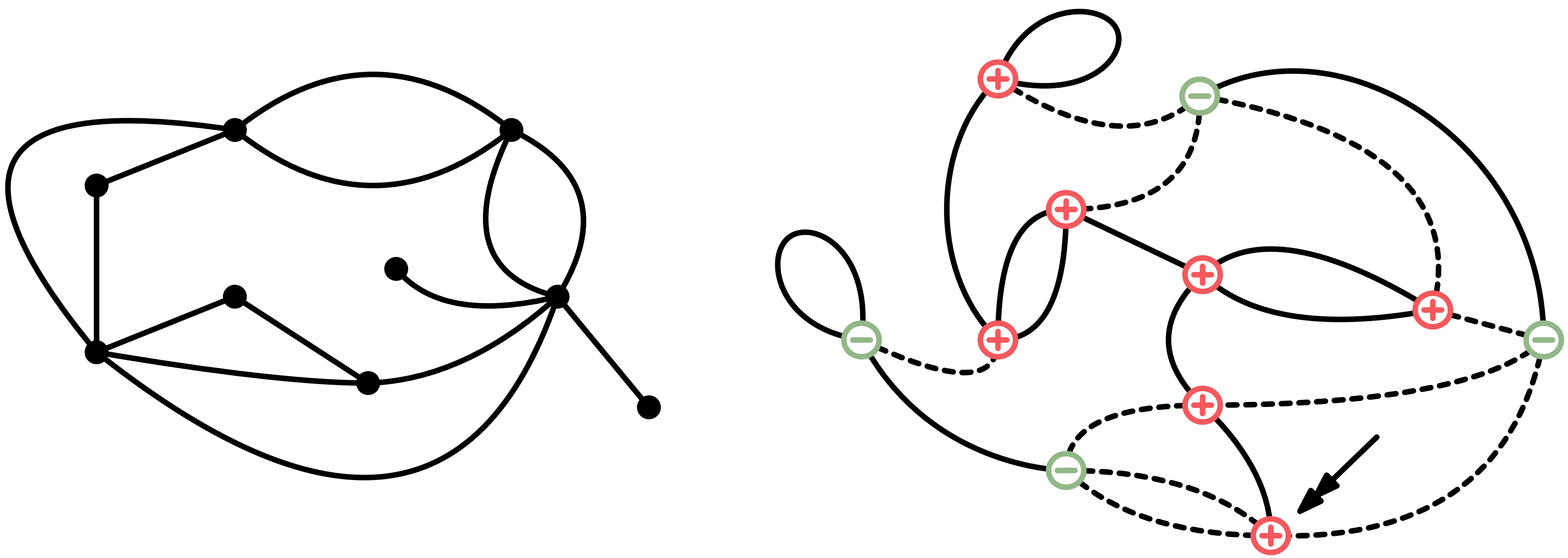
THE ISING MODEL ON RANDOM PLANAR MAPS

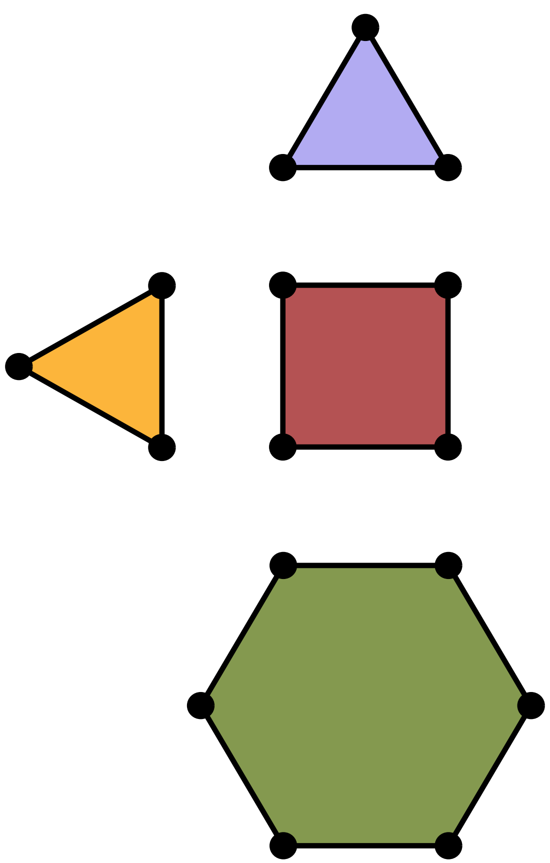
Nicolas Tokka (Univ. Paris-Nanterre, MODAL'X)
Supervised by **Marie Albenque** and **Laurent Ménard**



I PLANAR MAPS

Planar maps

① A planar map = polygons .



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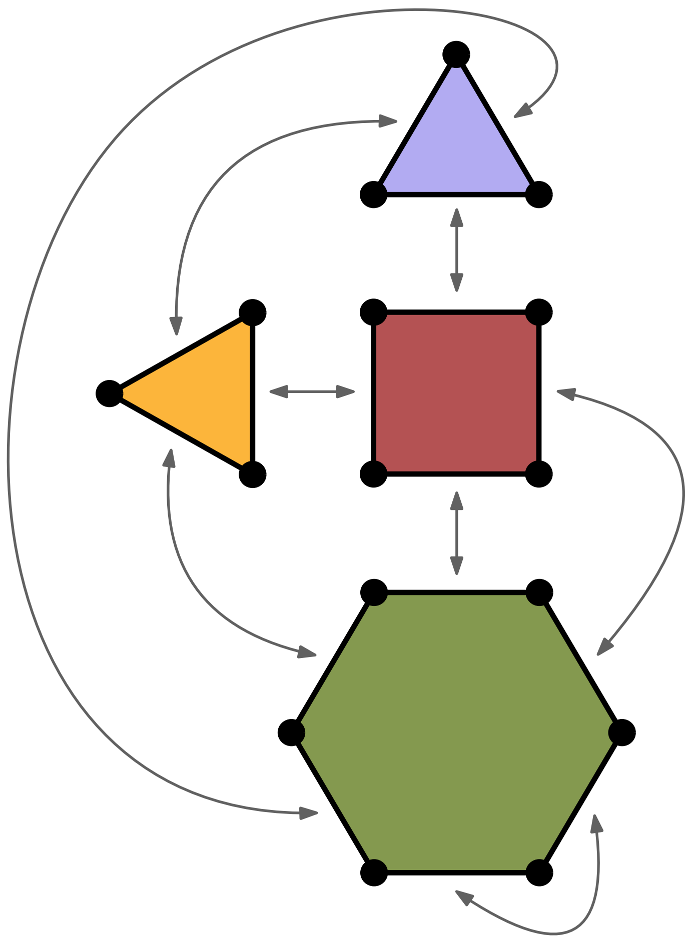

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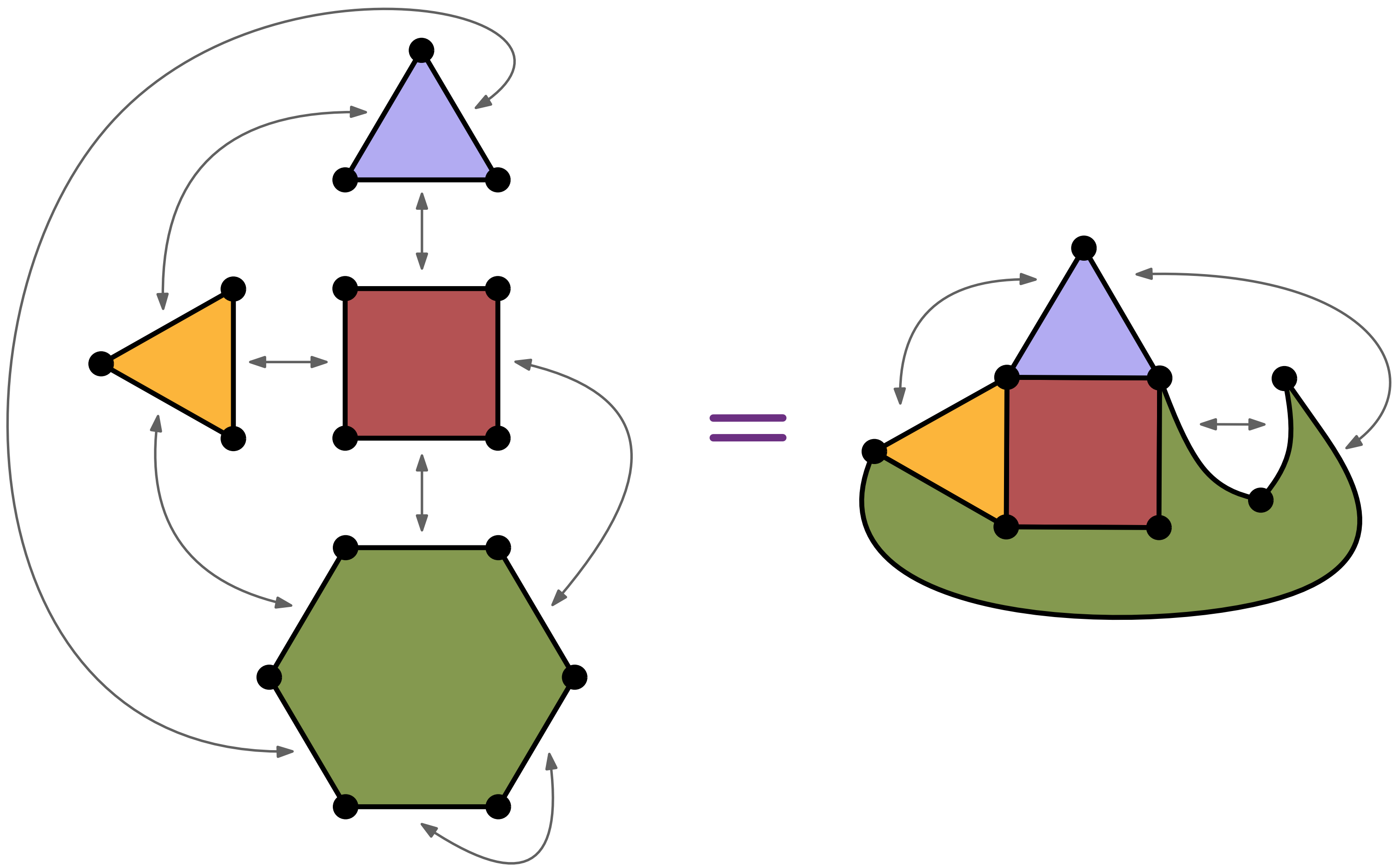

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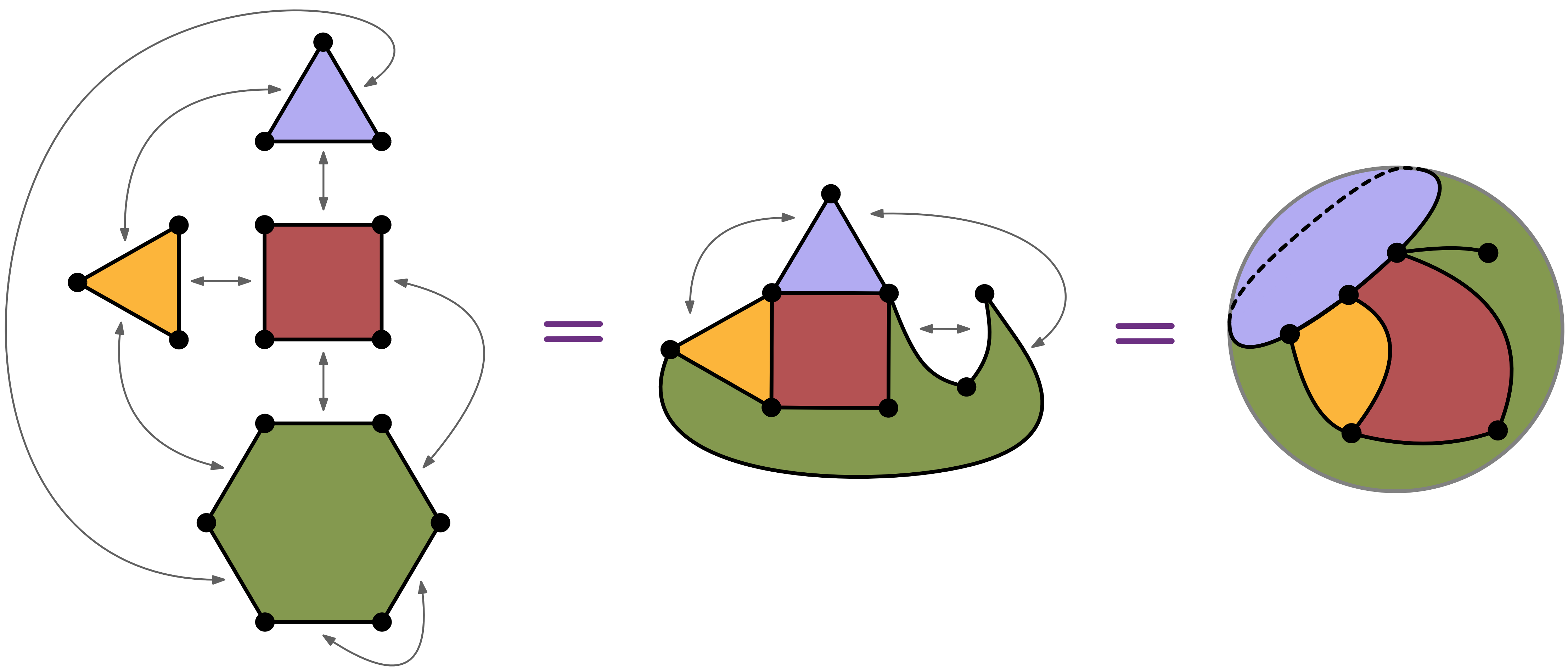
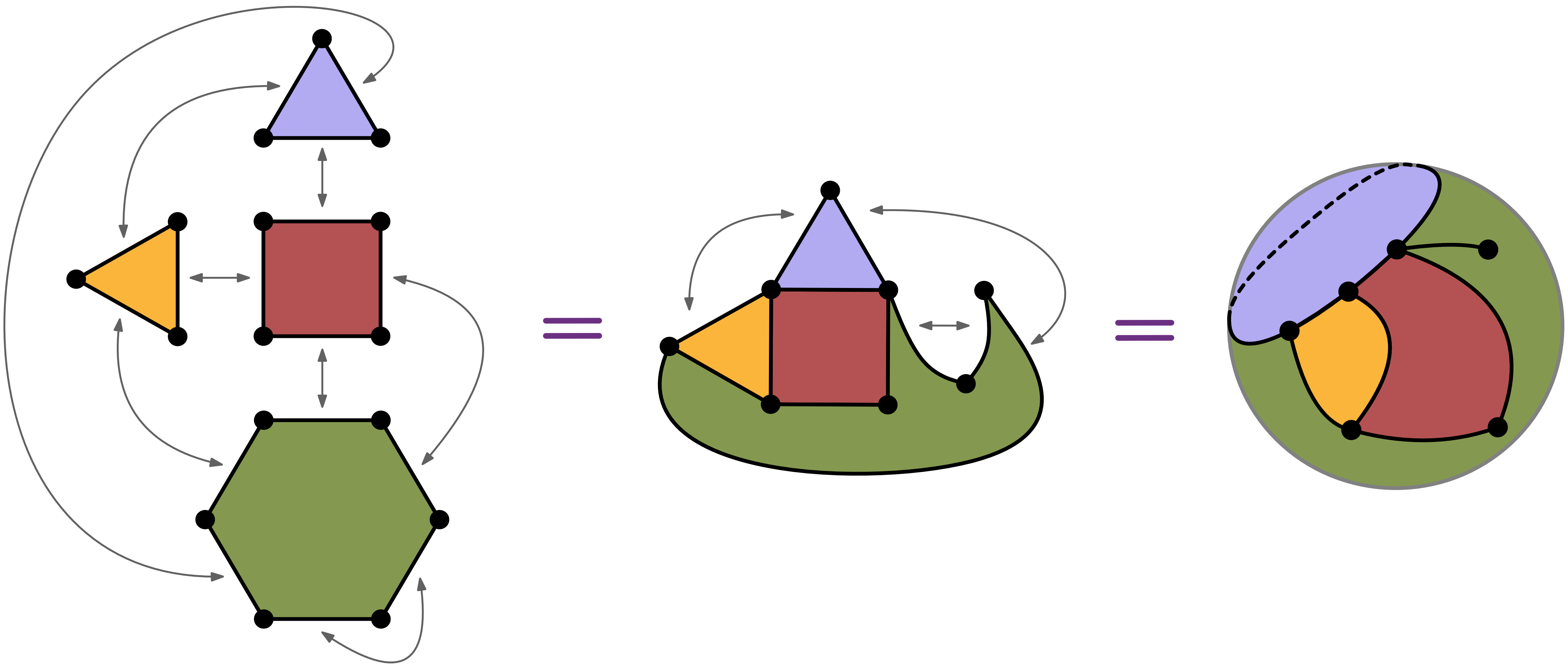


Planar maps

① A planar map = A gluing of polygons whose resulting surface is the sphere $\mathbb{S}^2$.



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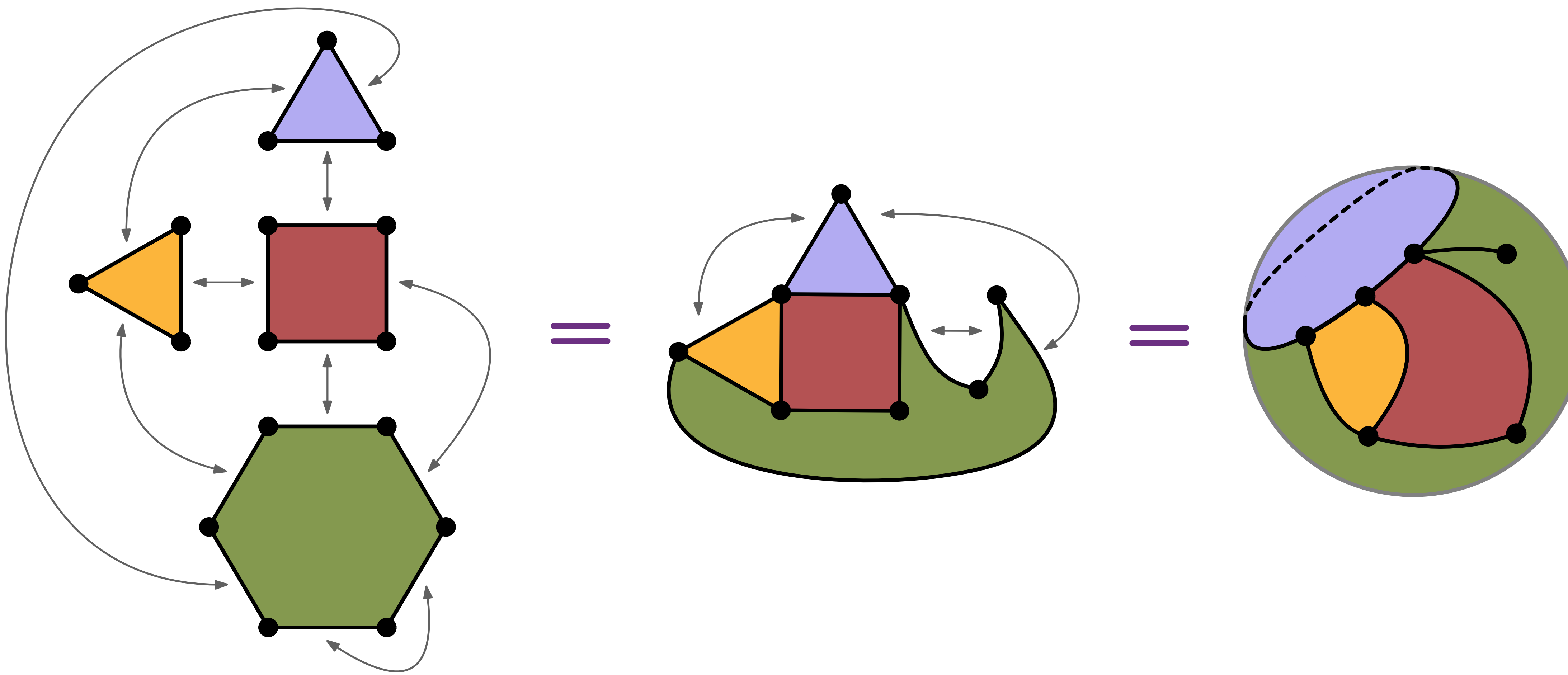
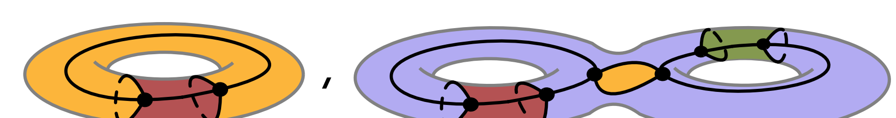


? A general map = A gluing of polygons:  …

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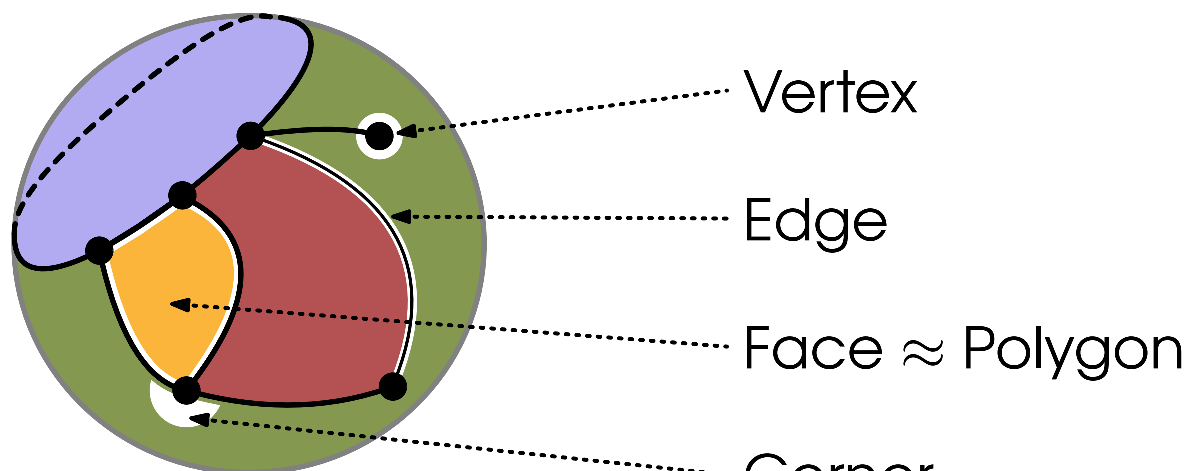

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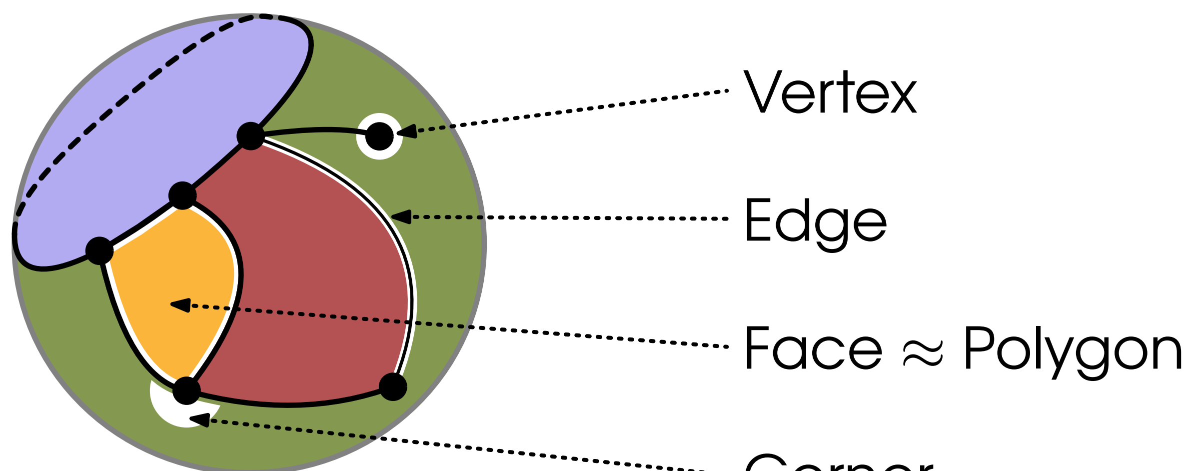
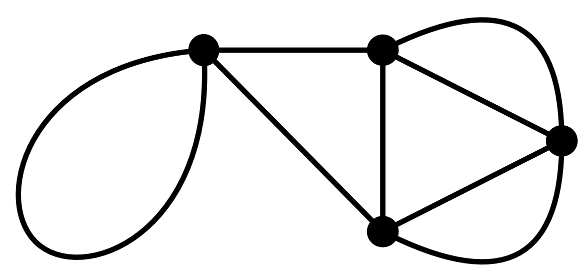
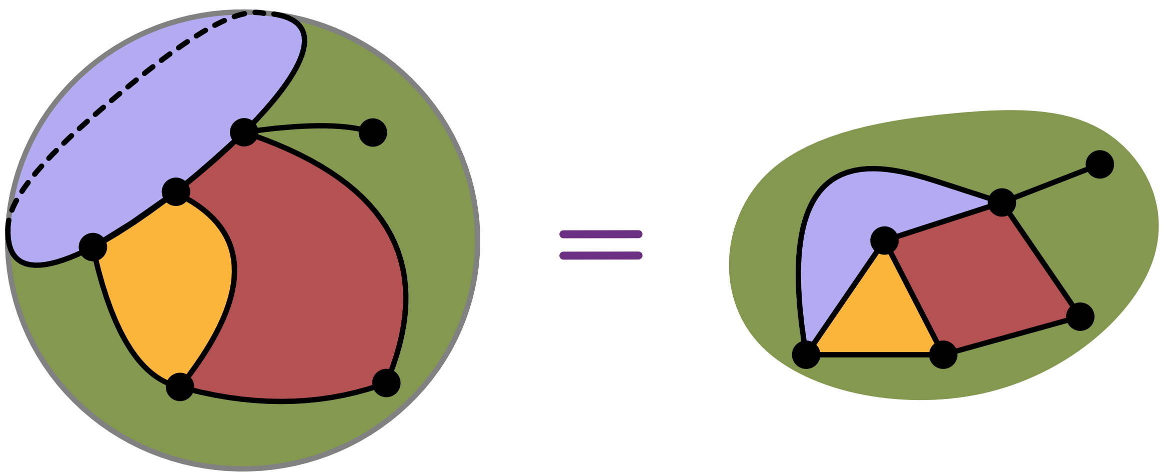


A quartic map = all vertices have degree 4.



Planar maps

② A planar map = Also a proper embedding of a connected (planar) graph in the sphere $\mathbb{S}^2$.



Remark: a map = planar graph + cyclic order of edges around each vertex.

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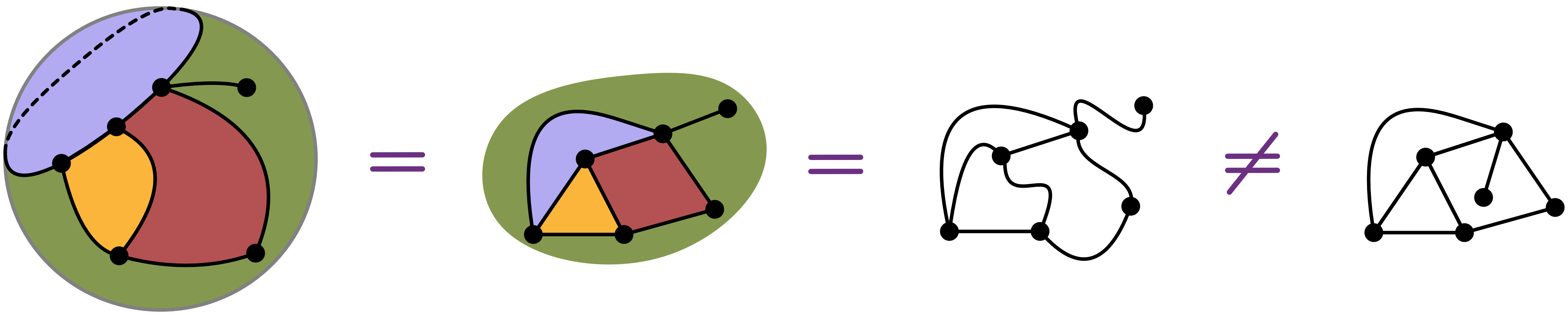

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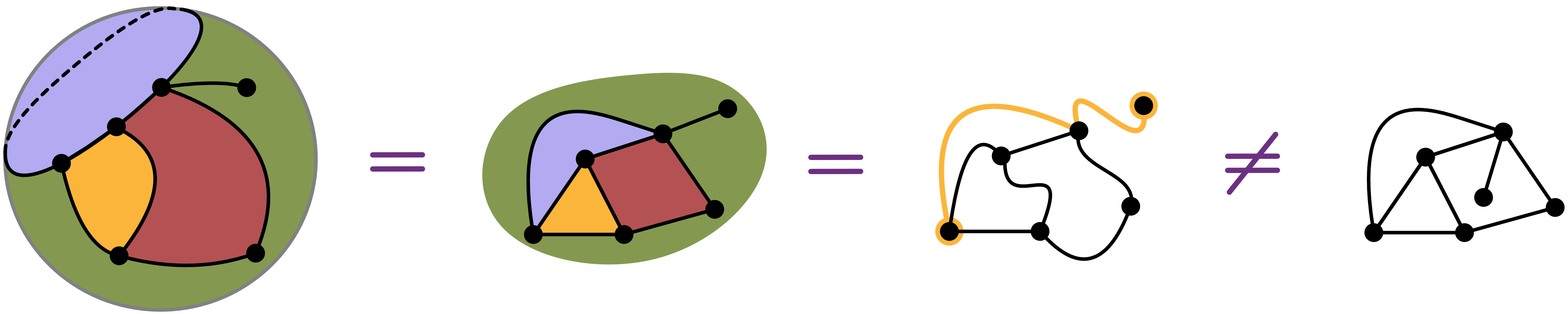


Remark: a map = planar graph + cyclic order of edges around each vertex.

Prop: A planar map m induces a metric space $\mathrm{m} \equiv (\mathrm{V}(\mathrm{m}), \mathrm{d}_{\mathrm{gr}})$:

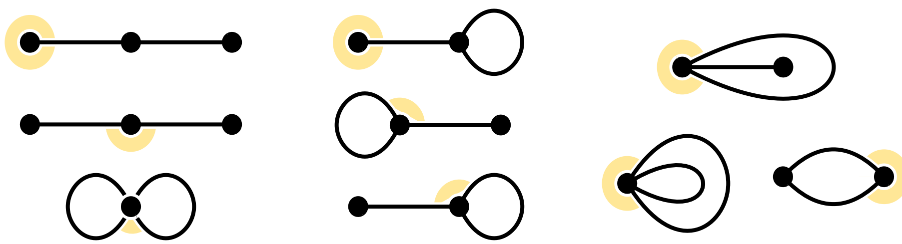
- $\mathrm{V}(\mathrm{m})$ is the set of vertices of m,
- d_{gr} is the graph distance of m.

History

Introduced in the 60′ by W. Tutte
[W. Tutte ′62, ′62, ′63, ′68...]

Ex: Planar maps with n edges

$$\frac{2\,3^n}{(n+1)(n+2)}\binom{2n}{n}$$



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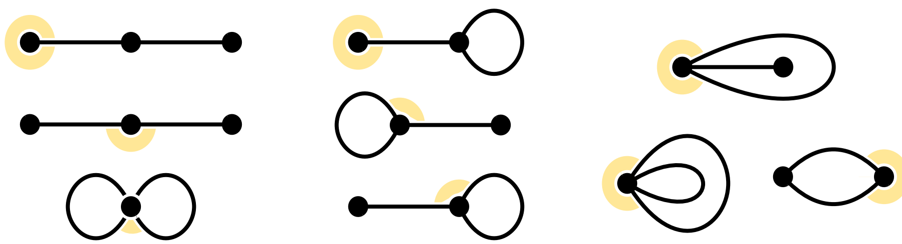

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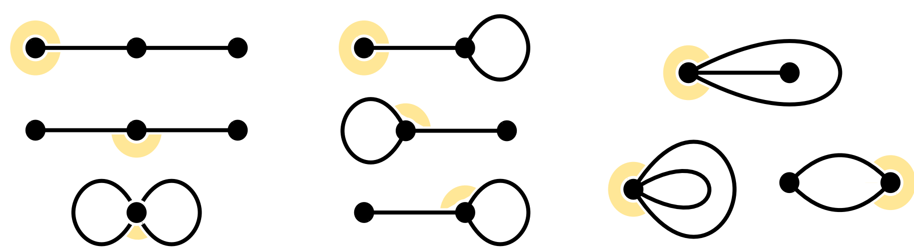

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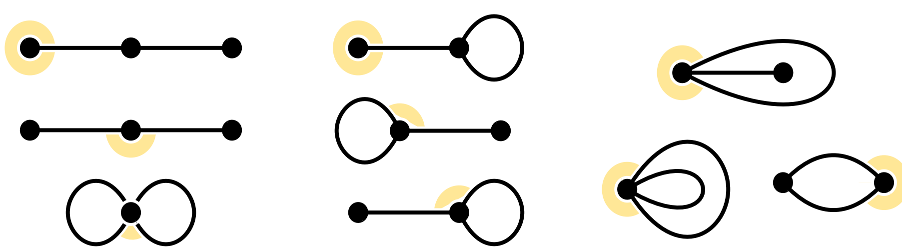

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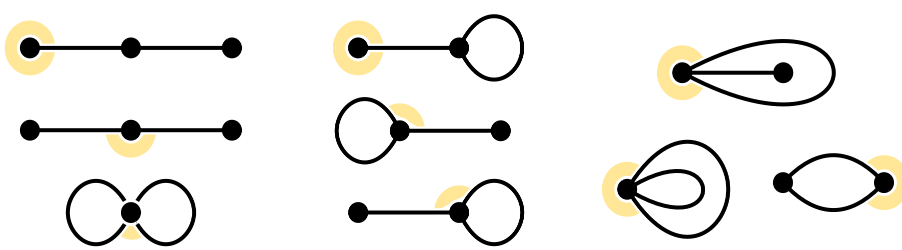

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Enumerative topology

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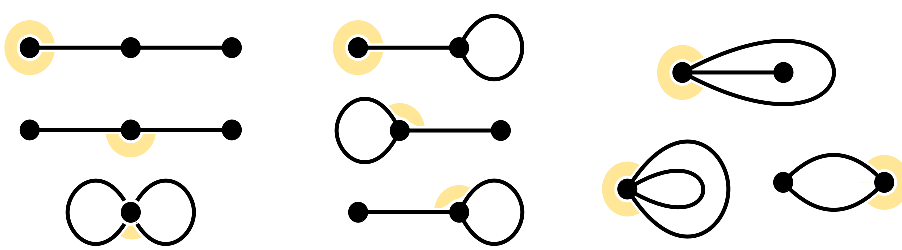

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Enumeration

① Some formulas

- Quadrangulations with n faces:
  [Tutte '60, Cori-Vauquelin '81, Schaeffer '98]

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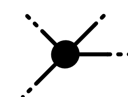
- Quartic maps with n vertices: 
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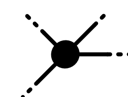
- Quartic maps with n vertices: 
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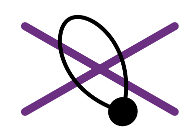
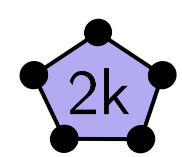
- Loopless planar maps with n edges: 
  [Tutte '62, Poulalhon-Schaeffer '06]

  $$\frac{2}{(3n+1)(3n+2)}\binom{4n+1}{n+1}.$$

- 2k-angulations with n faces: 

  [Bouttier-Di francesco-Guitter '04]

  $$\frac{2\binom{2k-1}{k}^n}{\big((k-1)n+1\big)\big((k-1)n+2\big)}\binom{kn+1}{n}.$$

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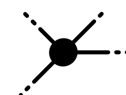
- Quartic maps with n vertices:  $\frac{2\,3^n}{(n+1)(n+2)}\binom{2n}{n}.$
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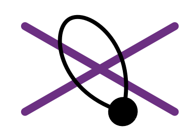
- Loopless planar maps with n edges:  $\frac{2}{(3n+1)(3n+2)}\binom{4n+1}{n+1}.$
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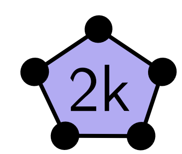
- 2k-angulations with n faces:  $\frac{2\binom{2k-1}{k}^n}{\big((k-1)n+1\big)\big((k-1)n+2\big)}\binom{kn+1}{n}.$
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② A large class of universality

$$|\mathrm{M}_n| \underset{n\to\infty}{\sim} C_1 \cdot (C_2)^n \cdot n^{-5/2}.$$

II RANDOM PLANAR MAPS

Random planar maps

① Uniform planar map

Let $\mathcal{M}_n :=$ { planar maps with n edges}, and $\mathrm{m}_n \sim \mathrm{Unif}(\mathcal{M}_n)$.

➞ $\mathrm{m}_n \equiv \big(\mathrm{V}(\mathrm{m}_n), \mathrm{d}_{\mathrm{gr}}\big)$ is random discrete metric space.

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② Question: What does it look like when n is large ?

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↙ Local limit

↘ Scaling limit

Random planar maps Scaling limit

① Rescaling ? Let $\mathrm{m}_n \sim \mathrm{Unif}(\mathcal{M}_n)$. Is there a sequence $(x_n)_n \in {\mathbb{R}_{>0}}^{\mathbb{N}}$ such that:

$$'' \left(\mathrm{V}(\mathrm{m}_n), \frac{\mathrm{d}_{\mathrm{gr}}}{x_n} \right) \underset{n\to\infty}{\longrightarrow} \textit{metric space}''$$

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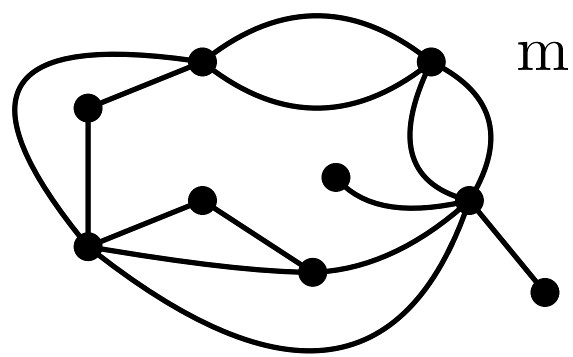
② Diameter Let m be a planar map. Its diameter is:

$$\mathrm{diam}(\mathrm{m}) := \max_{u,v \in \mathrm{V}(\mathrm{m})} \mathrm{d_{gr}}(u, v)$$



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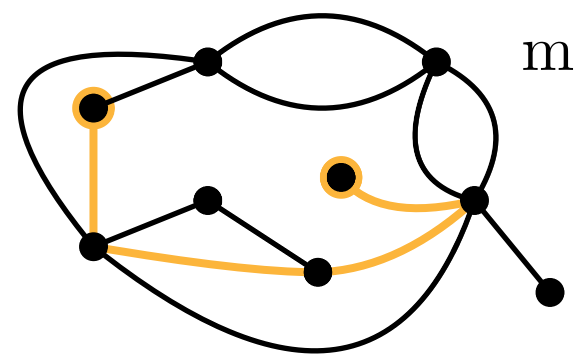

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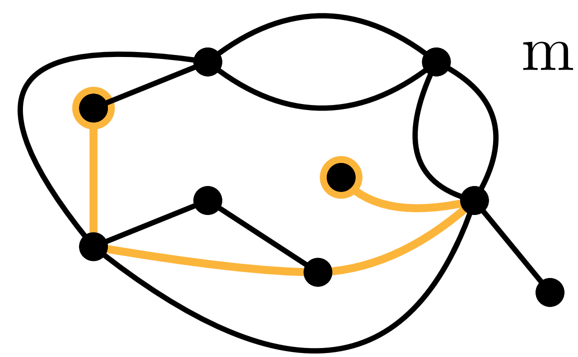

$\mathrm{diam}(\mathrm{m}) = 4$

Prop: Let $\mathrm{m}_n \sim \mathrm{Unif}(\mathcal{M}_n)$, then $\mathrm{diam}(\mathrm{m}_n)$ is of order $n^{1/4}$.

[Chassaing-Schaeffer '02]

Random planar maps

③ Convergence

THEOREM: [Miermont '13 $(p=4)$, Le Gall '13 $(p=3, p$ even$)$, Addario Berry-Albenque '21 $(p \geq 5$ odd$)$]

Let $\mathrm{m}_n \sim \mathrm{Unif}(\{p-$ angulations with n faces$\}$, there exists a random compact metric space $(\mathrm{m}_\infty, \mathrm{d})$ such that:

$$\left(\mathrm{V}(\mathrm{m}_n), C_p\, n^{-1/4}\, \mathrm{d}_{\mathrm{gr}}\right) \xrightarrow[n\to\infty]{} (\mathrm{m}_\infty, \mathrm{d}) \quad \text{in distribution,}$$

in the Gromov-Hausdorff sense. The law of $(\mathrm{m}_\infty, \mathrm{d})$ is called the **Brownian sphere**.

Random planar maps Scaling limit

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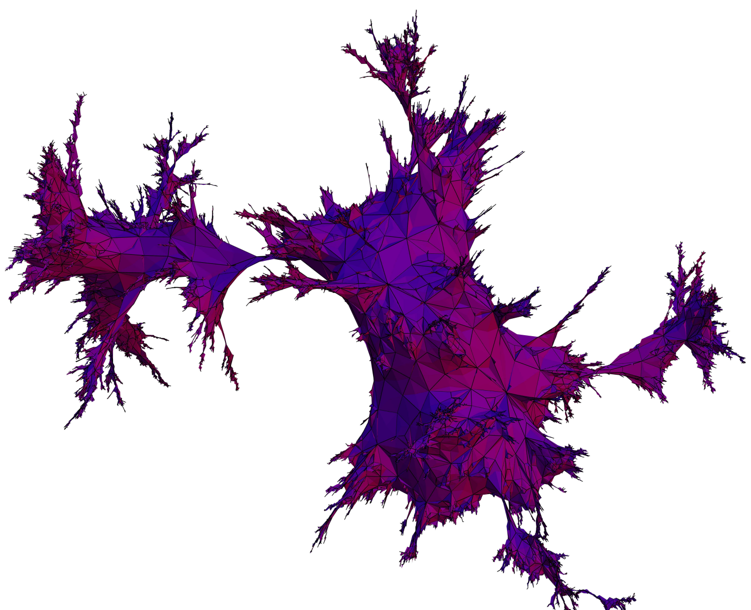
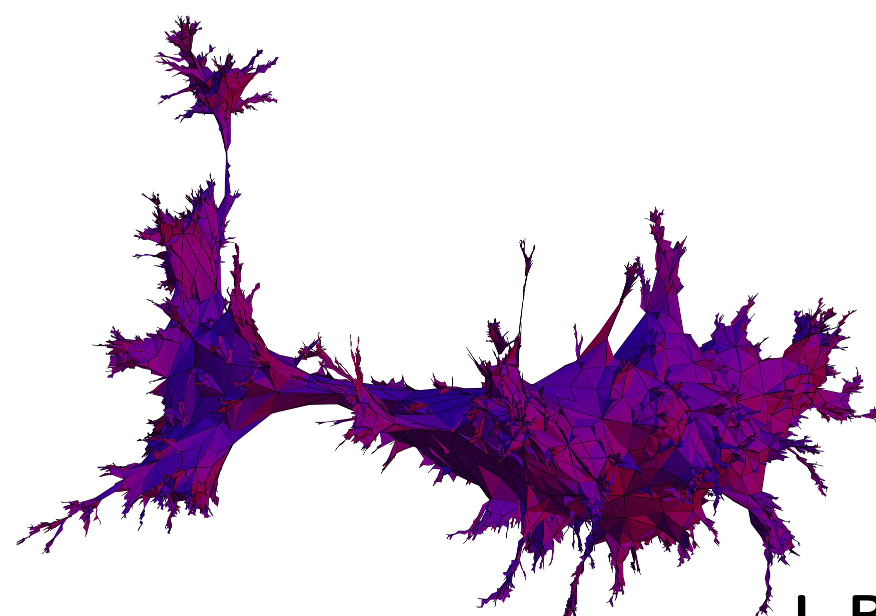
④ Does not depend on p, and same limit for other families of maps : universality.

⑤ Two fundamental properties





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THEOREM: [Le Gall-Paulin '08, Le Gall '07]

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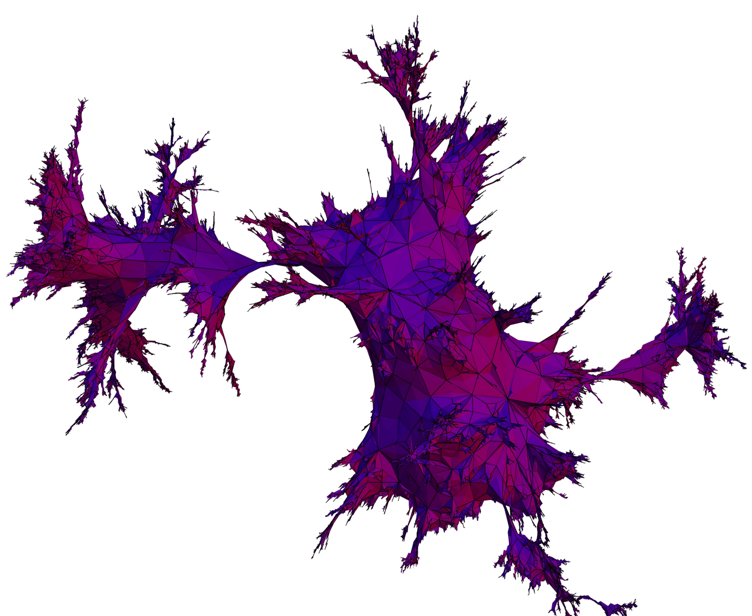

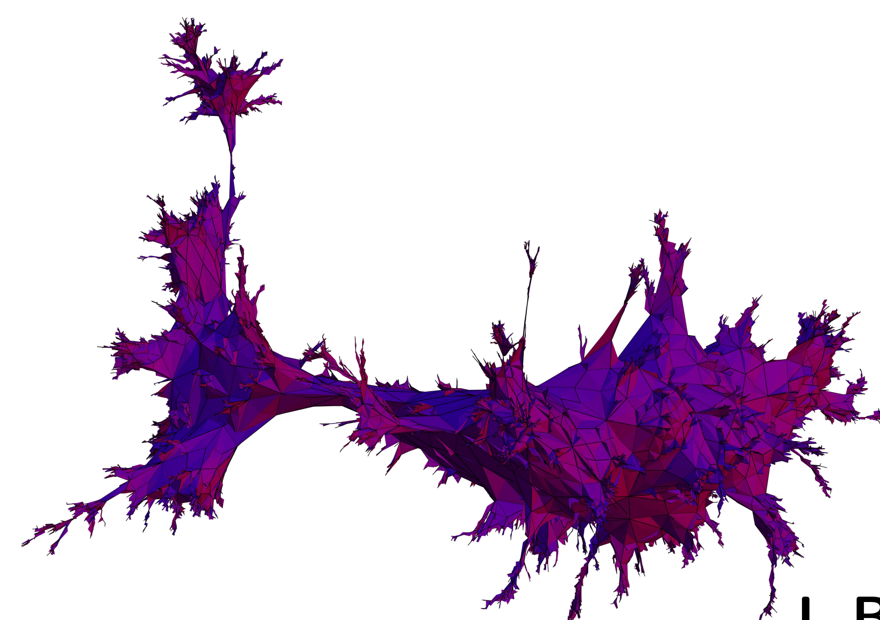

J. Bettinelli

Random planar maps Scaling limit

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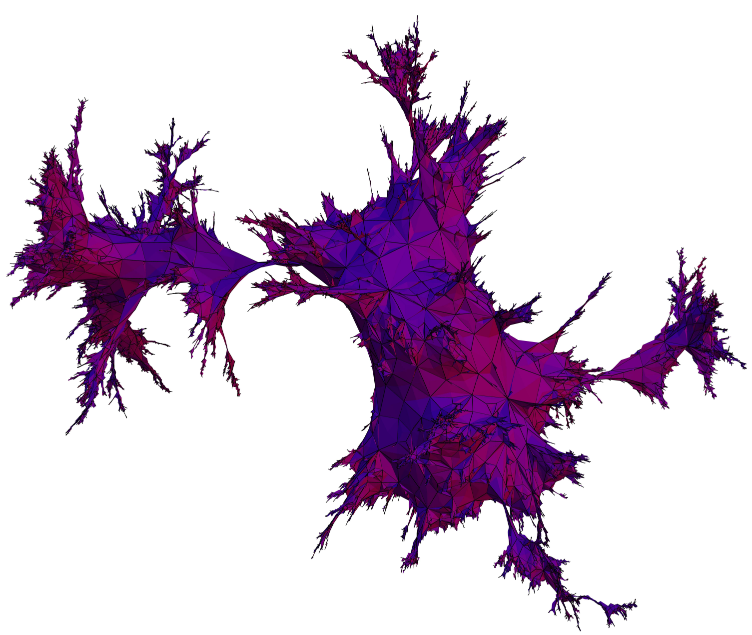

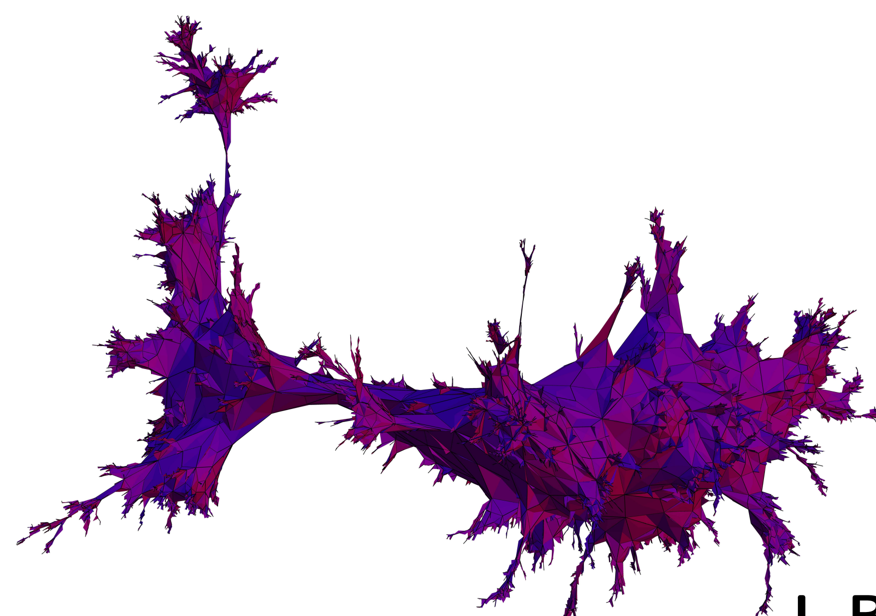

J. Bettinelli

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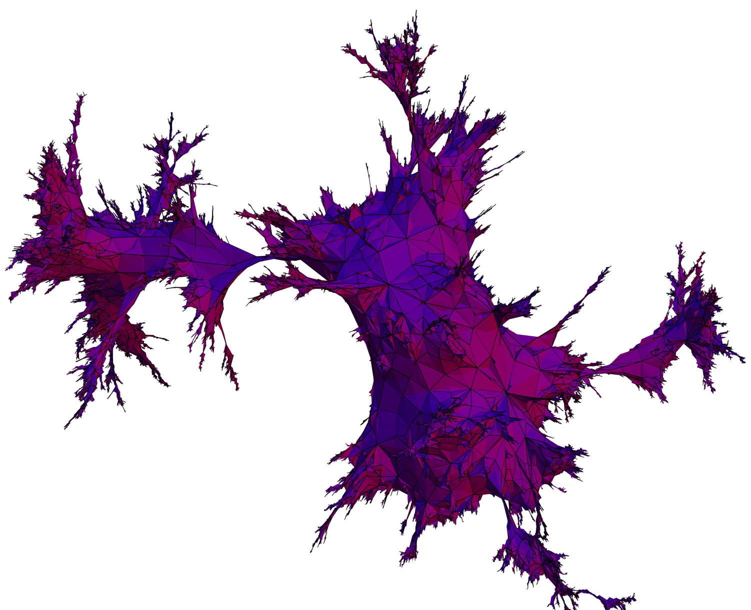

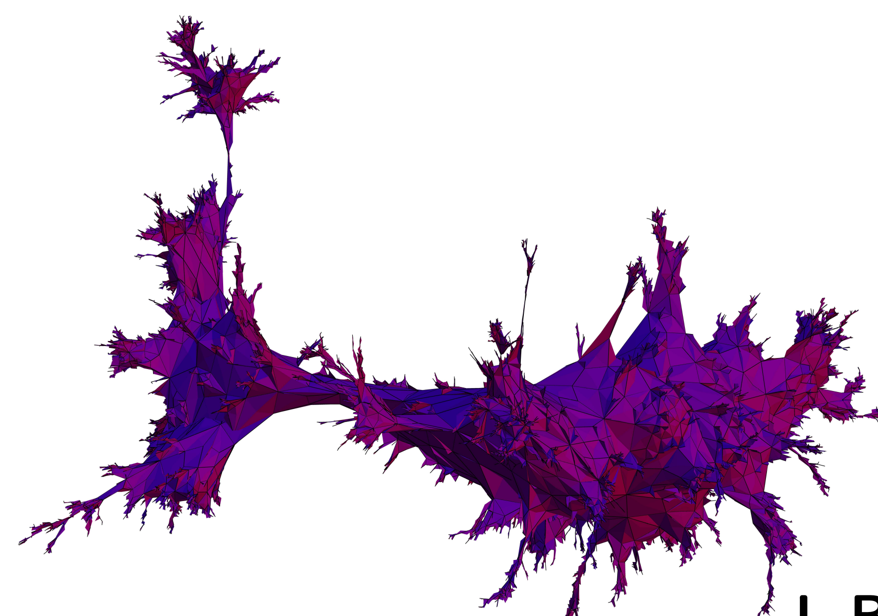

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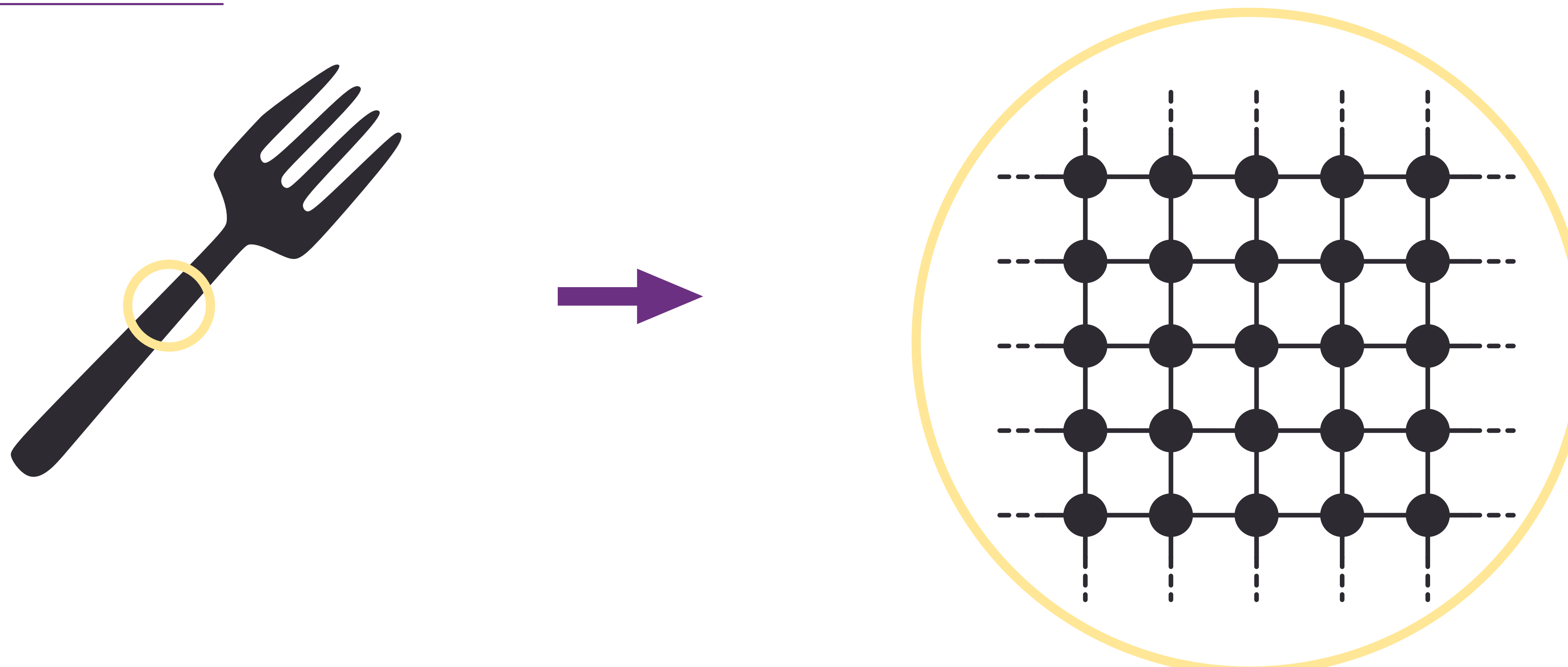
III THE ISING MODEL

Physics & spins

① Motivations



Human scale

Microscopic scale

Physics & spins

① Motivations

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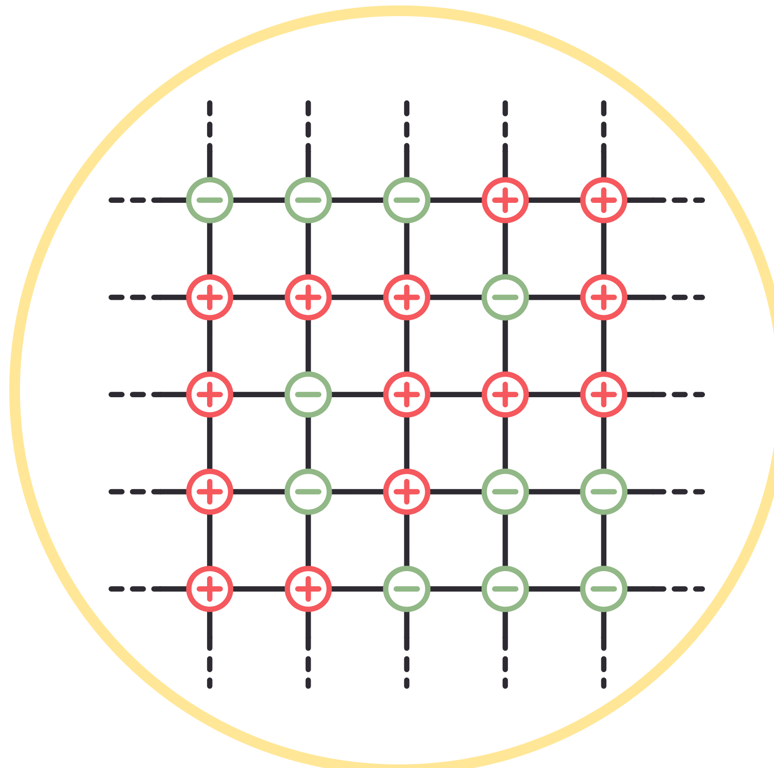


Microscopic scale

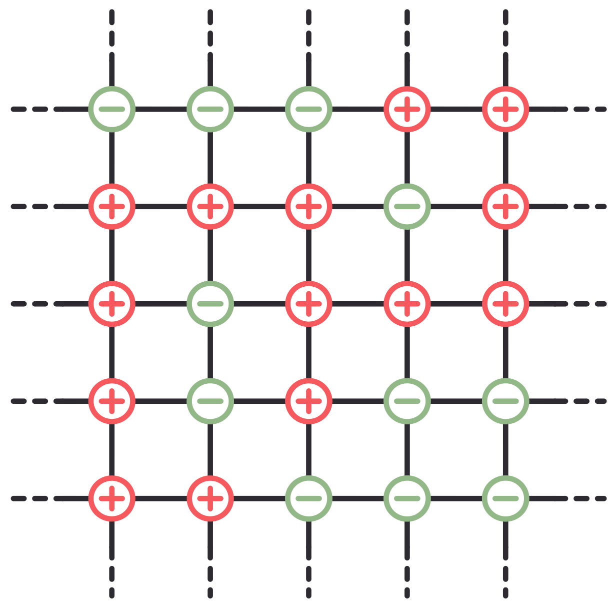
Atoms ● ≈ small magnets, with two possible orientations ⊕ and ⊖

→ magnetic moment / spins.

The Ising model

② Model

Consider a finite graph, and $\sigma : V \to \{\ominus, \oplus\}$ a spin configuration on its vertices.



[Lenz, Ising 25], [Onsager 44]

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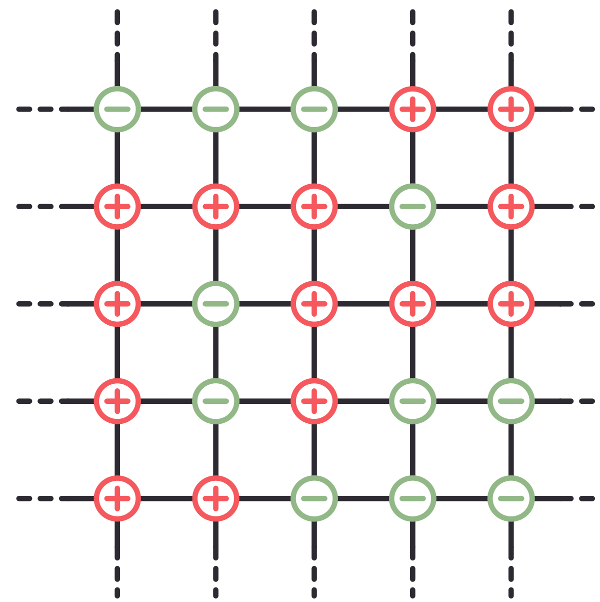

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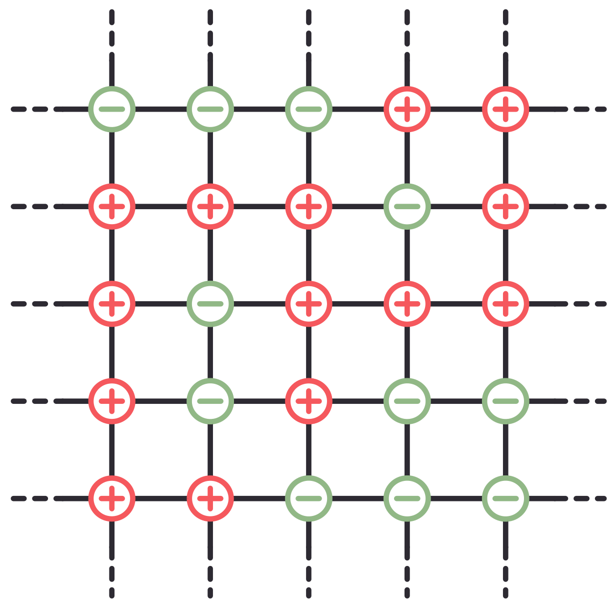

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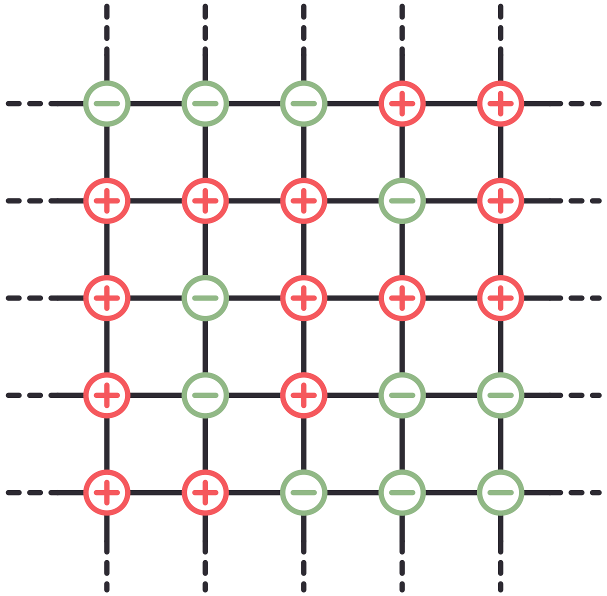
Quantity of interest: the **magnetization**

$$M(\nu, c) := \frac{1}{|V|}\mathbb{E}\left(\sigma_\oplus - \sigma_\ominus\right)$$



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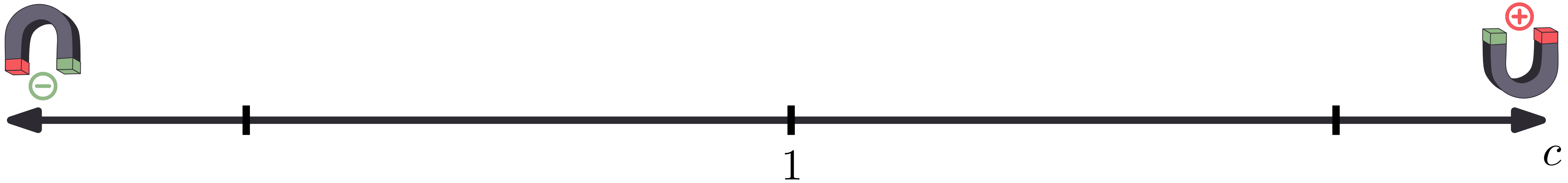
The Ising model

③ Physics experiment



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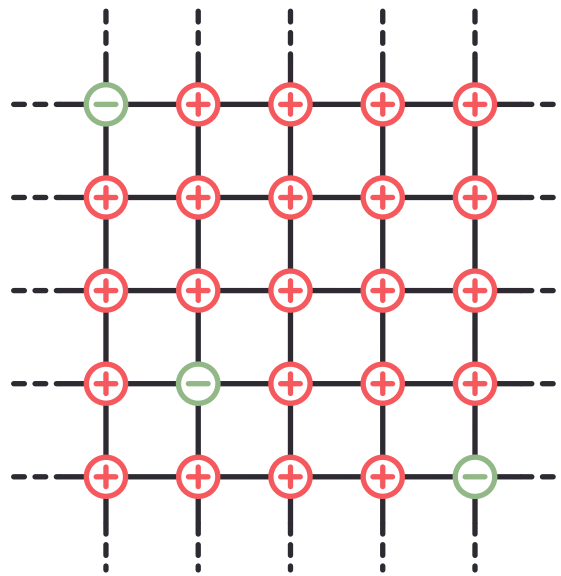

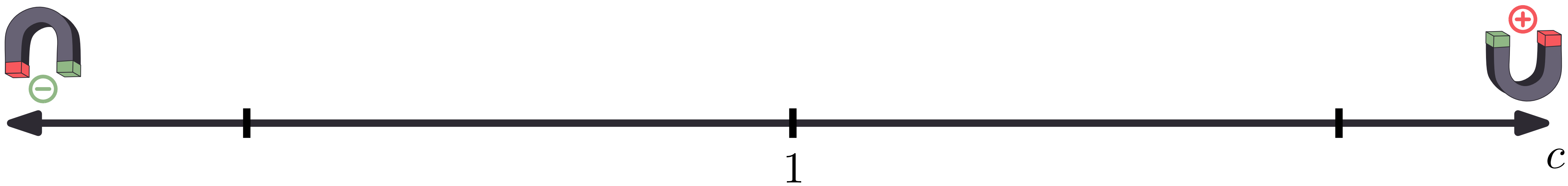

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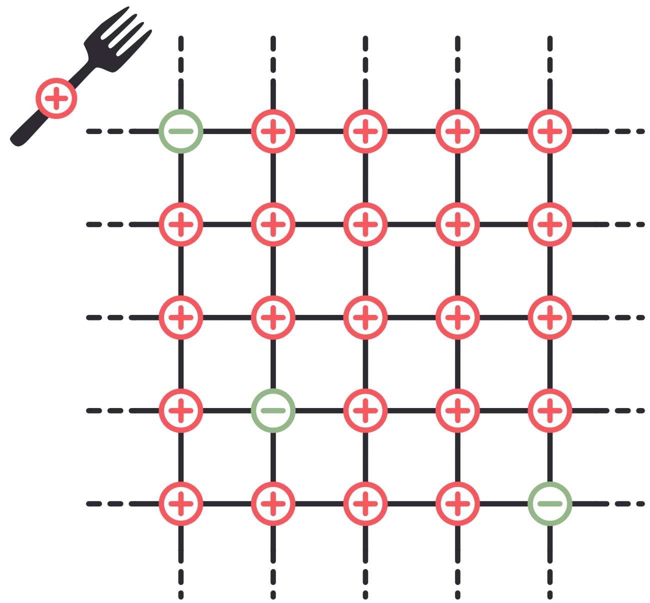
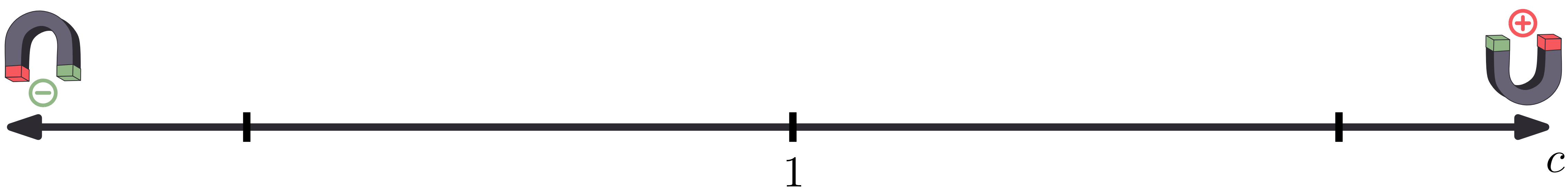


Proper magnetization

$M(\nu, c) > 0$



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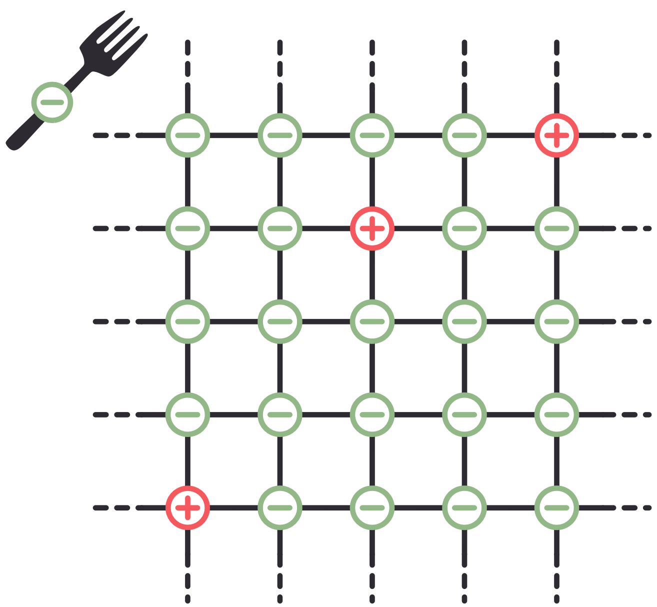
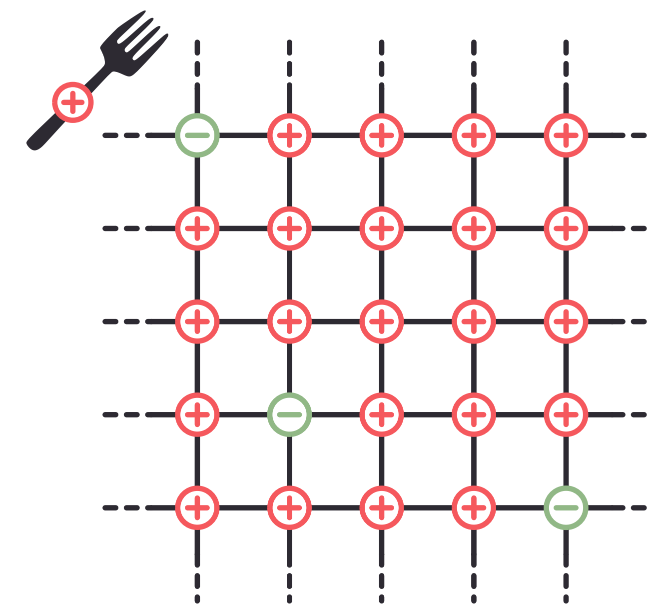
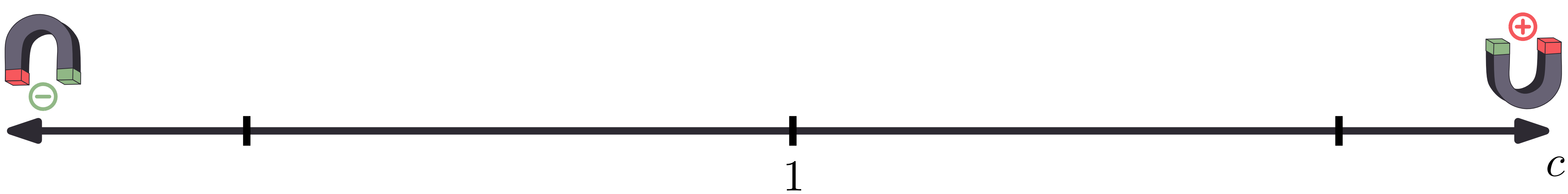


Proper magnetization
$M(\nu, c) < 0$



Proper magnetization
$M(\nu, c) > 0$



The Ising model

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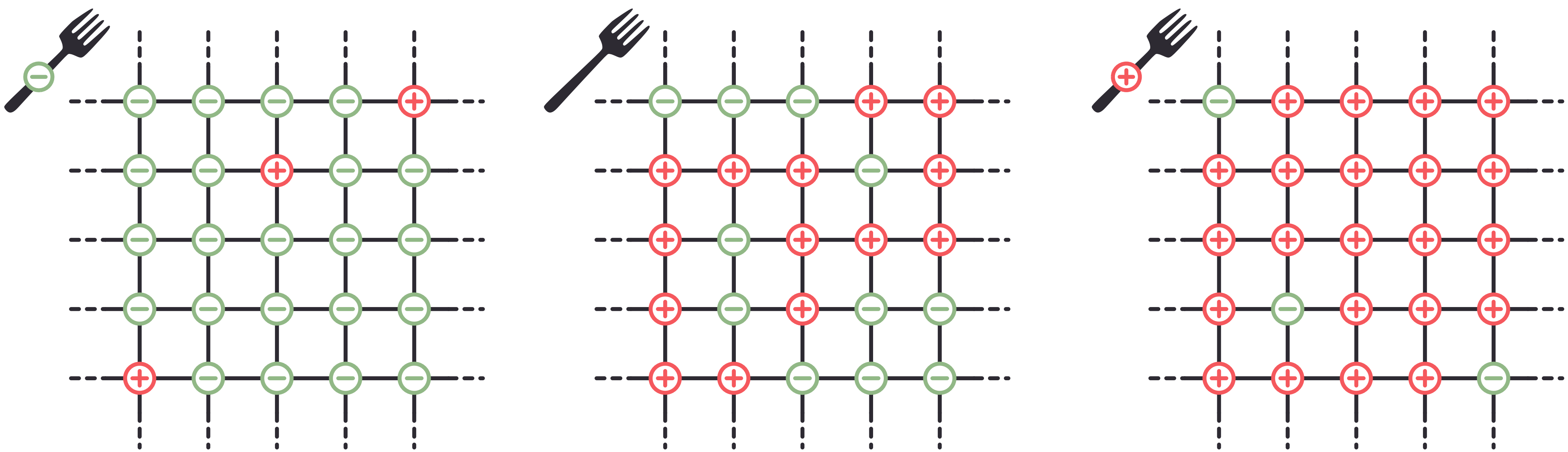
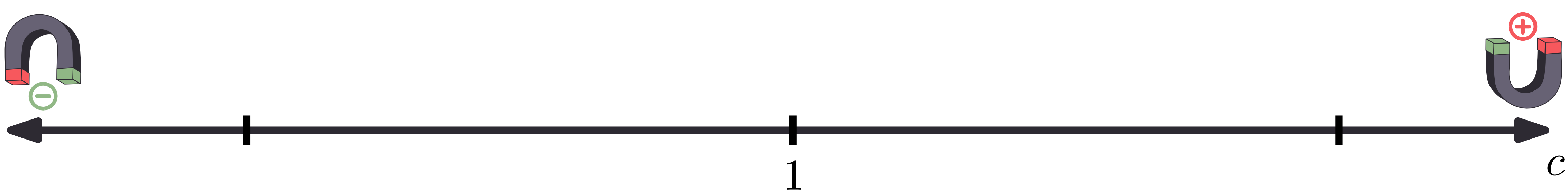


Proper magnetization	No proper magnetization	Proper magnetization
$M(\nu, c) < 0$	$M(\nu, c) = 0$	$M(\nu, c) > 0$



The Ising model

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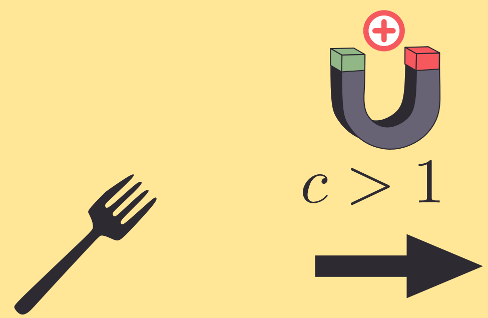

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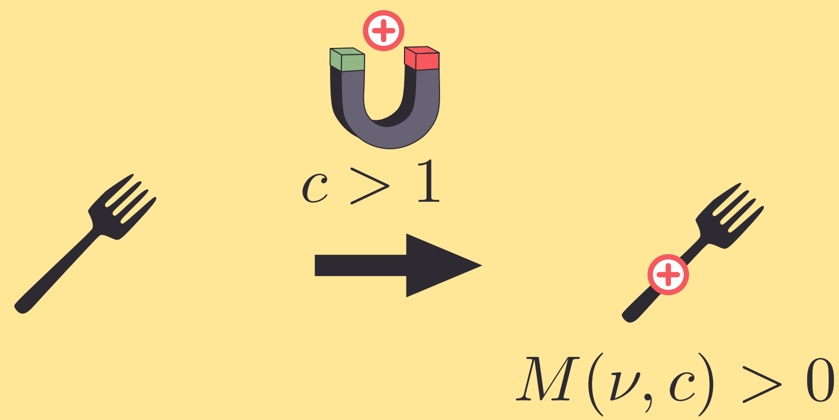

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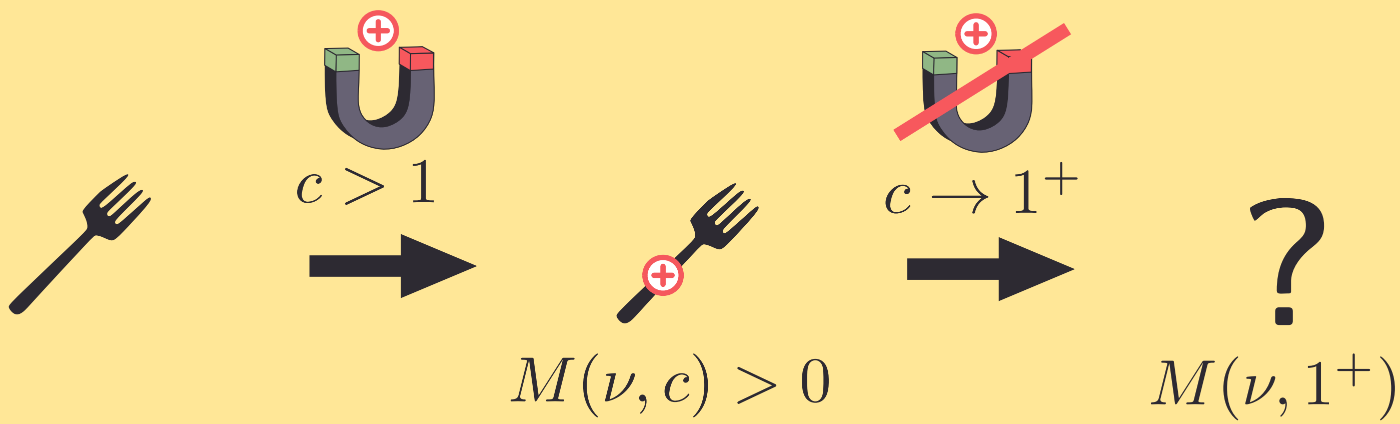

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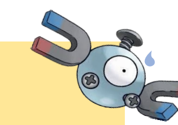

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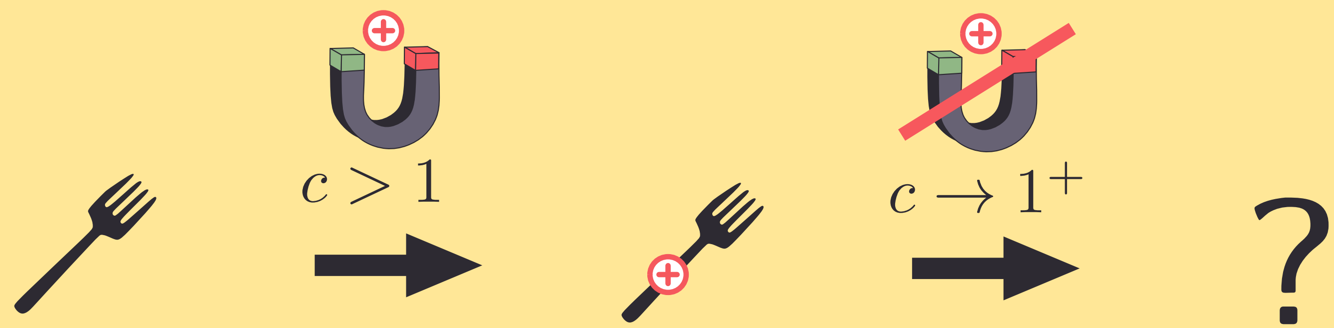
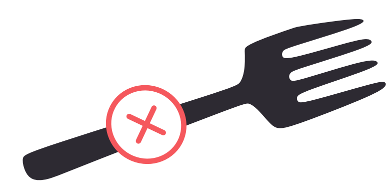


$M(\nu, c) > 0$ $\qquad$ $M(\nu, 1^+)$

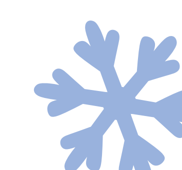


No magnetization

$M(\nu, 1^+) = 0$

Spontaneous magnetization

$M(\nu, 1^+) > 0$



$\nu_\star$

Curie temperature

ν

The Ising model

④ Success of the model

i) Simple interaction model : local rules,

ii) Exhibit a phase transition [Ising 25], [Onsager 44], [Yang 52]

iii) Many generalizations : Heisenberg model, Potts model...

iv) Studied a lot, and on large families of lattices ans graphs [Aizenman, Duminil-Copin, Sidoravicius...]

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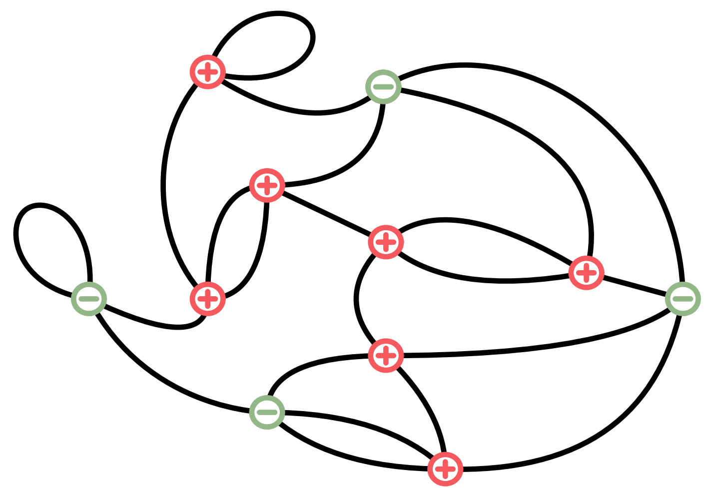
⑤ And what about planar maps ?

→ Planar maps $\equiv$ our model of random lattice



The Ising model on planar maps

① Model: Ising model on quartic planar maps

A spin configuration on a planar map m, with n vertices:

$$\sigma : \mathrm{V}(\mathrm{m}) \to \{-1, +1\}$$

The probability on pairs (m, σ):

$$\mathbb{P}_n^{\nu,c}\Big(\{(\mathrm{m}, \sigma)\}\Big) \propto \nu^{m(\mathrm{m},\sigma)} c^{\sigma_\oplus - \sigma_\ominus}$$

$m(\mathrm{m}, \sigma) \equiv$ the number of monochromatic edges,

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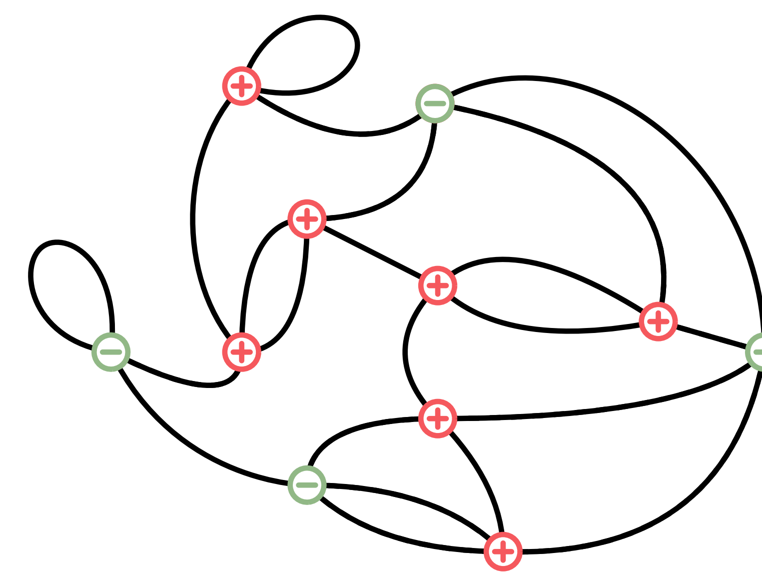

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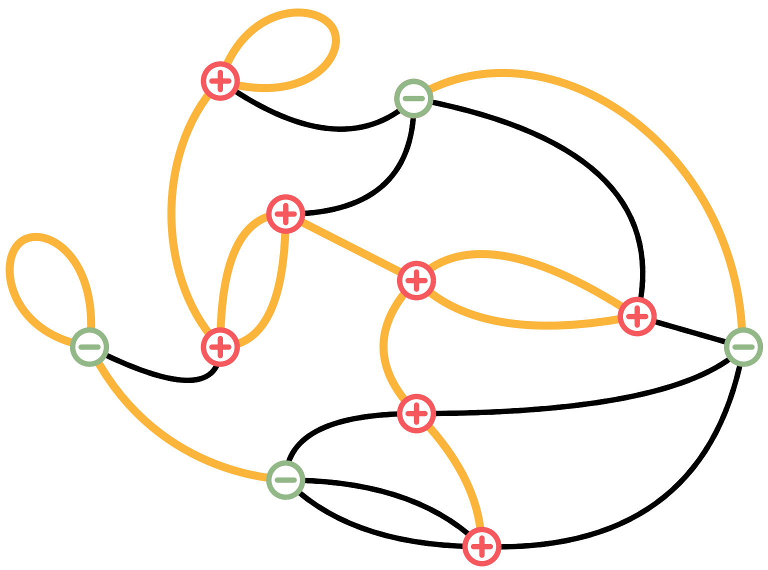

$m(\mathrm{m}, \sigma) = 12,$
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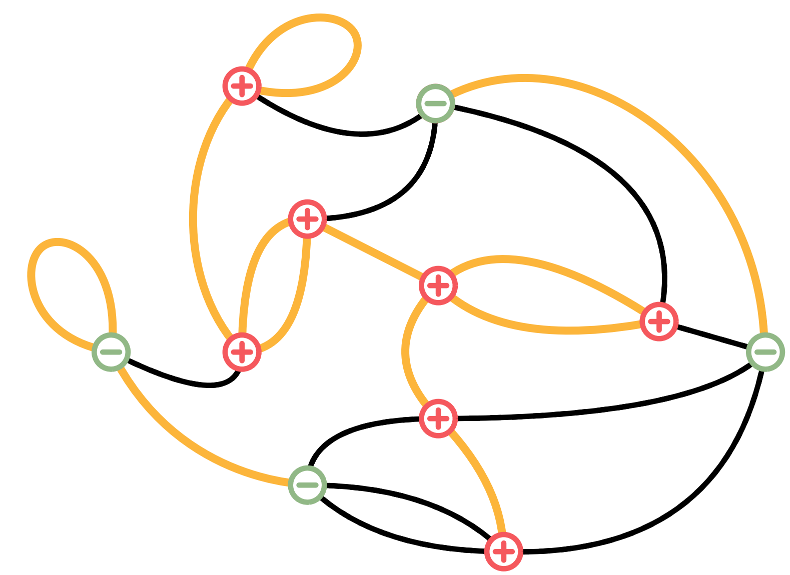

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Remark: The marginal on the maps is not the uniform law.

The Ising model on planar maps

② Quantities of interest & statistical physics

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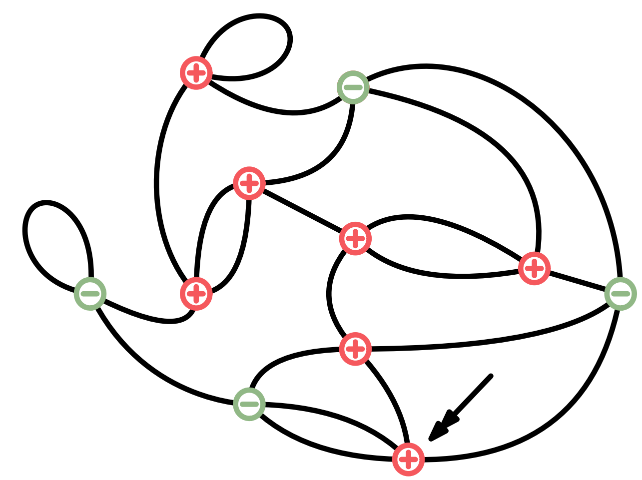

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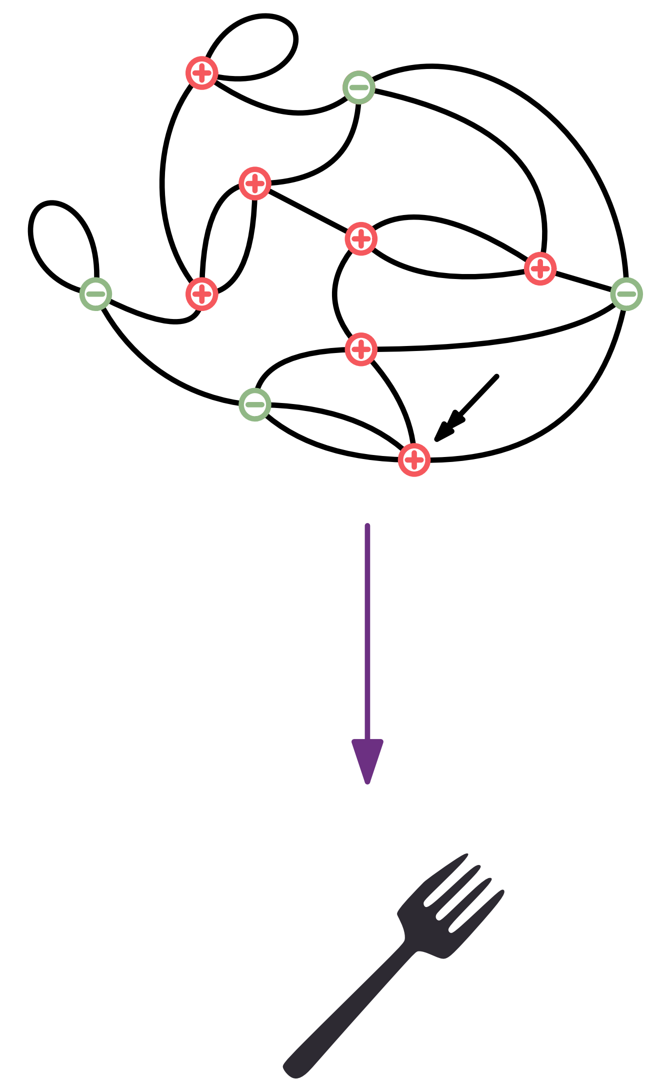

$n \to \infty$

The Ising model on planar maps

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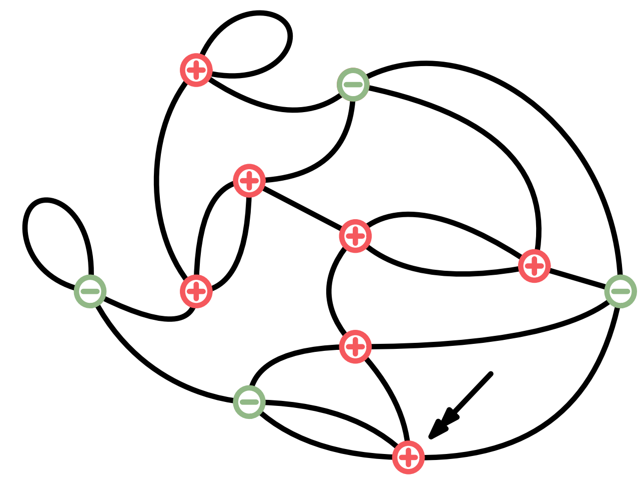

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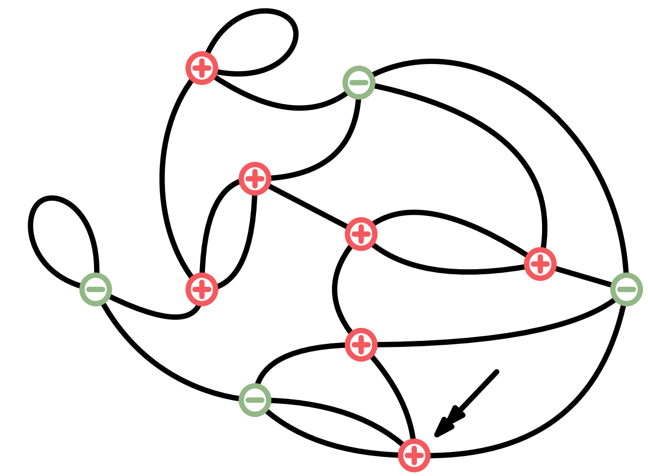

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The thermodynamic limit

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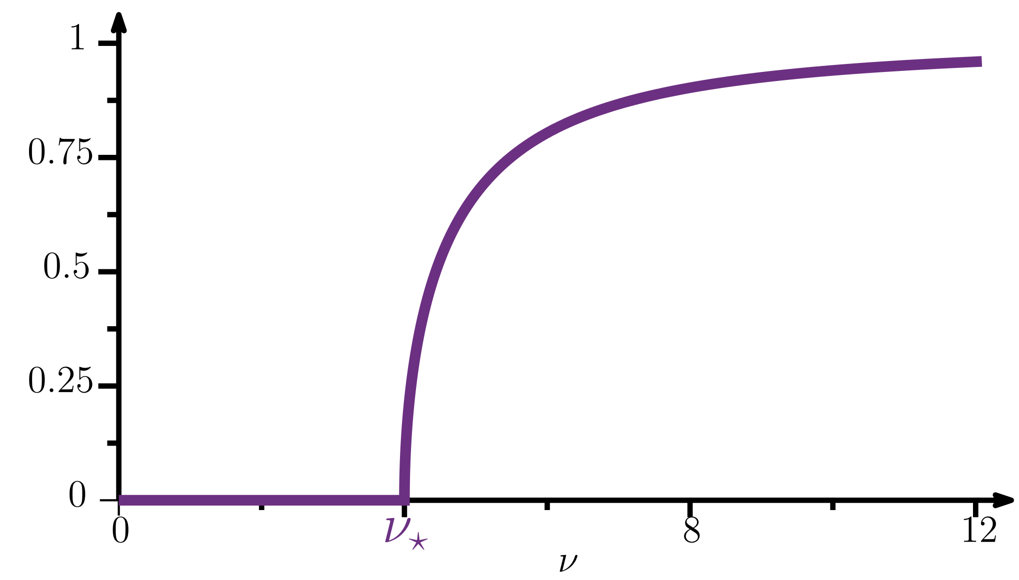
Infinite spontaneous magnetization: $M_1(\nu) := \lim_{c\to 1^+} M_n(\nu, c)$

The Ising model on planar maps

PROPERTY: [Boulatov, Kazakov '87, T. '25]

i) The infinite spontaneous magnetization $M_1(\nu) := \lim_{c \to 1^+} M(\nu, c)$ verifies:

$$\nu_\star := \inf \{\nu > 0 \mid M_1(\nu) > 0\} = 4$$



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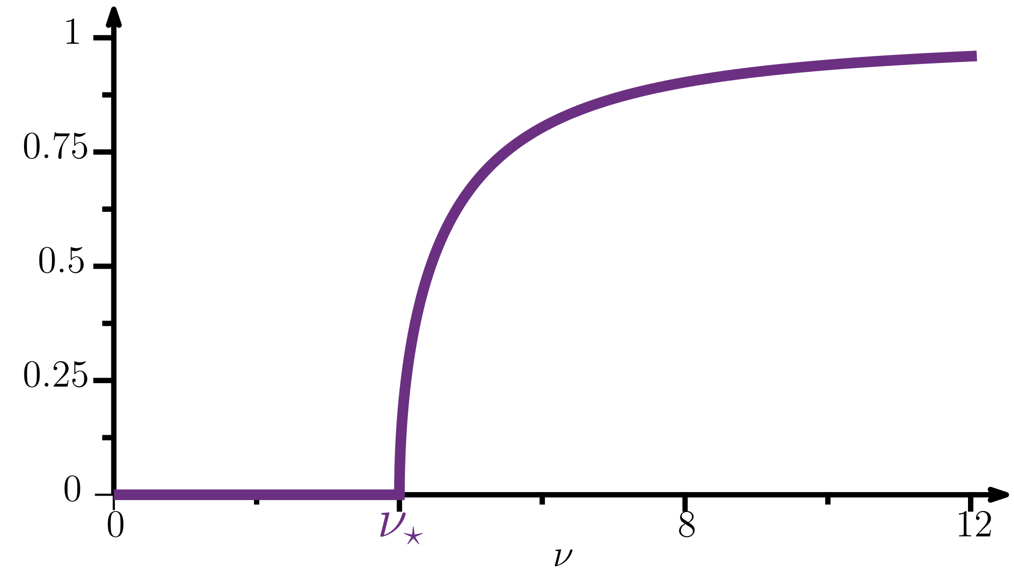

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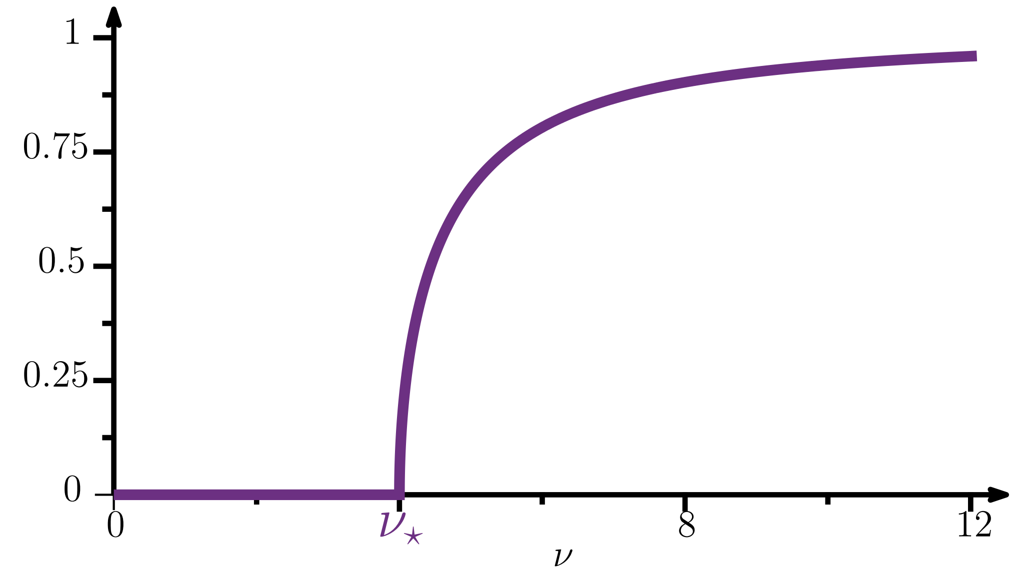

ii) The critical exponents of the model are $\alpha = -1$, $\beta = \frac{1}{2}$, $\gamma = 2$, $\delta = 5$:

$$M_1(\nu) \sim C \cdot \left(\frac{\nu}{\nu_\star} - 1\right)^{\mathbf{1/2}}, \quad \chi(\nu, 1) \sim C \cdot \left(\frac{\nu}{\nu_\star} - 1\right)^{\mathbf{2}}, \quad M(\nu_\star, c) \sim C \cdot (c-1)^{1/\mathbf{5}}.$$

The Ising model on planar maps

③ Ideas of proof

We need to study the probability to sample the pair (m, σ):

$$\mathbb{P}_n^{\nu,c}\Big(\{(\mathrm{m}, \sigma)\} \Big) \propto \nu^{m(\mathrm{m},\sigma)} c^{\sigma_\oplus - \sigma_\oplus}$$

The Ising model on planar maps

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We need to study the probability to sample the pair (m, σ):

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We need study the normalization constant a.k.a **partition function**:

$$\mathcal{Z}_n(\nu, c) := \sum_{(\mathrm{m},\sigma)\in\mathcal{T}_n} \nu^{m(\mathrm{m},\sigma)} c^{\sigma_\oplus - \sigma_\ominus}.$$

The Ising model on planar maps

③ Ideas of proof : Bijections & analytic combinatorics

PROPERTY: [Albenque, Ménard, T. '25]

The partition function $\mathcal{Z}_n(\nu, c)$ verifies:

$$\mathcal{Z}(\nu, c, z) = \frac{Pol_2(S(\nu,c,z),\nu,c,z)}{z^2(1-\nu^2)(1+3c^2(1-\nu^2)S(\nu,c,z))},$$

The Ising model on planar maps

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where $S(\nu, c, z)$ is the unique formal power series in $\mathbb{Q}(\nu, c)[\![z]\!]$ having constant term 0 and satisfying the following lagragian equation:

$$z = \frac{S\left(1 - 243c^6(1-\nu^2)^5S^6 + 135c^4(1-\nu^2)^3S^4 - 3c^2(1-\nu^2)(3\nu^2+7)S^2 - 3\nu^2(c^2+1)S\right)}{(1-3c(1-\nu^2)S)^2(1+3c(1-\nu^2)S)^2}.$$

The Ising model on planar maps

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Key property: The power series S has positive coefficients.

The Ising model on planar maps

③ Ideas of proof

PROPERTY: [Bernardi-Bousquet Mélou '11 $(c=1)$, T. '25 $(c \neq 1)$]

The normalization constant verifies the following:

For all $\nu > 0$, there exists $\varepsilon_\nu > 0$, such that for all $c \in [1-\varepsilon_\nu, 1+\varepsilon_\nu]$, when $n \to \infty$:

$$\mathcal{Z}_n(\nu, c) = [z^n]\, \mathcal{Z}(\nu, c, z) \sim \begin{cases} \daleth(\nu, 1) \cdot \mu_{\nu,1}{}^{-n}\, n^{-\mathbf{7/3}} & \text{for } (\nu, c) = (\nu_\star, 1), \\ \daleth(\nu, c) \cdot \mu_{\nu,c}{}^{-n}\, n^{-\mathbf{5/2}} & \text{otherwise.} \end{cases}$$

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~~Universality~~

Remark: This help to study the magnetization since $M(\nu, c) = \lim_{n\to\infty} c\partial_c \log \mathcal{Z}_n(\nu, c)$.

What's next ?

A new class of Universality & new limiting metric spaces ?

i) Yes, the exponent $7/3$ appears for several statistical physics model on maps at critical point (Ex: Hard particule model, dimer model... [Many people])

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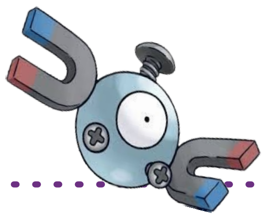

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Thank you for your attention !