EXPLICIT FORMULAE FOR NON-LOCAL INTEGRABLE PDES AND APPLICATIONS : THE REAL LINE SETTING

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Abstract. This is the second part of a mini–course in collaboration with E. Lenzmann. First we introduce the specific structure of the Hardy space on the upper half–plane. Then we focus on the link between Lax pairs and explicit fromulae, with some emphasis on the explicit formula for the Benjamin–Ono equation. Finally, we briefly explain how to use these formulae, with some applications to the long–time behavior of the solutions.

1. The Hardy space setting and the Lax pair structure

1.1. **The Hardy space on the upper half–plane.** We will use the following closed subspace of $L^2(\mathbb{R})$,

$$\begin{aligned} L^2_+(\mathbb{R}) &:= \{f \in L^2(\mathbb{R}) : \forall \xi < 0, \hat f(\xi) = 0\} \\ &= \left\{ f \text{ holomorphic on } \mathbb{C}_+ : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < +\infty \right\} \end{aligned}$$

The associated orthogonal projector was already introduced in the previous section. It is the Riesz–Szegő projector given by

$$\widehat{\Pi f}(\xi) = \mathbf{1}_{\xi \geq 0} \hat f(\xi) \ ,$$

or by

$$\Pi f(z) = \frac{1}{2i\pi} \int_{\mathbb{R}} \frac{f(x)}{x-z} \, dx \ , \ z \in \mathbb{C}_+ .$$

Next, we introduce an important class of operators acting on $L^2_+(\mathbb{R})$. Given $b \in L^\infty(\mathbb{R})$, we define the Toeplitz operator of symbol b,

$$T_b : L^2_+ \to L^2_+ \ , \ f \mapsto T_b f := \Pi(bf).$$

We notice that T_b is bounded and that $T_b^* = T_{\bar b}$. Let us consider a very simple example.

Example 1.1. *Assume that*

$$b(x) = \frac{1}{x+p}$$

for some $p \in \mathbb{C} \setminus \mathbb{R}$. Then

$$T_b f(x) = \begin{cases} \frac{f(x)}{x+p} & \text{if } p \in \mathbb{C}_+ \ , \\ \\ \frac{f(x) - f(-p)}{x+p} & \text{if } p \in \mathbb{C}_- \ . \end{cases}$$

We continue this subsection by a crucial lemma related to the Toeplitz operators.

Lemma 1.2. *For $a, b \in H^1, f \in L^2_+$, we have*

$$(T_{ab} - T_a T_b) f = \Pi\Big(\Pi(a)(\mathrm{Id} - \Pi)\big\{(\mathrm{Id} - \Pi)(b) f\big\}\Big)$$

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Proof. The main observation is the following : if f, g have only positive (resp. negative) frequencies, then fg has only positive (resp.negative) frequencies.

$$T_{ab}f - T_aT_bf = \Pi\big(abf\big) - \Pi\big(a\Pi\big(bf\big)\big) = \Pi(aU), \quad U := \big(\mathrm{Id} - \Pi\big)(bf)$$
$$\Pi\big(aU\big) = \Pi(\Pi(a)U) + \Pi\big(\big(\mathrm{Id} - \Pi\big)(a)U\big) = \Pi\big(\Pi(a)U\big).$$

Finally, write

$$bf = \Pi(b)f + (\mathrm{Id} - \Pi\big)(b)f,$$

and $\Pi(b)f \in L^2_+$, so that

$$U = (\mathrm{Id} - \Pi\big)(bf) = (\mathrm{Id} - \Pi\big)\big\{(\mathrm{Id} - \Pi\big)(b)f\big\}.$$

□

We now come to the crucial structure of the Hardy space on the upper–half plane that will be used for all our explicit formulae on the line. Consider the Lax–Beurling semigroup on $L^2_+(\mathbb{R})$ [Lax59] defined by

$$S(\eta)f(x) := \mathrm{e}^{i\eta x}f(x)\ ,\ \eta \geq 0,$$

and characterized by

$$\widehat{S(\eta)f}(\xi) = \hat{f}(\xi - \eta).$$

Notice that the positivity of η is necessary to keep the Fourier transform of $S(\eta)f$ supported into $[0, \infty[$ for every $f \in L^2_+(\mathbb{R})$. The infinitesimal generator of this semigroup is the multiplication by x on $L^2_+(\mathbb{R})$, which we denote by X. It is important here to observe that, contrary to what happens on $L^2(\mathbb{R})$, $S(\eta)$ is not a unitary operator on $L^2_+(\mathbb{R})$ if $\eta > 0$, because its range is included into the space of functions whose Fourier transform is supported in $[\eta, \infty)$. Hence X is not a selfadjoint operator on $L^2_+(\mathbb{R})$. In fact, one can easily check that the adjoint operator X^* has a bigger domain than the domain of X, namely is such that

$$\mathrm{Dom}(X^*) = \{f \in L^2_+(\mathbb{R}) : \exists \lambda_f \in \mathbb{C} : Xf + \lambda_f \in L^2(\mathbb{R})\}, \tag{1.1}$$

$$X^*f(x) = xf(x) + \lambda_f. \tag{1.2}$$

There is a way of understanding the above characterization of $\mathrm{Dom}(X^*)$ using the Fourier transformation. Indeed, we have

$$\mathrm{e}^{-i\eta X^*} = S(\eta)^* = T_{\mathrm{e}^{-i\eta x}}, \eta \geq 0,\ \widehat{S(\eta)^*f}(\xi) = \hat{f}(\xi + \eta), \xi \geq 0,$$

and therefore, from the semigroup theory, a function $f \in L^2_+(\mathbb{R})$ belongs to $\mathrm{Dom}(X^*)$ if and only if $\hat{f} \in H^1(0, \infty)$. Moreover,

$$\forall \xi > 0, \widehat{X^*f}(\xi) = i\frac{d}{d\eta}\hat{f}(\xi + \eta)_{|\eta=0} = i\frac{d}{d\xi}\hat{f}(\xi). \tag{1.3}$$

In particular, $\hat{f}$ is right continuous at 0, and we may define

$$I_+(f) := \hat{f}(0^+) = \lim_{\epsilon \to 0^+}\langle f, \chi_\epsilon\rangle_{L^2}, \tag{1.4}$$

where χ_ϵ is defined by

$$\chi_\epsilon(x) := (1 - i\epsilon x)^{-1}\ ,\ \epsilon > 0. \tag{1.5}$$

From the Fourier viewpoint, the only difference between $\mathrm{Dom}(X^*)$ and $\mathrm{Dom}(X)$ is that the jump $I_+(f)$ may not be zero if $f \in \mathrm{Dom}(X^*)$. Then, taking the Fourier transform of both sides of equation (1.2), we obtain the following distributional identity in $\mathcal{D}'(\mathbb{R})$,

$$\widehat{X^*f} = i\partial_\xi\hat{f} + 2\pi\lambda_f\delta_0,$$

where $i\partial_\xi \hat f$ denotes the distributional derivative of $\hat f$ on $\mathbb{R}$, while $\widehat{X^*f}$ denotes the L^2 function on $\mathbb{R}$ defined by $i\partial_\xi \hat f$ on $(0,\infty)$ and by 0 on $(-\infty,0)$. We infer that

$$2\pi\lambda_f = -i\hat f(0^+) = -iI_+(f).$$

This leads to the following variant of the Cauchy integral formula.

Proposition 1.3. *For every $z\in\mathbb{C}_+$, for every $f\in L^2_+(\mathbb{R})$,*

$$(X^*-z\mathrm{Id})^{-1}f(x) = \frac{f(x)-f(z)}{x-z}\ ,\ f(z)=\frac{1}{2i\pi}I_+((X^*-z\mathrm{Id})^{-1}f).$$

Proof. The function $g_z : x\mapsto \frac{f(x)-f(z)}{x-z}$ belongs to $L^2_+(\mathbb{R})$ and satisfies

$$(X-z)g_z+f(z)=f,$$

hence $g_z=(X^*-z)^{-1}f$ and $2\pi f(z)=-iI_+(g_z)$, which leads to the claim.
Here is another proof using the inverse Fourier transform. Recall that χ_ϵ has been defined in (1.5).

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_0^{+\infty} \mathrm{e}^{iz\xi}\hat f(\xi)\,d\xi = \frac{1}{2\pi}\int_0^{+\infty}\mathrm{e}^{iz\xi}\lim_{\epsilon\to 0^+}\langle S(\xi)^*f,\chi_\epsilon\rangle\,d\xi \\ &= \frac{1}{2\pi}\int_0^{+\infty}\mathrm{e}^{iz\xi}\lim_{\epsilon\to 0^+}\langle \mathrm{e}^{-i\xi X^*}f,\chi_\epsilon\rangle_{L^2}\,d\xi \\ &= \frac{1}{2i\pi}\lim_{\epsilon\to 0^+}\langle (X^*-z)^{-1}f,\chi_\epsilon\rangle_{L^2}\ . \end{aligned}$$

□

Example 1.4. *Here is a very elementary example of a vector in* $\mathrm{Dom}(X^*)$*. Given $p\in\mathbb{C}_+$, the function f_p defined by*

$$f_p(x)=\frac{1}{x+p}$$

clearly belongs to $L^2_+(\mathbb{R})$, and the elementary identity

$$xf_p(x)-1=-f_p(x)$$

proves that $f_p\in\mathrm{Dom}(X^*)$ and that $X^*f_p=-pf_p, I_+(f_p)=-2\pi i$.

1.2. **The Lax pair and conservation laws.** We come to the definition of operators which constitute a Lax pair for the Benjamin–Ono equation,

$$\partial_t u+\partial_x(u^2)=\partial_x|D|u. \tag{1.6}$$

For $u\in H^2_{\mathrm{real}}(\mathbb{R})$, we define, with $D:=-i\partial_x$,

$$L_u := D-T_u\in\mathcal{L}(H^1_+,L^2_+),\ \ B_u:=i(T_{|D_x|u}-T_u^2)\in\mathcal{L}(H^k_+,H^k_+), k=0,1. \tag{1.7}$$

Notice that L_u can also be seen as an unbounded selfadjoint operator on $L^2_+(\mathbb{R})$, with domain $H^1_+(\mathbb{R})$, while B_u is bounded and antiselfadjoint. The following result is the starting point of our analysis.

Theorem 1.5. *If $u\in C(I,H^2_{\mathrm{real}})$ solves (1.6) on a time interval I if and only if, then*

$$\frac{dL_{u(t)}}{dt}=[B_{u(t)},L_{u(t)}]. \tag{1.8}$$

Proof. Observe that $T_u^* = T_u$ and $B_u^* = -B_u$. We have

$$\frac{d}{dt}L_{u(t)} = -T_{\partial_t u(t)} = -T_{\partial_x |D_x| u(t)} + 2T_{u(t)\partial_x u(t)} =: (1)$$

Since $[\partial_x, T_b] = T_{\partial_x b}$ and $D_x = \frac{1}{i}\partial_x$, we infer

$$(1) = i\,[T_{|D_x|u}, D_x] + 2T_{u\partial_x u} = i\,[T_{|D_x|u}, D_x - T_u] + 2T_{u\partial_x u} + i[T_{|D_x|u}, T_u].$$

Consequently,

$$\frac{d}{dt}L_{u(t)} = i\,[T_{|D_x|u}, L_u] + 2T_{u\partial_x u} + i[T_{|D_x|u}, T_u].$$

Next we define

$$(2) := i[T_{|D_x|u}, T_u]f = i\Big(T_{|D_x|u}T_u - T_{u|D_x|u}\Big)f + i\Big(T_{u|D_x|u} - T_u T_{|D_x|u}f\Big).$$

Apply Lemma 1.2 with $a = |D_x|u, b = u$, then with $a = u, b = |D_x|u$. We infer

$$\begin{aligned}(2) = &- i\Pi\Big(\Pi(|D_x|u)(\mathrm{Id} - \Pi)\big\{(\mathrm{Id} - \Pi)(u)f\big\}\Big) +\\ &i\Pi\Big(\Pi(u)(\mathrm{Id} - \Pi)\big\{(\mathrm{Id} - \Pi)(|D_x|u)f\big\}\Big).\end{aligned}$$

Since

$$\Pi(|D_x|v)(x) = \frac{1}{i}(\Pi\partial_x v)(x)\ ,\ \ (\mathrm{Id} - \Pi)(|D_x|v)(x) = i(\mathrm{Id} - \Pi)\partial_x v(x),$$

we obtain

$$(2) = -\Pi\Big(\Pi(\partial_x u)(\mathrm{Id} - \Pi)\big\{(\mathrm{Id} - \Pi)(u)f\big\}\Big) - \Pi\Big(\Pi(u)(\mathrm{Id} - \Pi)\big\{(\mathrm{Id} - \Pi)(\partial_x u)f\big\}\Big).$$

Applying again Lemma 1.2, we obtain

$$\begin{aligned}(2) = i[T_{|D_x|u}, T_u]f &= -\Big(T_{u\partial_x u} - T_{\partial_x u}T_u\Big)f - \Big(T_{u\partial_x u} - T_u T_{\partial_x u}\Big)f,\\ &= -2T_{u\partial_x u}f + T_{\partial_x u}T_u f + T_u T_{\partial_x u}f.\end{aligned}$$

Then observe that

$$\begin{aligned}[T_u^2, D_x] &= T_u[T_u, D_x] + [T_u, D_x]T_u = -\frac{1}{i}T_u T_{\partial_x u} - \frac{1}{i}T_{\partial_x u}T_u,\\ &= i(T_u T_{\partial_x u} + T_{\partial_x u}T_u),\end{aligned}$$

so that

$$i[T_{|D_x|u}, T_u] = -2T_{u\partial_x u} - i[T_u^2, D_x] = -2T_{u\partial_x u} - i[T_u^2, L_u].$$

Finally,

$$\frac{d}{dt}L_u = i\,[T_{|D_x|u}, L_u] - i[T_u^2, L_u] = [B_u, L_u].$$

Conversely, assume that (1.8) holds, and apply it to χ_ϵ given by (1.5). We observe that, for every $b \in H^1(\mathbb{R})$,

$$T_b\chi_\epsilon = \Pi b + o(1), \tag{1.9}$$

in $H^1 + (\mathbb{R})$ as $\epsilon \to 0$. Consequently, $L_u\chi_\epsilon = -\Pi u + o(1)_H^1$ and

$$B_u\chi_\epsilon = -iL_u^2\chi_\epsilon + o(1)_{L^2}. \tag{1.10}$$

This leads to

$$\partial_t \Pi u = (B_u + iL_u^2)\Pi u, \tag{1.11}$$

which, after an elementary calculation, is also what one obtains when one applies Π to both sides of (1.6). Since u is real valued, it is characterized by Πu through $u = \Pi u + \overline{\Pi u}$, and therefore (1.11) is equivalent to (1.6). $\square$

Remark 1.6. *A similar identity to (1.8) would hold if one adds to B_u any function of L_u. A special choice is*

$$P_u = B_u + iL_u^2,$$

which is still an antiselfadjoint operator and has the further interesting property that the version (1.11) of (BO) can be rewritten as :

$$\partial_t \Pi u = P_u \Pi u.$$

However, we preferred to keep the previous definition of B_u, because it has the advantage that this B_u is a bounded operator on $L^2_+(\mathbb{R})$ and $H^1_+(\mathbb{R})$, which makes further constructions simpler — see in particular Proposition 2.2 below.

As a direct consequence of the Lax pair, we infer an infinite family of conservation laws.

Theorem 1.7. *If u is a $H^{\max(2,k/2)}$ solution of (BO), then*

$$E_k(u) := \langle L_u^k(\Pi u), \Pi u\rangle$$

is conserved.

Proof. We have

$$\begin{aligned}
&\frac{d}{dt}E_k(u(t))\\
&= \langle L^k_{u(t)}\partial_t \Pi u(t), \Pi u(t)\rangle + \langle L^k_{u(t)}\Pi u(t), \partial_t \Pi u(t)\rangle + \langle \partial_t[L^k_{u(t)}]\Pi u(t), \Pi u(t)\rangle\\
&= \langle L_u^k(B_u\Pi u + iL_u^2\Pi u), \Pi u\rangle + \langle L_u^k \Pi u, B_u\Pi u + iL_u^2\Pi u\rangle + \langle (B_uL_u^k - L_u^kB_u)\Pi u, \Pi u\rangle\\
&= 0\ .
\end{aligned}$$

□

Remark 1.8. *The simplicity of the expression of conservation laws in Theorem 1.7 is striking if we compare them for instance to the conservation laws of the KdV equation. This fact is directly related to the identity (1.11), which do not have simple equivalent in the KdV case.*

We come to some commutator identities relating X^* and the partners of our Lax pairs, which are the crux of the explicit formulae in the next section.

Lemma 1.9. *For every $b \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, for every $f \in \mathrm{Dom}(X^*)$, we have $T_bf \in \mathrm{Dom}(X^*)$ and*

$$X^*T_bf - T_bX^*f = \frac{i}{2\pi}I_+(f)\Pi b\ .$$

Proof. We start from the identity

$$xf(x) = X^*f(x) + \frac{i}{2\pi}I_+(f),$$

and we multiply both sides by $b(x)$,

$$xb(x)f(x) = b(x)X^*f + \frac{i}{2\pi}I_+(f)b(x).$$

This leads to

$$\Pi(xbf) = T_bX^*f + \frac{i}{2\pi}I_+(f)\Pi b.$$

In order to conclude the proof, we just need to check that $\Pi(Xbf) = X^*\Pi(bf)$. For this, we observe that, for every $g \in L^2(\mathbb{R})$ such that $xg \in L^2(\mathbb{R})$, we have $g \in L^1(\mathbb{R})$ and

$$X\Pi g = \Pi(xg) + \frac{i}{2\pi}\int_{\mathbb{R}} g(x)\, dx,$$

in particular $\Pi(Xg) = X^*\Pi g$. Indeed, the above identity, after applying the Fourier transform, is nothing but the distributional formula

$$i\partial_\xi(\mathbf{1}_{(0,\infty)}\hat g) = \mathbf{1}_{(0,\infty)} i\partial_\xi \hat g + i\hat g(0+)\delta_0,$$

from distribution theory. The proof is completed by applying this observation to $g = bf$. □

2. Explicit formulae

2.1. The Benjamin–Ono case. Recall the Lax operators L_u, B_u for (1.6) have been defined respectively by (1.7). Here are important consequences of Lemma 1.9.

$$[X^*, B_u] = -2L_u - i[X^*, L_u^2], \tag{2.1}$$

Indeed, using Lemma 1.9, we have, for every $f \in \mathrm{Dom}(X^*) \cap \mathrm{Dom}(L_u^2)$,

$$\begin{aligned}
[X^*, B_u]f &= i([X^*, T_{|D|u}]f - T_u[X^*, T_u]f - [X^*, T_u]T_u f) \\
&= \frac{i}{2\pi}(iI_+(f)(D\Pi u - T_u\Pi u) - iI_+(T_u f)\Pi u) \\
&= \frac{i}{2\pi}(iI_+(f)L_u\Pi u + iI_+(L_u f)\Pi u) \\
&= i(L_u(if - [X^*, L_u]f) + iL_u f - [X^*, L_u]L_u f) \\
&= -2L_u f + i[L_u^2, X^*]f\ .
\end{aligned}$$

Theorem 2.1 ([Gé23]). *The following representation formula holds. The solution $u \in C(\mathbb{R}, H^2_{\mathrm{real}}(\mathbb{R}))$ of the Benjamin–Ono equation with $u(0) = u_0 \in H^2_{\mathrm{real}}(\mathbb{R})$ is given by $u(t,x) = \Pi u(t,x) + \overline{\Pi u(t,x)}$ with*

$$\forall z \in \mathbb{C}_+\ ,\ \Pi u(t,z) = \frac{1}{2i\pi} I_+((X^* - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0). \tag{2.2}$$

Proof. As a first step, we use the Lax pair provided by Theorem 1.5. The following proposition holds in a more general context, and has been proved by Lax in his original paper [Lax68].

Proposition 2.2. *Define the family of bounded operators $\{U(t)\}_{t\in\mathbb{R}}$ on $L^2_+(\mathbb{R})$ by the linear ODE*

$$U'(t) = B_{u(t)}U(t)\ ,\ U(0) = \mathrm{Id}.$$

Then $U(t)$ is unitary and

$$L_{u(t)} = U(t)L_{u(0)}U(t)^*.$$

Proof. Notice that $U(t)$ is well defined as the solution to a Cauchy problem for a linear ordinary differential equation in $\mathscr{L}(L^2_+)$. Since $B_{u(t)}$ is antiselfadjoint, we have

$$\frac{d}{dt}U(t)^*U(t) = U(t)^*\left(-B_{u(t)} + B_{u(t)}\right)U(t) = 0$$

so that $U(t)^*U(t) = U(0)^*U(0) = \mathrm{Id}$. Furthermore,

$$\frac{d}{dt}U(t)U(t)^* = B_{u(t)}U(t)U(t)^* - U(t)U(t)^*B_{u(t)}, U(0)U(0)^* = \mathrm{Id}\ .$$

The unique solution of this linear Cauchy problem is Id. Hence $U(t)U(t)^* = \mathrm{Id}$, and $U(t)$ is unitary. Finally,

$$\frac{d}{dt}U(t)^*L_{u(t)}U(t) = U(t)^*\left(-B_{u(t)}L_{u(t)} + \frac{d}{dt}L_{u(t)} + L_{u(t)}B_{u(t)}\right)U(t) = 0,$$

so that

$$U(t)^*L_{u(t)}U(t) = U(0)^*L_{u(0)}U(0) = L_{u(0)},$$

and the proposition follows. □

Remark 2.3. *Additional properties of $U(t)$. In view of the formula (1.7) giving B_u, it is clear that $B_u : H^1_+(\mathbb{R}) \to H^1_+(\mathbb{R})$ of $u \in H^2(\mathbb{R})$, and that $t \mapsto B_{u(t)}$ is a continuous function valued in bounded operators on H^1_+. Furthermore, from Lemma 1.9, this map is also continuous with values in bounded operators on the Banach space* $\mathrm{Dom}(X^*)$*, endowed with the graph norm. Consequently, $U(t)$ is also continuous with valued bounded operators on $H^1_+(\mathbb{R})$ and on* $\mathrm{Dom}(X^*)$.

Let us conclude the proof of the explicit formula for the BO solution. Since $u(t,.)$ is real valued,

$$u(t,x) = \Pi u(t,x) + \overline{\Pi u(t,x)}.$$

Now we apply the variant (1.3) of the Cauchy formula,

$$f(z) = \frac{1}{2i\pi} \lim_{\epsilon\to 0^+} \langle (X^* - z\mathrm{Id})^{-1} f, \chi_\epsilon\rangle_{L^2}$$

to $f := \Pi u(t,.)$, and we let the unitary operator $U(t)^*$ act on both sides of this inner product,

$$\Pi u(t,z) = \frac{1}{2i\pi} \lim_{\epsilon\to 0^+} \langle (U(t)^* X^* U(t) - z\mathrm{Id})^{-1} U(t)^* \Pi u(t), U(t)^*\chi_\epsilon\rangle. \tag{2.3}$$

Then we calculate the quantities $U(t)^*\Pi u(t), U(t)^*\chi_\epsilon, U(t)^*X^*U(t)$ in terms of u_0 only, using the ODE $U'(t) = B_{u(t)}U(t)$, the crucial identities (**??**), (2.1), and (1.10). We obtain, using (**??**),

$$\frac{d}{dt} U(t)^*\Pi u(t) = U(t)^*(\partial_t \Pi u(t) - B_u \Pi u(t)) = iU(t)^* L_u^2 \Pi u(t) = iL_{u_0}^2 U(t)^* \Pi u(t),$$

hence

$$U(t)^*\Pi u(t) = \mathrm{e}^{itL_{u_0}^2} \Pi u_0 \ . \tag{2.4}$$

Similarly, in L^2_+, locally uniformly in the time variable, we have, using (1.10),

$$\frac{d}{dt} U(t)^*\chi_\epsilon = -U(t)^* B_u \chi_\epsilon = iU(t)^* L_u^2 \chi_\epsilon + o(1),$$

hence

$$U(t)^*\chi_\epsilon = \mathrm{e}^{itL_{u_0}^2}\chi_\epsilon + o(1). \tag{2.5}$$

Finally, using (2.1),

$$\begin{aligned}\frac{d}{dt} U(t)^* X^* U(t) &= U(t)^*[X^*, B_u]U(t)\\ &= U(t)^*(-2L_u - i[X^*, L_u^2])U(t)\\ &= -2L_{u_0} - i[U(t)^*X^*U(t), L_{u_0}^2],\end{aligned}$$

hence

$$\mathrm{e}^{-itL_{u_0}^2} U(t)^* X^* U(t) \mathrm{e}^{itL_{u_0}^2} = -2tL_{u_0} + X^*. \tag{2.6}$$

It remains to plug identities (2.4), (2.5) and (2.6) in (2.3). We obtain

$$\begin{aligned}\Pi u(t,z) &= \frac{1}{2i\pi} \lim_{\epsilon\to 0^+} \langle (U(t)^*X^*U(t) - z\mathrm{Id})^{-1} \mathrm{e}^{itL_{u_0}^2}\Pi u_0, \mathrm{e}^{itL_{u_0}^2}\chi_\epsilon + o(1)\rangle\\ &= \frac{1}{2i\pi} \lim_{\epsilon\to 0^+} \langle (-2tL_{u_0} + X^* - z\mathrm{Id})^{-1}\Pi u_0, \chi_\epsilon\rangle\\ &= \frac{1}{2i\pi} I_+((X^* - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0),\end{aligned}$$

which is (2.2). ☐

Remark 2.4. *Of course the formula (2.2) makes sense for less regular initial data. Assuming $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is enough to give a meaning to the right hand side of the formula, since, by formula (1.3),*

$$X^* - 2tD = \mathrm{e}^{-itD^2} X^* \mathrm{e}^{itD^2},$$

and

$$X^* - 2tL_{u_0} = \mathrm{e}^{-itD^2}(X^* + A_t)\mathrm{e}^{itD^2},$$

where

$$A_t := 2t\mathrm{e}^{itD^2} T_{u_0} \mathrm{e}^{-itD^2}$$

is bounded and selfadjoint. Therefore $i(X^* + A_t)$ is also maximally accretive, and the resolvent of $X^* + A_t$ is well defined in the upper half plane, so that the explicit formula reads

$$\Pi u(t,z) = \frac{1}{2i\pi} I_+((X^* + A_t - z)^{-1} \mathrm{e}^{itD^2} \Pi u_0). \tag{2.7}$$

Xi Chen [Che25a] recently used this observation to extend this formula to L^2 data, by taking advantage of the dispersion inequality for the free Schrödinger group e^{itD^2}. See also [KLV24] for a different approach in the context of commuting flows.

The proof of Theorem 2.1 shows that the operator $U(t)$ itself can be represented by an explicit formula.

Corollary 2.5. *Under the assumptions of Theorem 2.1, we have, for every $f \in L^2_+(\mathbb{R})$,*

$$\forall z \in \mathbb{C}_+ \ , \ U(t)\mathrm{e}^{itL^2_{u_0}} f(z) = \frac{1}{2i\pi} I_+((X^* - 2tL_{u_0} - z\mathrm{Id})^{-1} f).$$

2.2. **Explicit formulae for other Hamiltonian equations.** The above method can be applied to other non–local Hamiltonian PDE and leads to similar explicit formulae. A first example is the Calogero–Moser DNLS equation,

$$\left\{ \begin{array}{lcl} i\partial_t u + \partial_x^2 u &=& \pm 2D\Pi(|u|^2)u, \ u = \Pi u, \ x,t \in \mathbb{R} \\ u(0,x) &=& u_0(x) \end{array} \right. \tag{2.8}$$

Here the sign choice $\pm$ corresponds to the defocusing/focusing equation. In this case, one can prove [GL24], [Gér26] the following Lax pair identity for a solution $u \in C(I, H^2_+(\mathbb{R})$ on the time interval I,

$$\frac{d}{dt} L_u = [B_u, L_u], \quad L_u := D \pm T_u T_{\overline{u}}, \quad B_u := \mp(T_u T_{\partial_x \overline{u}} - T_{\partial_x u} T_{\overline{u}}) + i(T_u T_{\overline{u}})^2. \tag{2.9}$$

Then the explicit formula reads [KLV25], [Gér26],

$$\forall t \in I, \quad \forall z \in \mathbb{C}_+ \ , \ u(t,z) = \frac{1}{2i\pi} I_+((X^* + 2tL_{u_0} - z\mathrm{Id})^{-1} u_0). \tag{2.10}$$

A second example is the cubic Szegő equation on the line,

$$\left\{ \begin{array}{lcl} i\partial_t u &=& \Pi(|u|^2 u), \ u = u(t,x), \ x,t \in \mathbb{R} \\ u(0,x) &=& u_0(x) \end{array} \right. \tag{2.11}$$

where $u \in C(\mathbb{R}, H^{\frac12}_+(\mathbb{R}))$, and where, for every $s \geq 0$,

$$H^s_+(\mathbb{R}) := \{ f \in L^2_+(\mathbb{R}), \ \xi \mapsto \xi^s \hat{f}(\xi) \in L^2_+([0,\infty[) \}$$

equipped with the norm

$$\|f\|^2_{H^s} := \frac{1}{2\pi} \int_{\mathbb{R}} (1+\xi^2)^s |\hat{f}|^2(\xi) d\xi.$$

The global wellposedness of (2.11) in $H^s_+(\mathbb{R})$ for every $s \geq 0$ has been established in [Poc11]. Moreover, it has been shown that (2.11) enjoys a Lax pair structure, which we briefly describe below. We recall that the Hankel operator of symbol u is defined on L^2_+ as $H_u(f) = \Pi(u\overline{f})$, when well defined, where Π is the orthogonal projector from L^2 into L^2_+, the so-called Szegő projector. Equivalently, one has

$$\forall f \in L^2(\mathbb{R}), \quad \Pi(f)(x) = \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{ix\xi}\hat{f}(\xi)d\xi$$

and

$$\forall f \in L^2_+(\mathbb{R}), \quad \widehat{H_u(f)}(\xi) = \frac{1}{2\pi}\int_0^\infty \hat{u}(\xi+\eta)\overline{\hat{f}(\eta)}d\eta.$$

It is well known that a Hankel operator H_u is a Hilbert-Schmidt operator on $L^2_+(\mathbb{R})$ whenever $u \in H^{\frac12}_+(\mathbb{R})$. Moreover, computing the trace of H_u^2, one gets

$$Tr(H_u^2) = \frac{1}{(2\pi)^2}\int_0^\infty \xi|\hat{u}|^2(\xi)d\xi =: \frac{1}{2\pi}\|u\|^2_{\dot{H}^{\frac12}}. \tag{2.12}$$

Here $\dot{H}^{\frac12}$ denotes the homogeneous Sobolev space. With this notation, for every $u_0 \in H^s_+(\mathbb{R})$ with $s > \frac12$, the solution $u \in C(\mathbb{R}, H^s_+(\mathbb{R}))$ (2.11) satisfies

$$\frac{dH_u}{dt} = [B_u, H_u], \tag{2.13}$$

where $B_u = -iT_{|u|^2} + \frac{i}{2}H_u^2$. Notice that this implies

$$\frac{dH_u^2}{dt} = [-iT_{|u|^2}, H_u^2], \quad \partial_t u - -iT_{|u|^2}u.$$

Applying the above method to this new equation leads to the following explicit formula, obtained by the author of these notes and A. Pushnitski (see [GP24]).

Theorem 2.6 ([GP24]). *Let $z \in \mathbb{C}_+$ and $u_0 \in H^{\frac12}_+(\mathbb{R})$. Then the solution of (2.11) is given by*

$$\forall z \in \mathbb{C}_+, \quad u(t,z) = \frac{1}{2i\pi}I_+((X^* + L_{u_0}(t) - z)^{-1}\mathrm{e}^{-itH^2_{u_0}}u_0). \tag{2.14}$$

Here

$$L_{u_0}(t) = \frac{1}{2\pi}\int_0^t \langle \cdot, \mathrm{e}^{-isH^2_{u_0}}u_0\rangle \mathrm{e}^{-isH^2_{u_0}}u_0 ds.$$

3. Applications to long–time behavior : user's guide

In this section, we explain how to use the explicit formulae derived in the previous section with applications to the long–time behavior. Again we give some emphasis to the Benjamin–Ono case.

3.1. **The first level : solitons.** The solitary waves for the Benjamin–Ono equation (1.6) were characterized in [AT91]. They are of the form

$$u(t,x) = R_p(x - c_pt), \quad R_p(y) := \frac{2\mathrm{Im}p}{|y+p|^2}, \quad c_p := \frac{1}{\mathrm{Im}p}, \quad p \in \mathbb{C}_+. \tag{3.1}$$

If $u_0 = R_p$, one can easily check that L_{u_0} has only one eigenvalue $\lambda_1 = -1/(2\mathrm{Im}p)$, and that $\Pi u_0 = i/(x+p)$ is the corresponding eigenfunction. Since we only observed in Example 1.4 that $X^*u_0 = -pu_0$, the resolvent term in the explicit formula (2.2) is collinear to u_0, and we finally recover the soliton identity,

$$\Pi u(t,x) = \Pi u_0(x - c_pt).$$

3.2. **The second level : multi–solitons.** The algebraic analysis in the previous subsection can be extended to a finite dimensional landscape if u_0 is a finite sum of finite functions R_{p_j}, $p_j \in \mathbb{C}_+, j = 1, \cdots, N$. For simplicity, let us assume here that the p_j are pairwise distinct. Then one can check that L_{u_0} preserves the N–dimensional subspace $\mathcal{E}$ spanned by the ΠR_{p_j}, which is also preserved by the action of X^*. Since obviously Πu_0 belongs to $\mathcal{E}$, the resolvent term in the explicit formula (2.2) can be calculated by inverting a $N \times N$ matrix, see [Sun21] where results by Matsuno [Mat79] are recovered. One gets

$$u(t,x) = \sum_{j=1}^{N} R_{p_j(t)},$$

where the complex numbers $p_j(t)$ are solutions of a finite dimensional integrable system, explicitely solved by this method. The long–time behaviour can also be easily described : up to a small error in all the Sobolev spaces, the solution can be decomposed as a sum of N solitons with different velocities.

3.3. **The third level : rational data.** Assume u_0 is a rational function in $L^2(\mathbb{R})$. Then the equation for $f := (X^* - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi \mathrm{u}_0$ take a particularly simple form. Again for simplicity, let us assume

$$u_0(x) = \frac{2c}{|x+p|^2}, p \in \mathbb{C}_+, c \in \mathbb{R}.$$

Then, in view of Example 1.1 and of the description (1.2), (1.1) of the action of X^*, the equation on f reads

$$xf(x) + \lambda_f f(x) + 2itf'(x) - ic\left(\frac{f(x)}{x+p} - \frac{f(x) - f(-\overline{p})}{x+\overline{p}}\right) = i\frac{c}{x+p}.$$

This equation is a simple first order linear ODE on f, which can be solved explicitly in terms of the two parameters λ_f and $f(-\overline{p})$. The additional condition that f belongs to the Hardy space imposes that f is holomorphic near the singular point $x = -\overline{p}$, and the finiteness of the L^2 norm of f. This leads to a system of the linear equations on the two parameters λ_f and $f(-\overline{p})$. Since the first unknown λ_f of this system is precisely $u(t,x)$, we infer that the solution can be explicitly described a the quotient of two 2×2 determinants involving integrals in the complex domain. The general approach for Πu_0 rational with N different poles in the lower half–plane is developed in [BGGM25], leading to a representation of the solution as a quotient of two $(N+1) \times (N+1)$ determinants. In the special case, where $c = -1, p = i$ — minus a soliton — this expression is used to prove that the solution scatters as $t \to \infty$.

3.4. **The fourth level : Benjamin–Ono Lax spectrum and soliton resolution.** First we discuss the spectrum of the Lax operator for the Benjamin–Ono equation. The following theorem is due to Y. Wu [Wu16, Wu17], and Blackstone–Gassot–Gérard–Miller, [BGGM25], Gassot–Gérard–Miller [GGM26]. More information about more general potentials u can be found in Badreddine–Killip–Vişan [BKV25].

Theorem 3.1. *Assume $u \in H^1(\mathbb{R})$ and $xu \in L^2(\mathbb{R})$.The operator L_u has only finitely many simple eigenvalues λ_j, $j = 1, \dots, N$, which are negative, with L^2– normalized eigenfunctions φ_j satisfying*

$$-2\pi\lambda_j = |\langle \Pi u, \varphi_j\rangle_{L^2}|^2 \ , \ \ \varphi_j \in \mathrm{Dom}(X^*), \qquad -\lambda_j I_+(\varphi_j) = \langle \varphi_j, \Pi u\rangle_{L^2}.$$

On $[0,\infty)$, L_u has absolutely continuous spectrum. More precisely, for every $\lambda > 0$, there exists unique L^∞ functions $m_-(.\lambda), m_+(.,\lambda)$ such that

$$(Dm_\pm(.,\lambda) - \Pi(um_\pm(.,\lambda)) = \lambda m_\pm(.,\lambda), \quad \mathrm{e}^{-i\lambda x} m_\pm(x,\lambda) \underset{x\to\pm\infty}{\longrightarrow} 1.$$

Functions $m_+(.,\lambda)$ and $m_-(.\lambda)$ are collinear

$$m_+(x,\lambda) = \ell(\lambda) m_-(x,\lambda), \quad |\ell(\lambda)| = 1.$$

Furthermore, the linear maps

$$f \in L^2_+(\mathbb{R}) \cap L^1(\mathbb{R}) \mapsto \tilde{f}^{\pm}(\lambda) = \int_{\mathbb{R}} f(x)\overline{m_{\pm}(x,\lambda)}\, dx$$

extend continuously from $L^2_+(\mathbb{R})$ to $L^2(0,\infty)$, leading to the following distorted Plancherel formula,

$$\forall f \in L^2_+(\mathbb{R}), \quad \|f\|^2_{L^2} = \sum_{j=1}^N |\langle f, \varphi_j\rangle|^2 + \frac{1}{2\pi}\int_0^\infty |\tilde{f}^{\pm}(\lambda)|^2\, d\lambda. \tag{3.2}$$

Combining Theorem 3.1 and the explicit formula (2.2) of Theorem 2.1, one can prove the following soliton resolution result.

Theorem 3.2 ([GGM26]). *Assume that u_0 is the sum of a Schwartz function and of a rational function such that $(1+|x|)u_0 \in L^2(\mathbb{R})$[1] . With the notation of Theorem 3.1, define*

$$p_j := -\langle X^*\varphi_j, \varphi_j\rangle \in \mathbb{C}_+, \quad c_{p_j} = -2\lambda_j, \quad j = 1, \cdots, N,$$

and define real valued functions $u^{\pm}_\infty \in \cap_{s\in\mathbb{R}} H^s(\mathbb{R})$ by

$$\forall \lambda > 0, \quad \hat{u}^{\pm}_\infty(\lambda) = \widetilde{\Pi u_0}^{\mp}(\lambda).$$

Then, as $t \to \pm\infty$, we have

$$u(t,x) - \sum_{j=1}^N R_{p_j}(x - c_{p_j} t) - \mathrm{e}^{t\partial_x|D|} u^{\pm}_\infty \to 0,$$

in all the H^s norms.

3.5. **Long–time asymptotics for some other equations.** For the defocusing Calogero–Moser derivative NLS equation (2.8), Xi Chen [Che25b] combined spectral theory for the Lax operator L_u given in (2.9) with the explicit formula (2.10) to prove long–time scattering in $L^2_+(\mathbb{R})$ for the almost critical data u_0 such that $(1+|x|)^\epsilon u_0 \in L^2$ for some $\epsilon > 0$, generalizing the result of [KK24]. In the focusing case, which is known to blow up in finite time [KKK24], it would interesting to use the explicit formula to describe a concrete class of initial data leading to such blow up solutions.

For the cubic Szegő equation (2.11), the explicit formula (2.14) was recently used in [GG26] to prove a soliton and breather resolution theorem in the space $H^{1/2}_+(\mathbb{R})$. In this case, the building blocks of the dynamics are traveling quasi–periodic breather solutions with special rational initial data. Here the interesting new phenomenon is that the velocity of the traveling patterns is nomore given by the Lax eigenvalues, but by the norming constants $\langle u_0, \varphi_j\rangle|^2/(2\pi)$, where φ_j are the L^2–normalized eigenfunctions.

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