$ is almost periodic if and only if its set of translates*

$$\mathrm{Trans}[f] = \{ f(\cdot + a) : a \in \mathbb{R} \}$$

is relatively compact in $C_b(\mathbb{R}; X)$ equipped with the usual sup*-norm $\|f\|_{\sup} = \sup_{t \in \mathbb{R}} \|f(t)\|_X$.*

As a next preliminary step, we observe that the spectrum of $\mathcal{L}_{u_0}$ is discrete (by compactness of resolvents) with eigenvalues

$$\lambda_0(\mathcal{L}_{u_0}) \leq \lambda_1(\mathcal{L}_{u_0}) \leq \lambda_2(\mathcal{L}_{u_0}) \leq \ldots \to +\infty.$$

In particular, the spectrum $\sigma(\mathcal{L}_{u_0}) = \sigma_d(\mathcal{L}_{u_0})$ is countable. For the corresponding strongly continuous unitary group

$$\boxed{\Omega(t) := \mathrm{e}^{-it\mathcal{L}_{u_0}} : L^2_+ \to L^2_+ \quad \text{with } t \in \mathbb{R}}$$

we easily obtain the following compactness property.

Lemma 8 (Compactness property of $\Omega(t)$)**.** *For every sequence (a_n) in $\mathbb{R}$, there exists a subsequence (a_n') such that*

$$\Omega(a_n') \to \Omega_\infty \textit{ strongly as operators,}$$

where $\Omega_\infty : L^2_+ \to L^2_+$ is some unitary map.

Proof. Let $(\lambda_n)_{n \geq 0}$ denote the sequence of eigenvalues of $\mathcal{L}_{u_0}$ from above. By functional calculus, we have

$$\Omega(a_n) = \mathrm{e}^{-ia_n\mathcal{L}_{u_0}} = \sum_{n=0}^{\infty} e^{-ia_n\lambda_n} P_n,$$

where P_n denotes the orthogonal projection onto the (finite-dimensional) eigenspace for $\mathcal{L}_{u_0}$ with eigenvalue λ_n. By Cantor's diagonal argument, we can find a subsequence (a_n') and a sequence of unimodular numbers $\zeta_k \in \mathbb{S}^1$ with $k \in \mathbb{N}$ such that

$$e^{-ia_n'\lambda_k} \to \zeta_k \in \mathbb{S}^1 \quad \text{as } n \to \infty \text{ for every } k \in \mathbb{N}.$$

We conclude that the unitary operators $\Omega(a_n')$ converge strongly as operators to $\Omega_\infty := \sum_{k=0}^{\infty} \zeta_k P_k$, which by functional calculus is a unitary map, too. ☐

Next, we establish the following convergence result for smooth solutions $u \in C(\mathbb{R}; H^\infty_+)$ of the defocusing (CS), which we recall can be written as in (4.3).

Lemma 9 (Convergence property)**.** *Let (t_n) be a sequence in $\mathbb{R}$ such that $\Omega(t_n) \to \Omega_\infty$ strongly as operators with some unitary map $\Omega_\infty : L^2_+ \to L^2_+$. Then we have*

$$u(t_n) \to u_\infty \textit{ in } H^s_+ \textit{ for every } s \geq 0,$$

where the limit $u_\infty \in H^\infty_+$ is given by

$$u_\infty(z) = \left\langle (\mathrm{Id} - z\Omega_\infty S^*)^{-1} u_0, 1 \right\rangle \quad \textit{for } z \in \mathbb{D}.$$

Proof. From the proof of Lemma 3 and strong operator convergence $\Omega(t_n) \to \Omega_\infty$, we infer that $u(t_n) \rightharpoonup u_\infty$ weakly in L^2_+, where u_∞ is given by the formula stated above. Using the a priori bounds (4.2) and Rellich compactness, we deduce that $u(t_n) \to u_\infty$ strongly in H^s_+ for every $s \geq 0$. ☐

Proof of Theorem 3

Suppose that $u \in C(\mathbb{R}; H^\infty_+)$ solves the defocusing (CS). Let $s \geq 0$ be given. We have $u \in C_b(\mathbb{R}; H^s_+)$ by continuity and a priori bounds. To show that $u \in C_b(\mathbb{R}; H^s_+)$ is almost periodic, we use that $u(t)$ is given by the explicit formula and apply Bochner's criterion. Thus, we have to prove that for every sequence (a_n) in $\mathbb{R}$, there exists a subsequence (a'_n) such that

$$u(\cdot + a'_n) \text{ is uniformly convergent on } \mathbb{R} \text{ as } n \to \infty. \tag{4.4}$$

Indeed, let (a_n) be a sequence in $\mathbb{R}$. By Lemma 8, we can find a subsequence (a'_n) such that

$$\Omega(a'_n) \to \Omega_\infty \quad \text{strongly as operators}$$

with some unitary map Ω_∞ on L^2_+. We now define

$$u_\infty(t, z) = \left\langle (\mathrm{Id} - z\Omega(t)\Omega_\infty S^*)^{-1} u_0, 1 \right\rangle \quad \text{for } (t, z) \in \mathbb{R} \times \mathbb{D}.$$

Evidently, we have that $\Omega(t)\Omega(a'_n) \to \Omega(t)\Omega_\infty$ strongly as operators for every $t \in \mathbb{R}$. Thus, by Lemma 9 applied to the sequence $(t_n) = (t + a'_n)$, we deduce that

$$u(t + a'_n) \to u_\infty(t) \text{ in } H^s_+ \text{ for any } t \in \mathbb{R}. \tag{4.5}$$

Moreover, it is easy to see that $u_\infty \in C_b(\mathbb{R}; H^s_+)$ thanks to the a priori bounds on $u(t)$. Now, we claim that we have in fact uniform convergence on $\mathbb{R}$, i.e.

$$\sup_{t \in \mathbb{R}} \|u(t + a'_n) - u_\infty(t)\|_{H^s} \to 0, \tag{4.6}$$

which would finish the proof of (4.4).

Indeed, pick a sequence of times (t_n) in $\mathbb{R}$ such that

$$\sup_{t \in \mathbb{R}} \|u(t + a'_n) - u_\infty(t)\|_{H^s} \leq \|u(t_n + a'_n) - u_\infty(t_n)\|_{H^s} + 2^{-n}. \tag{4.7}$$

Applying Lemma 8 again, we can extract a subsequence (t'_n) such that $\Omega(t'_n) \to \tilde{\Omega}_\infty$ strongly as operators with some unitary map $\tilde{\Omega}_\infty$. Thanks to the *group property* of $\{\Omega(t)\}_{t \in \mathbb{R}}$, we deduce

$$\Omega(t'_n + a'_n) = \Omega(t'_n)\Omega(a'_n) \to \tilde{\Omega}_\infty \Omega_\infty \quad \text{strongly as operators.}$$

Invoking Lemma 9, we infer

$$u(t'_n + a'_n) \to u_\infty \quad \text{in } H^s_+,$$

where the limit satisfies

$$u_\infty(z) = \left\langle (\mathrm{Id} - z\tilde{\Omega}_\infty \Omega_\infty S^*)^{-1} u_0, 1 \right\rangle \quad \text{for } z \in \mathbb{D}. \tag{4.8}$$

On the other hand, from the explicit expression for $u_\infty(t)$, we see that

$$u_\infty(t'_n, z) = \left\langle (\mathrm{Id} - z\Omega(t'_n)\Omega_\infty S^*)^{-1} u_0, 1 \right\rangle \quad \text{for } z \in \mathbb{D}.$$

Since $\Omega(t'_n)\Omega_\infty \to \tilde{\Omega}_\infty \Omega_\infty$ strongly as operators, we can use Lemma 9 once again to conclude

$$u(t'_n) \to u_\infty \quad \text{in } H^s_+,$$

where the limit u_∞ is again given by (4.8). Hence we have found that

$$\|u(t'_n + a'_n) - u_\infty(t'_n)\|_{H^s} \to 0.$$

By passing to the subsequence (t'_n) in (4.7) and taking the limit $n \to \infty$, we deduce that (4.6) holds.

The proof of Theorem 3 is now complete. □

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