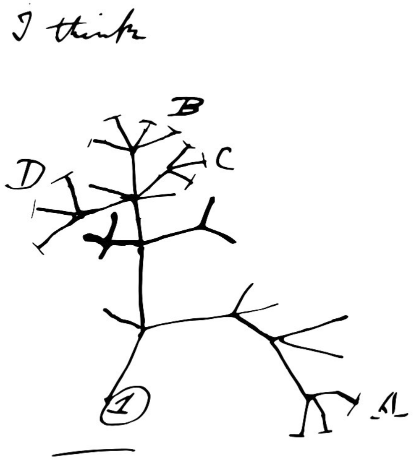
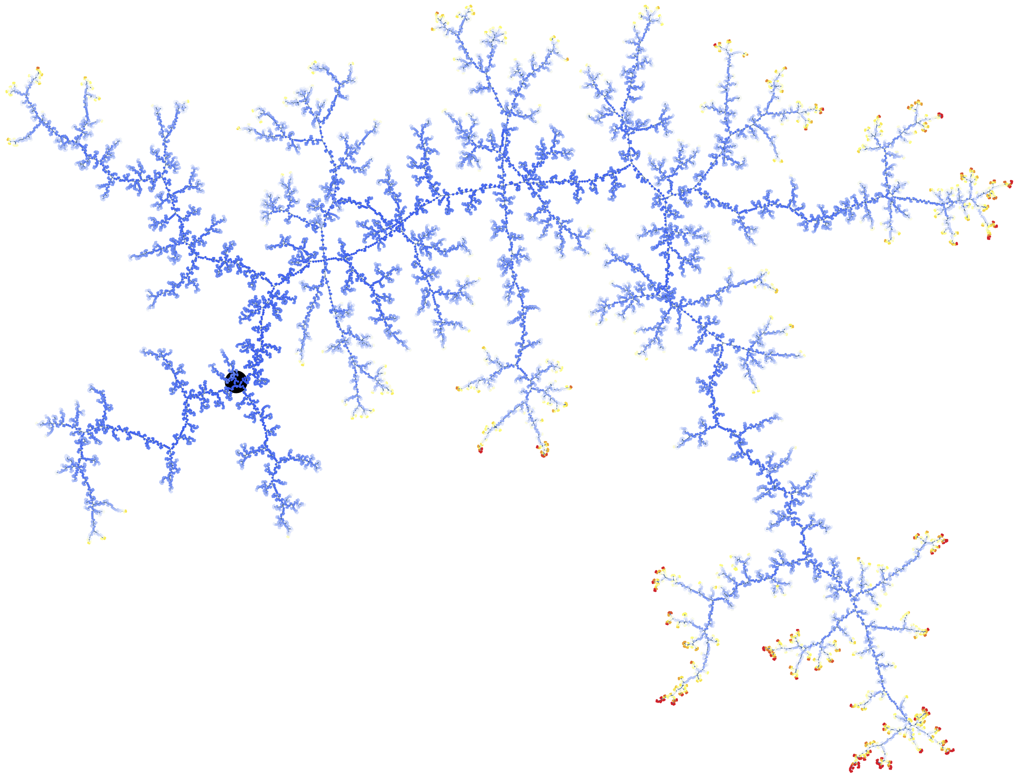
The Brownian Continuum Random Tree

From 1990's to October 10th 2025…



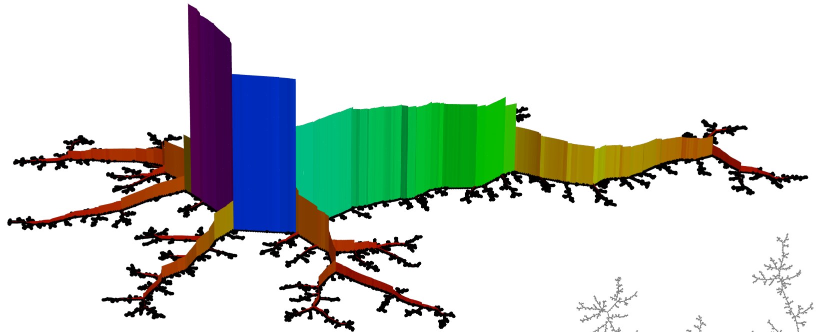


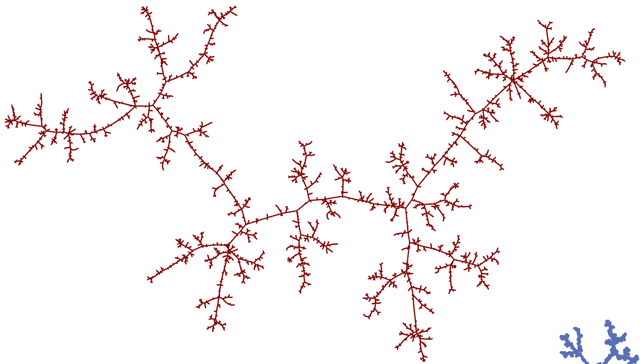
Part I

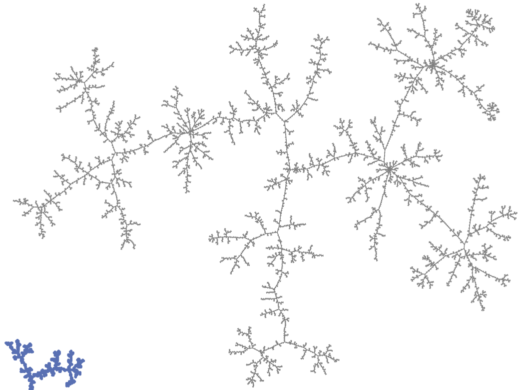
What is a (random) tree ?

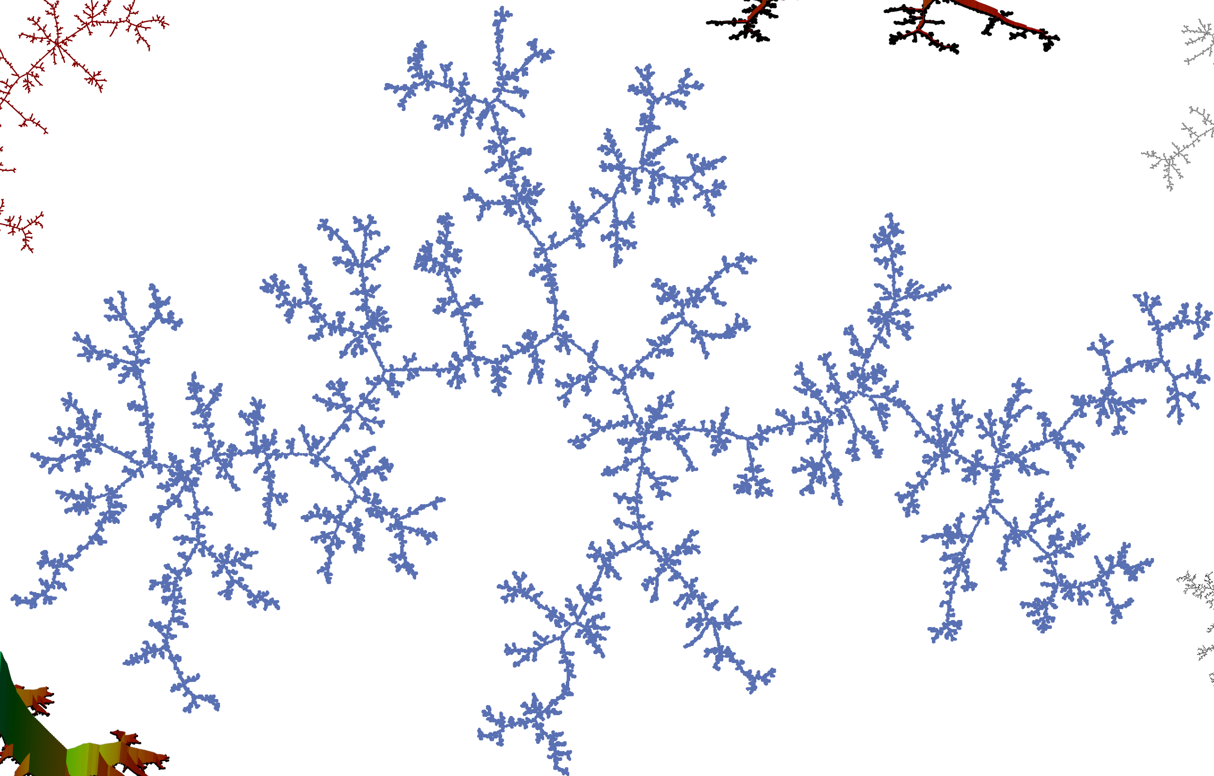
Aldous, Le Gall . . . 90's

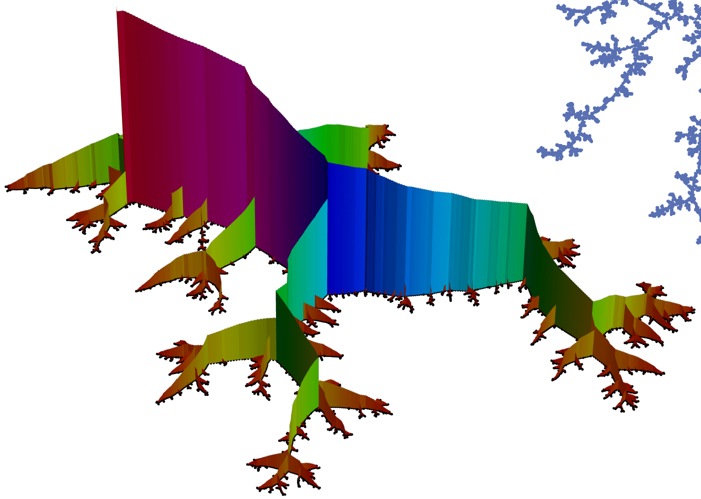
Gallery

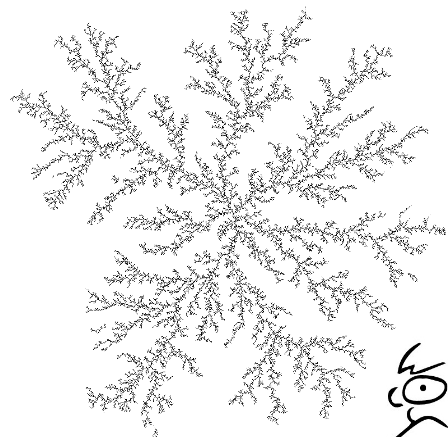


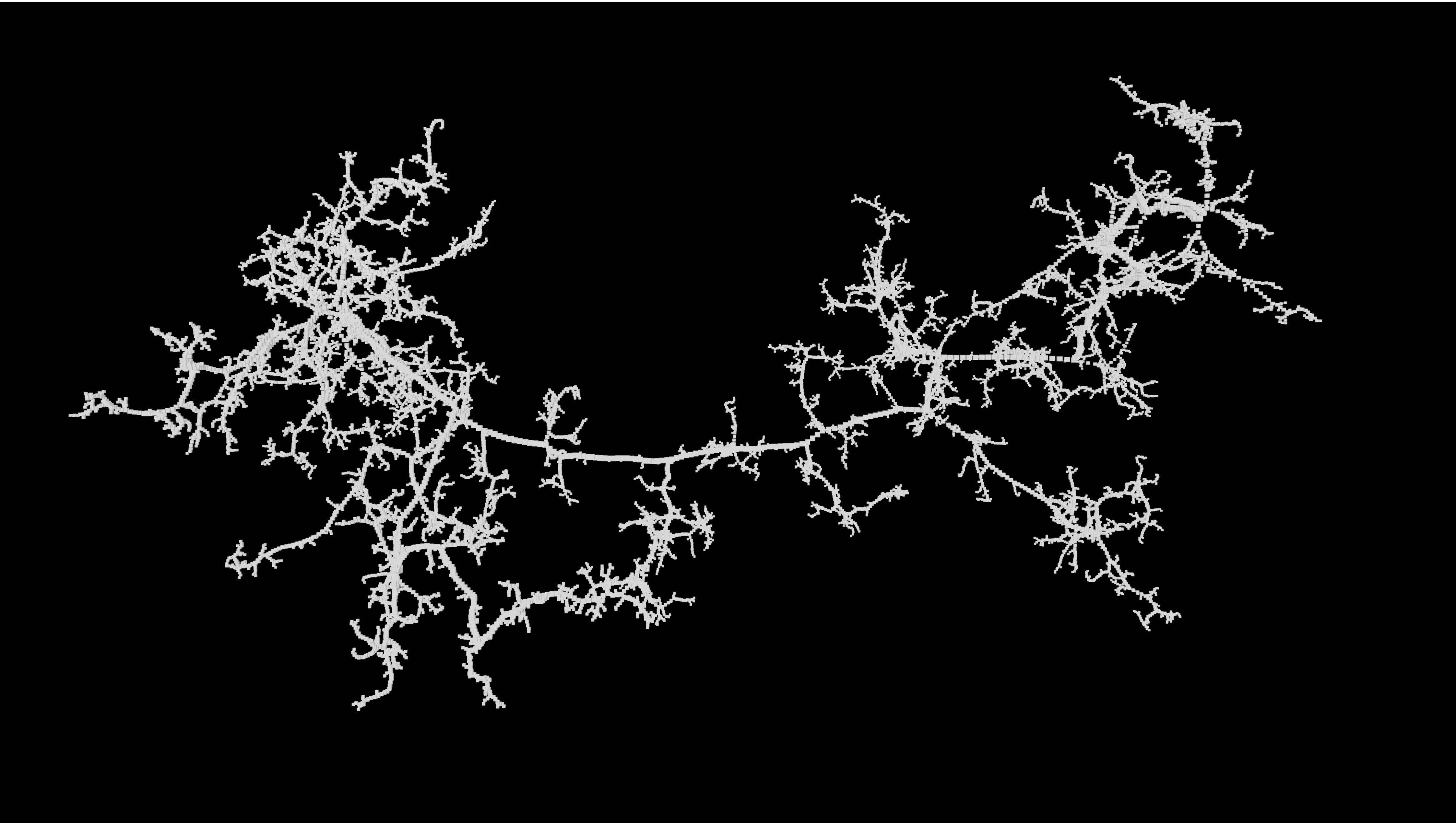


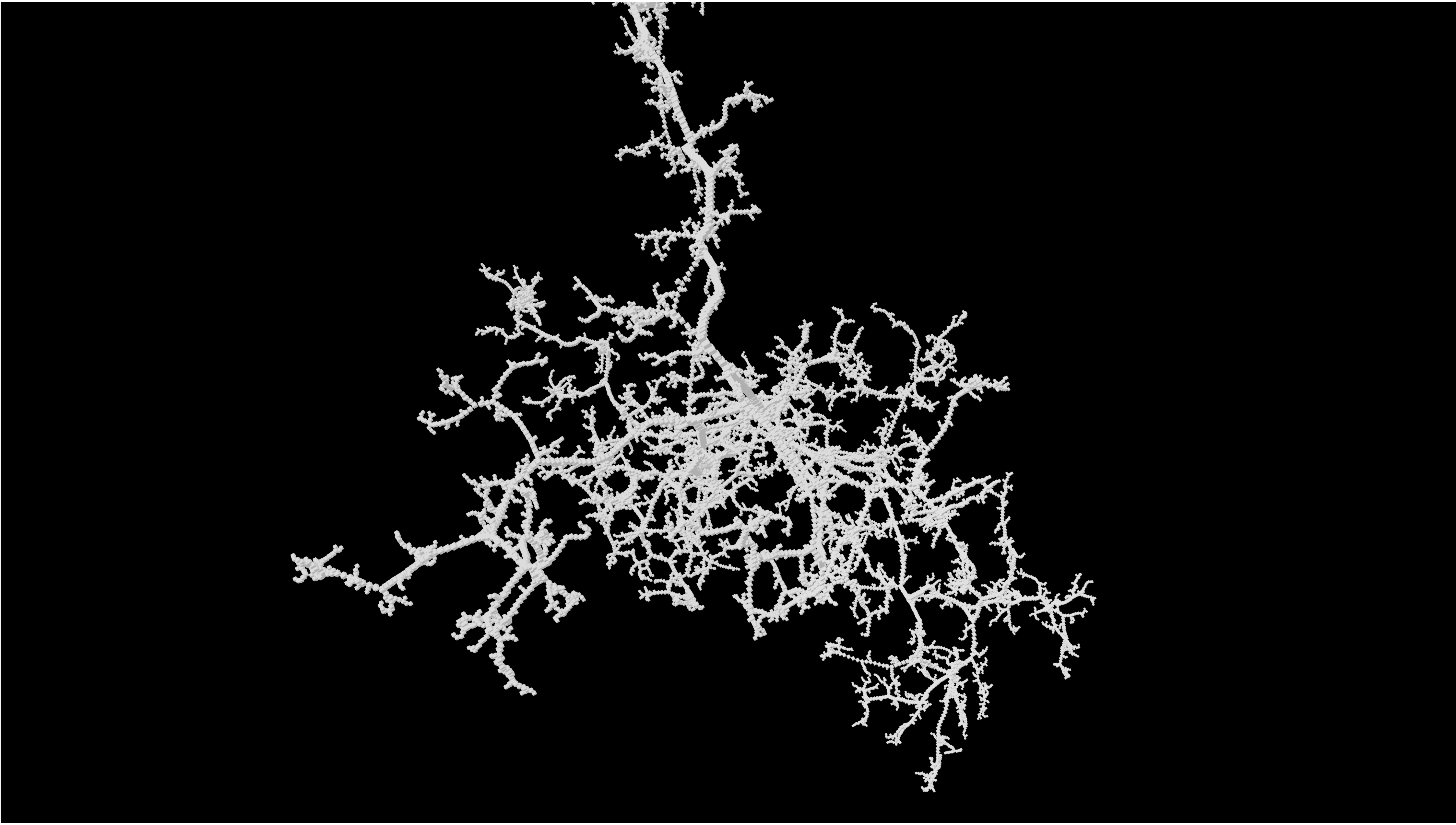






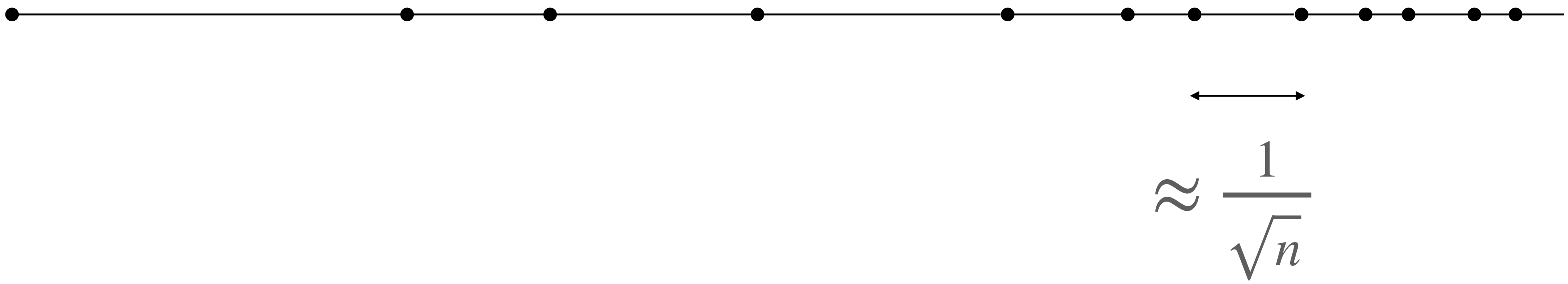






The stick-breaking construction

Let $\mathscr{P}$ be a Poisson process on $\mathbb{R}_+$ with intensity $t\,\mathrm{d}t$



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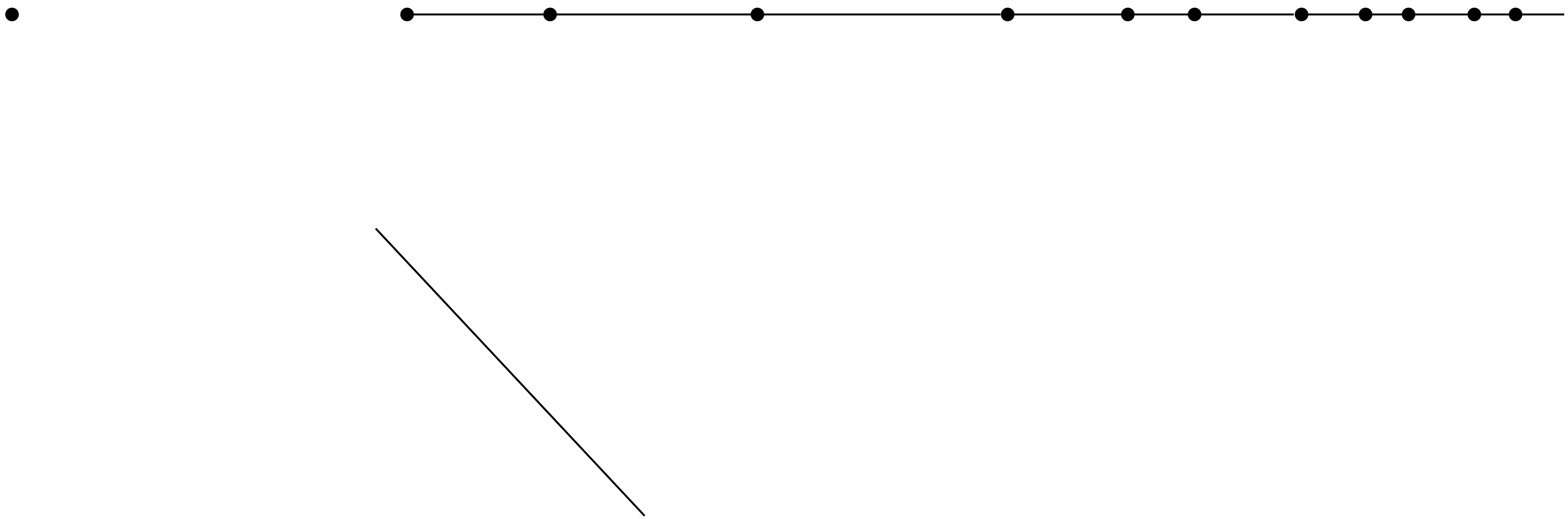

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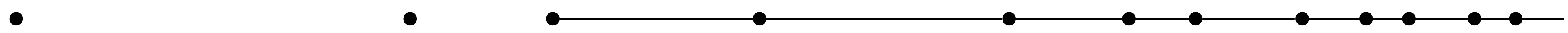

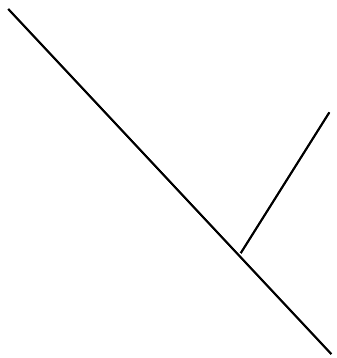

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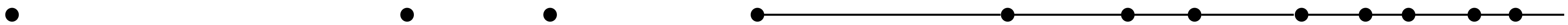

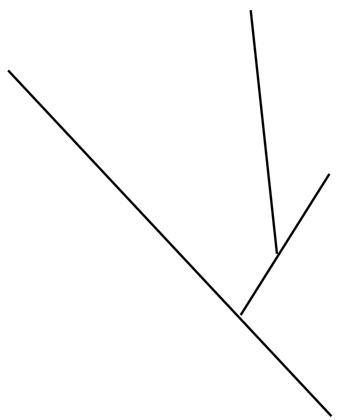

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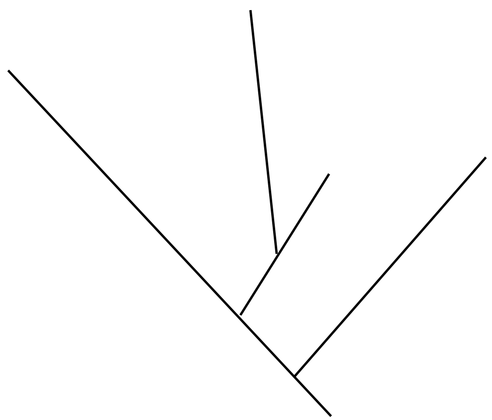

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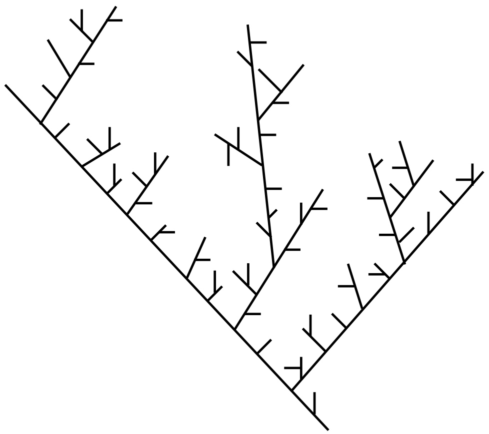
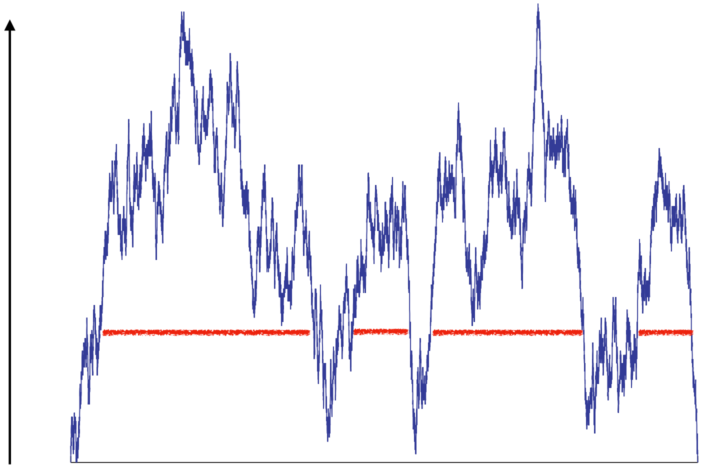
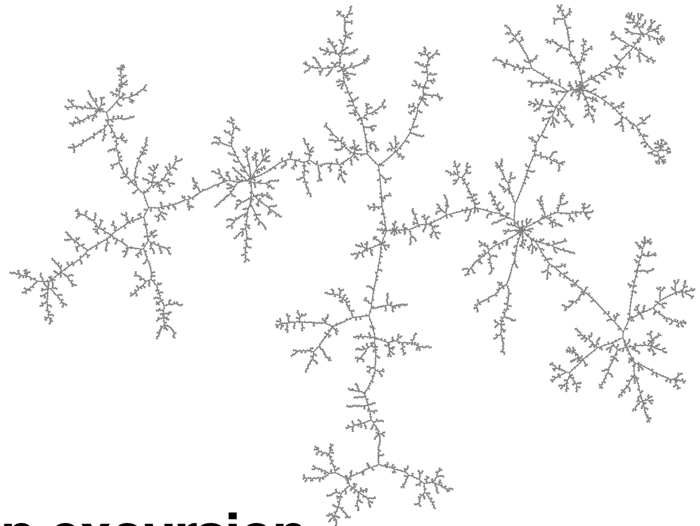


Glue recursively at random, take the completion

Aldous (91): this has the law of $\mathscr{T}$

Constructions of the CRT (continued)





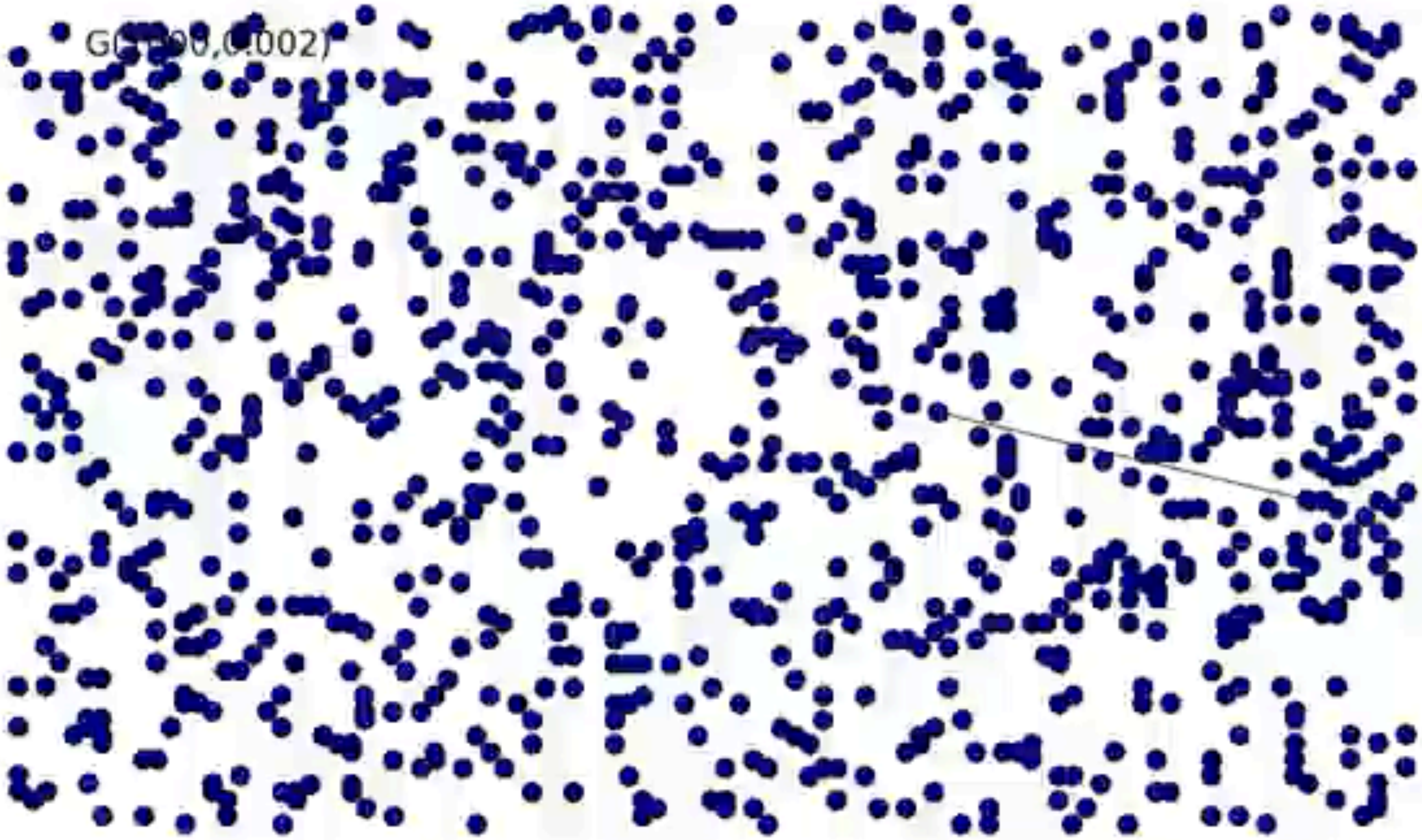
Le Gall & Duquesne 98: $\mathscr{T}$ can encoded by a Brownian excursion

Bertoin 2002: $\mathscr{T}$ can be seen as a the continuous tree coding for the genealogy of particles undergoing fragmentation.

Albenque & Goldschmidt 2012: $\mathscr{T}$ is the unique fixed point of a recursive distributional equation into three parts.

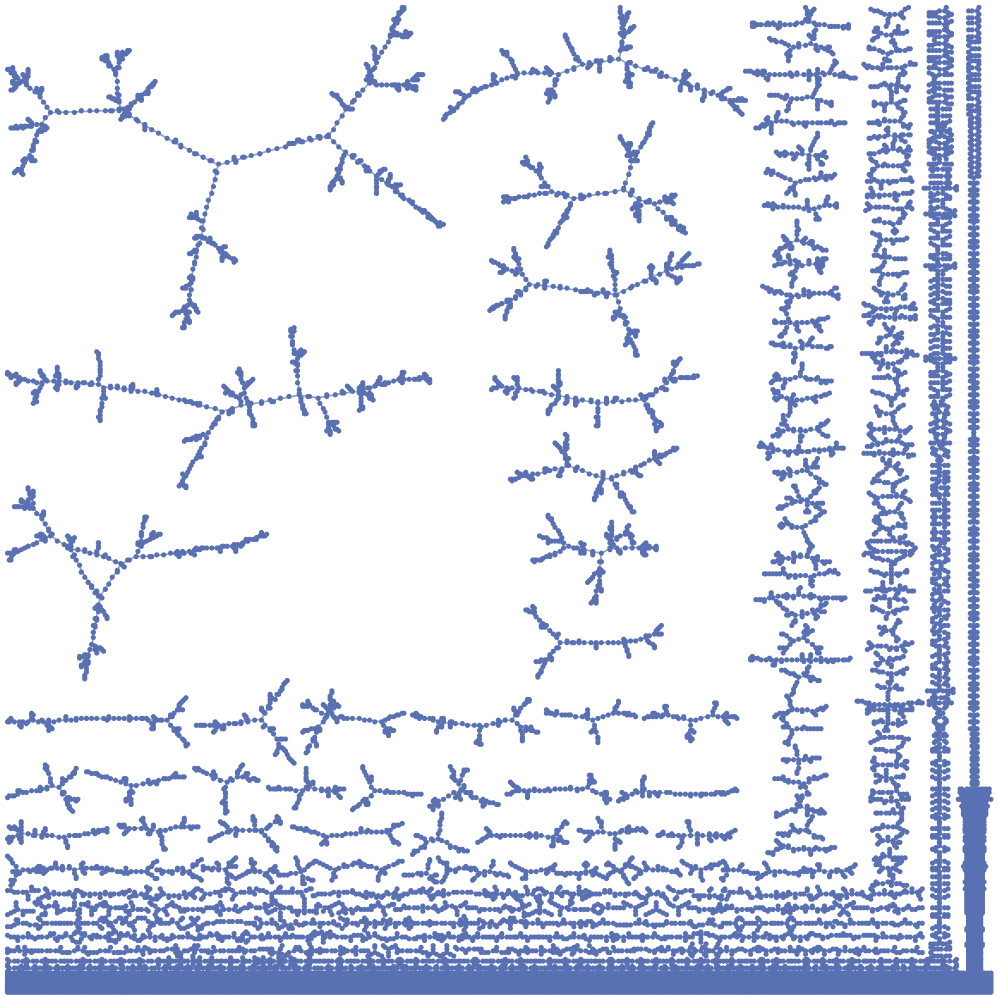
Bertoin, C. & Riera 2024: Self-similar Markov trees theory…

Critical Percolation Clusters



Youtube : The evolution of the G(n,p) random graph

Critical Percolation Clusters

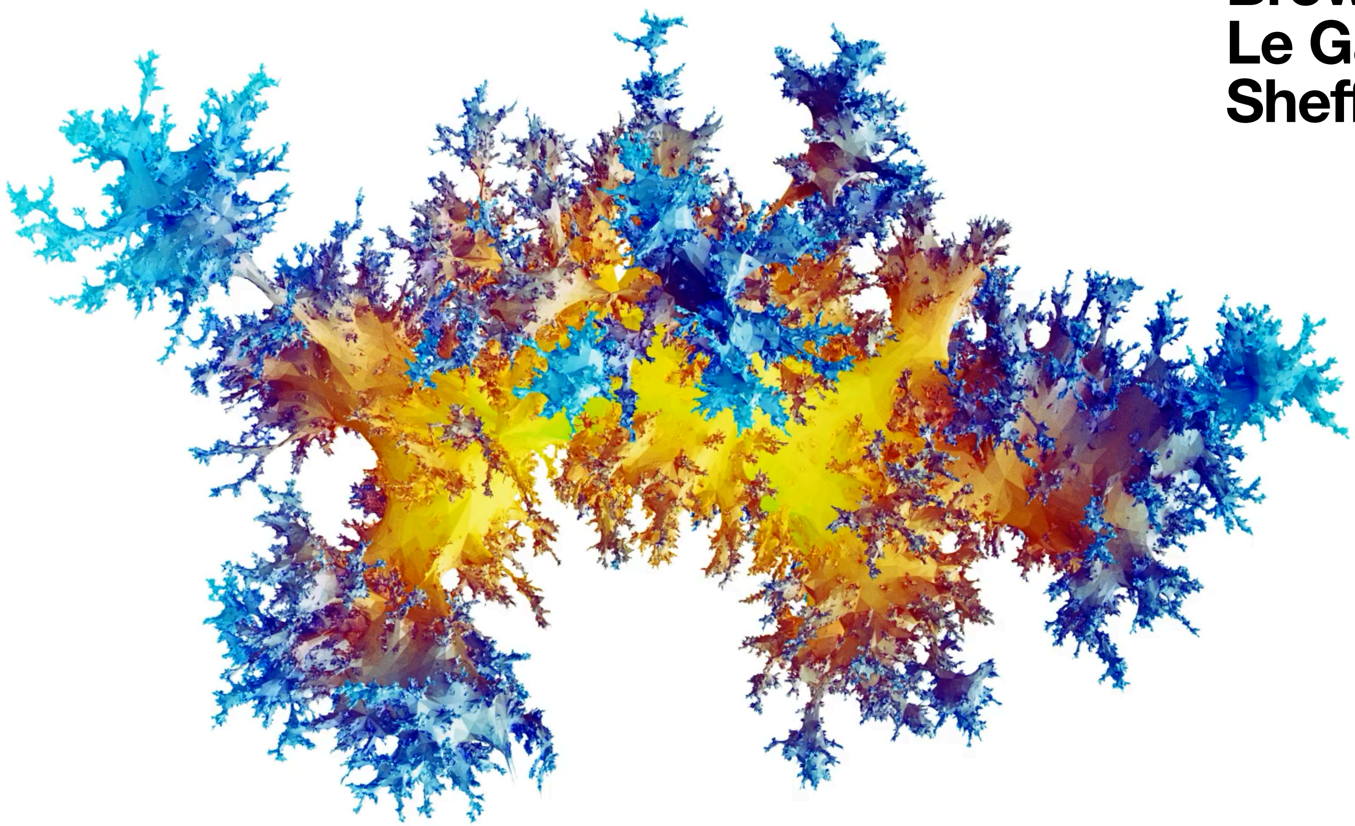


$G(n, m)$ with $m = \frac{n}{2}$

Rescaled large components
close to being Brownian CRT

Building blocks in Liouville Quantum Gravity

Brownian Sphere:
Le Gall, Miermont, Duplantier,
Sheffield, Miller, Gwynne…



Simulation by B. Stufler

Part II

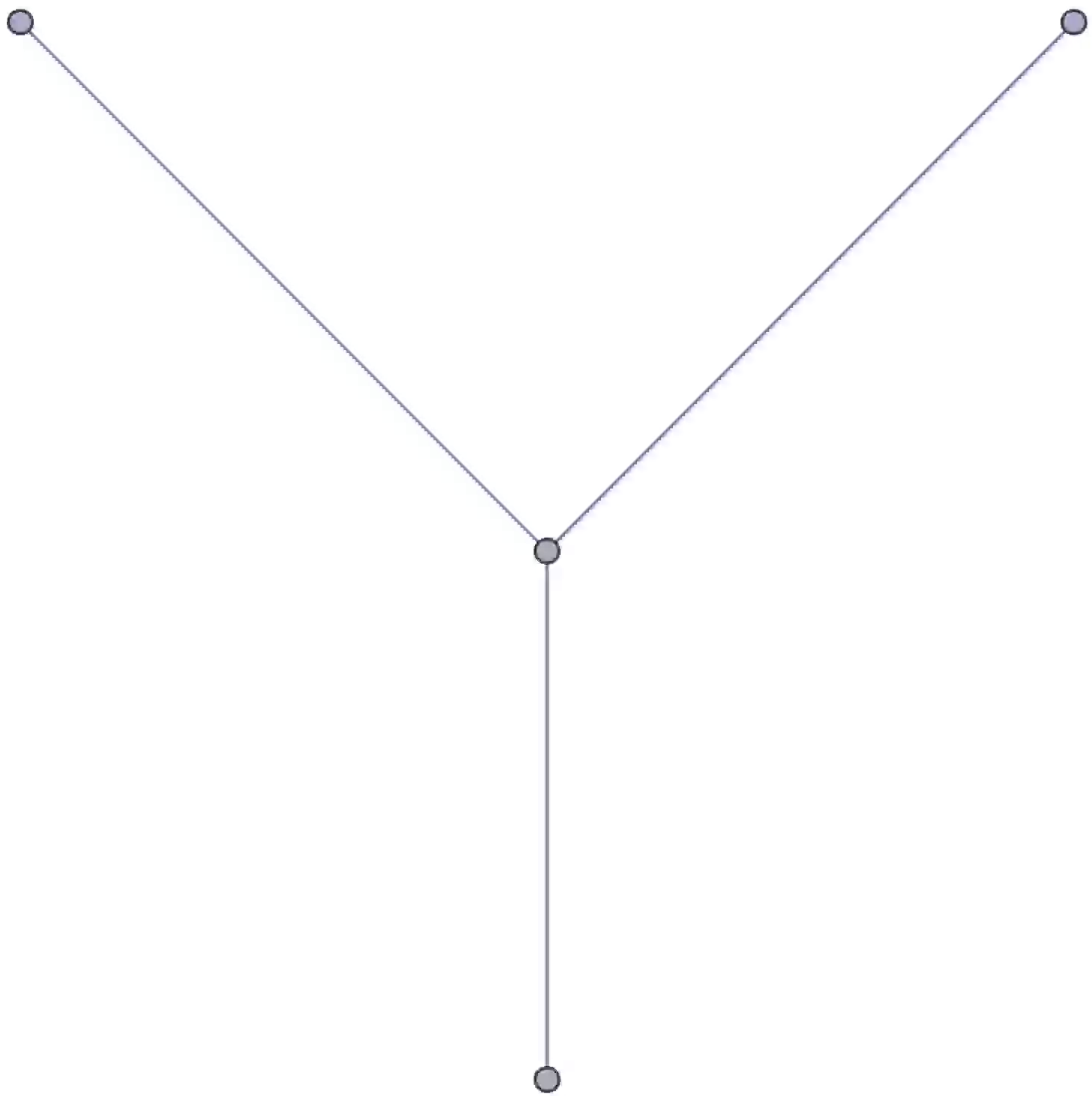
Rémy's algorithm

Rémy 85'

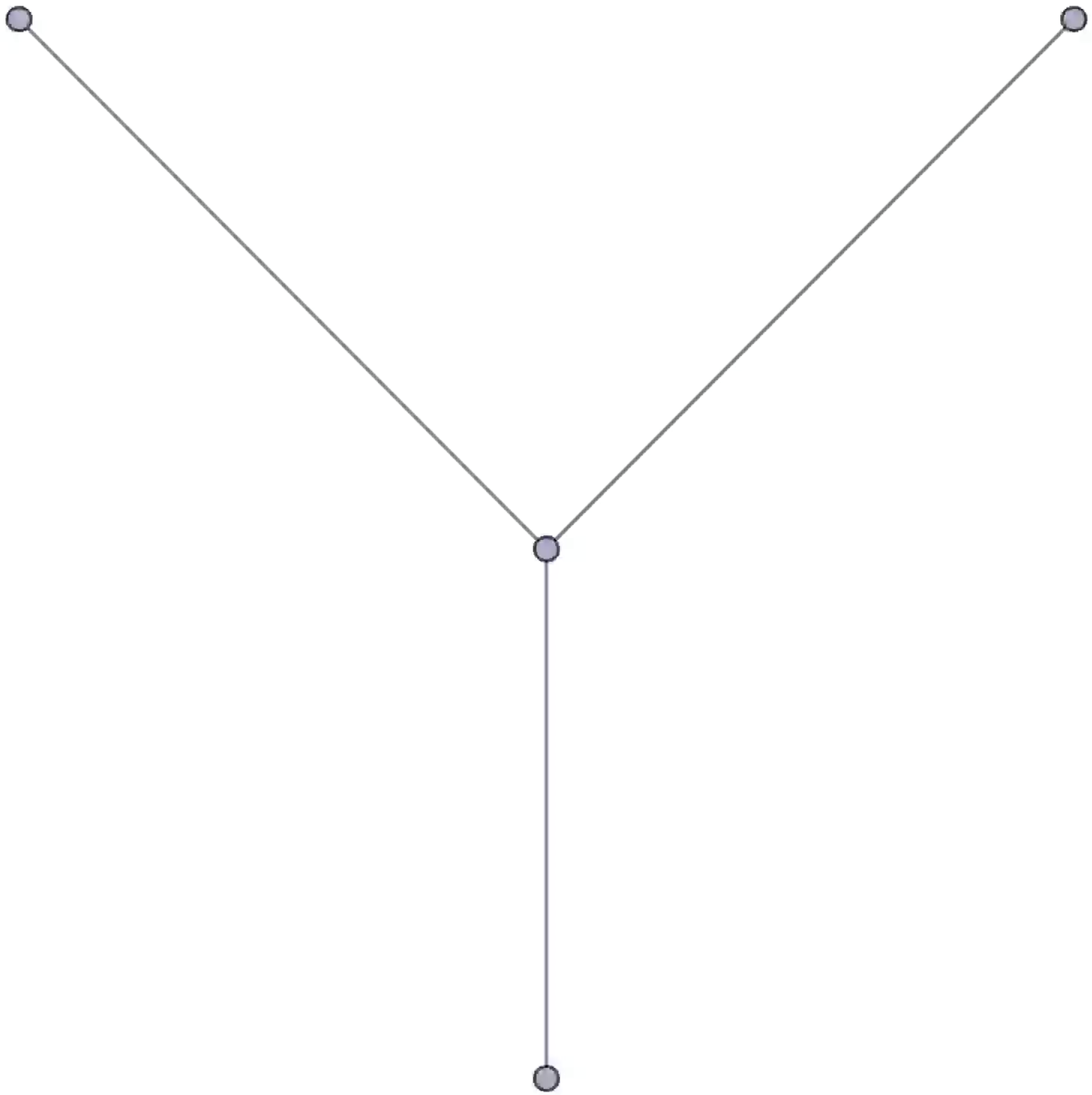
Pitman 06'

Peköz, Röllin, Ross, 14'

Rémy's alg.
200



Rémy’s alg.
500



Part III

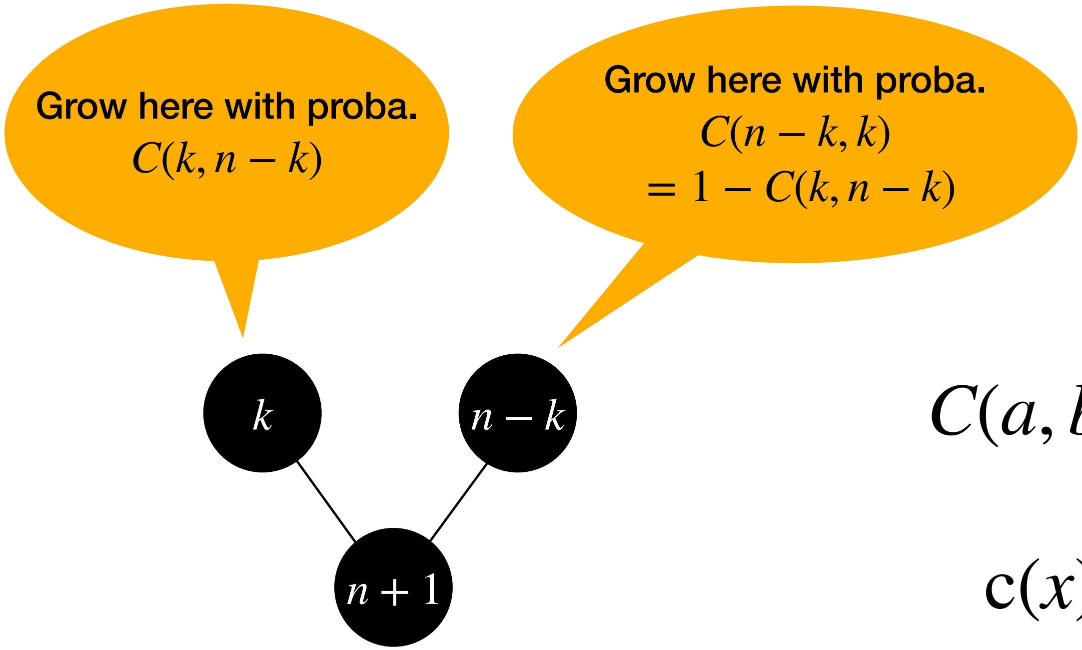
Leaf growth algorithm

With W. Fleurat, A. Twigt,
And A. Caraceni, C. Marzouk, R. Stephenson

Mecanism

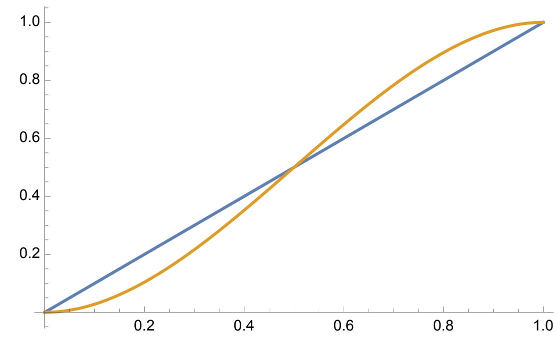
$$C(a,b) = \frac{(2a+1)(2a+2)(2a+3+3(2b+1))}{(2a+2b+2)(2a+2b+3)(2a+2b+4)}$$

Best of three matches play



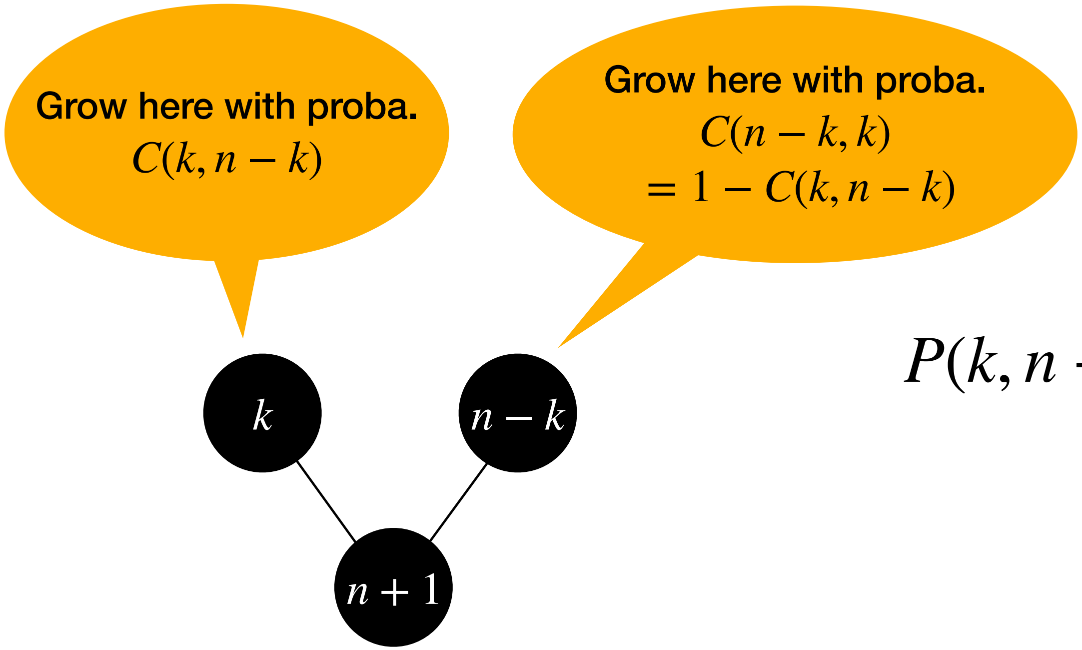
$$C(a,b) \approx \mathrm{c}\left(\frac{a}{a+b}\right)$$

$$\mathrm{c}(x) = x^2(3-2x)$$



A posteriori clear

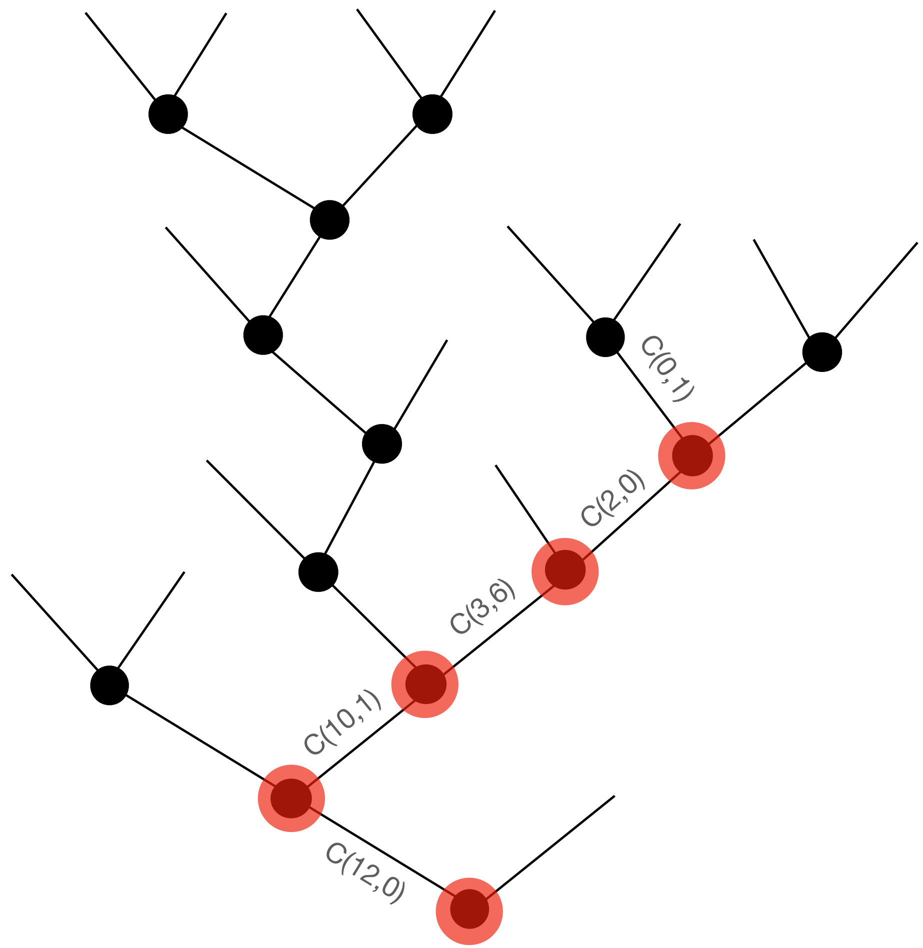
$$C(a,b)???$$



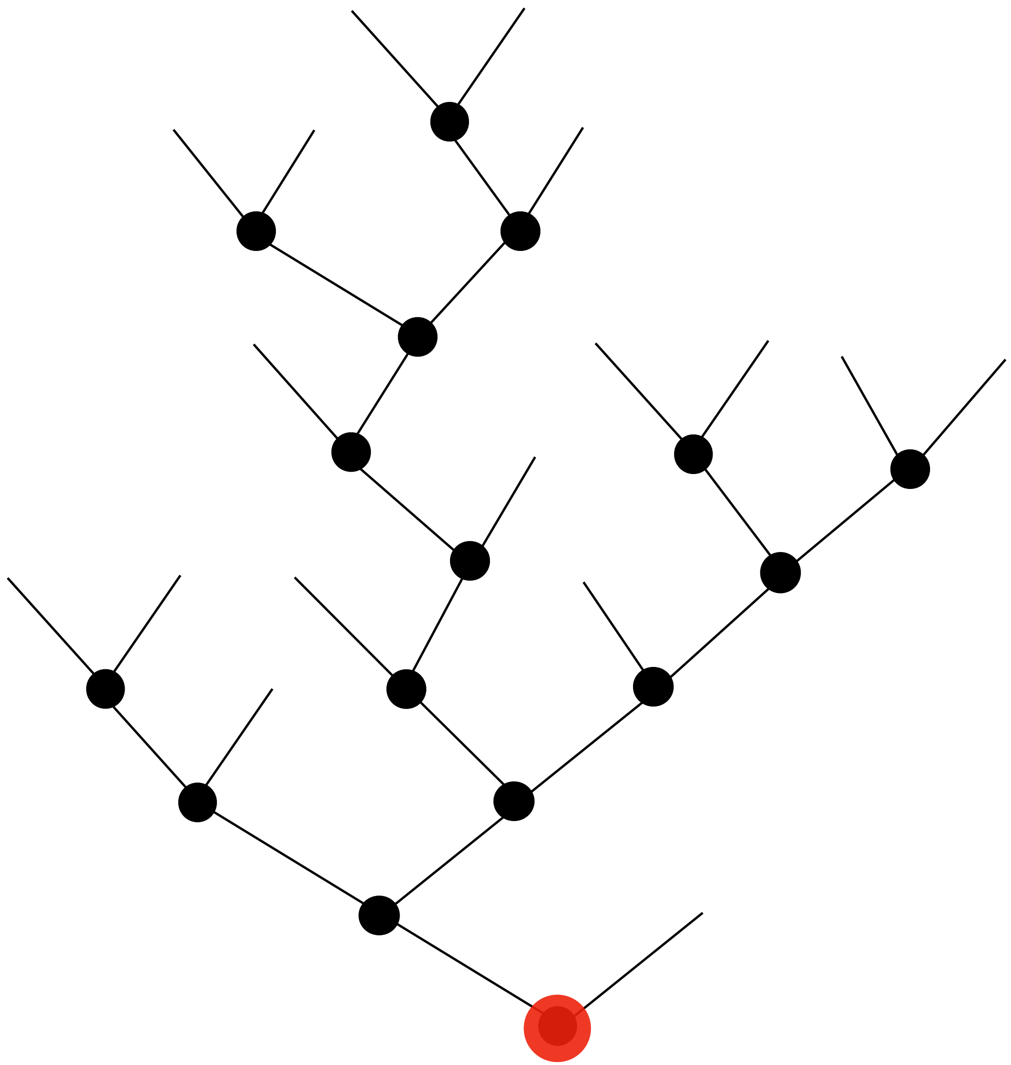
$$P(k, n-k) = P(k, n-k-1) \cdot C(\underline{n-k-1}, k) + P(k-1, n-k) \cdot C(\underline{k-1}, n-k)$$

+ boundary conditions

$$P(n, n-k) = \frac{\mathrm{Cat}(k) \cdot \mathrm{Cat}(n-k)}{\mathrm{Cat}(n)}$$

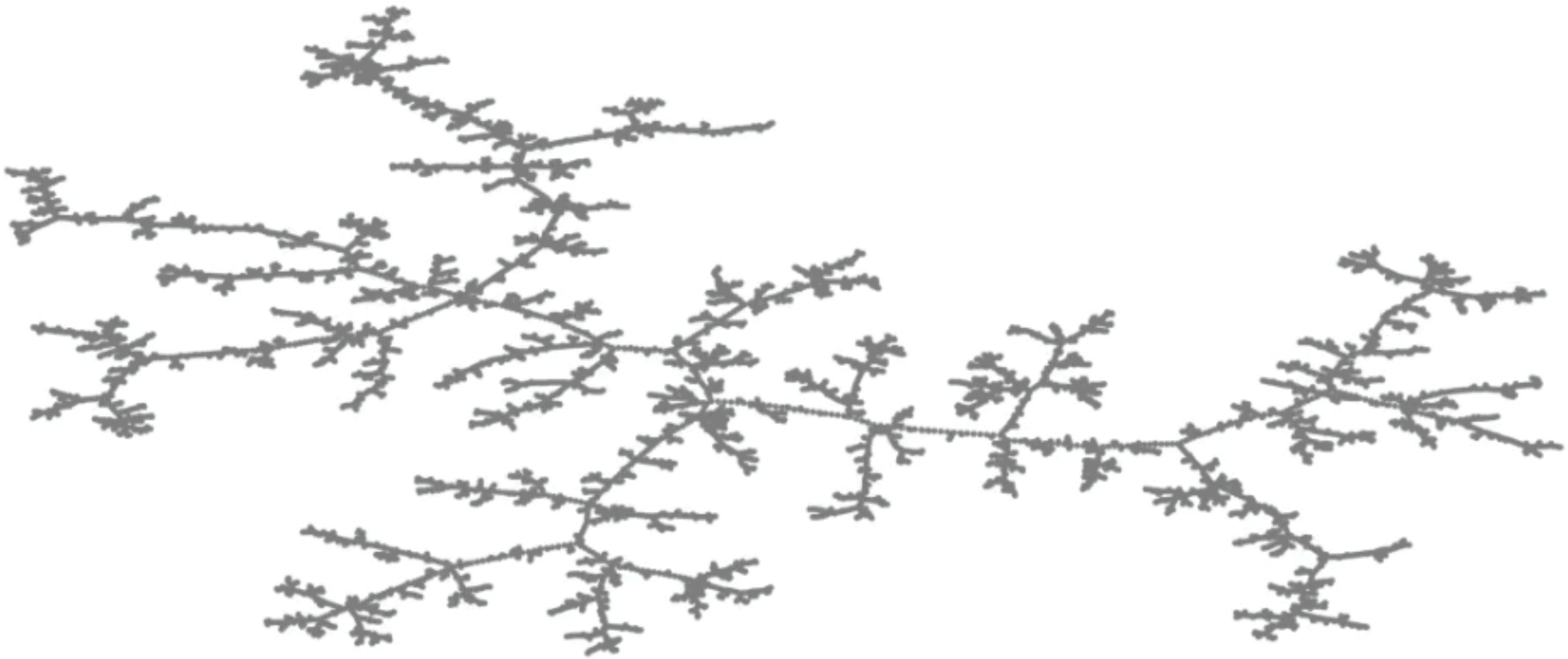
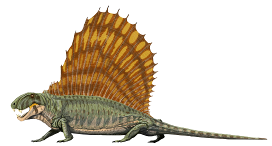


C(0,1)
C(2,0)
C(3,6)
C(10,1)
C(12,0)



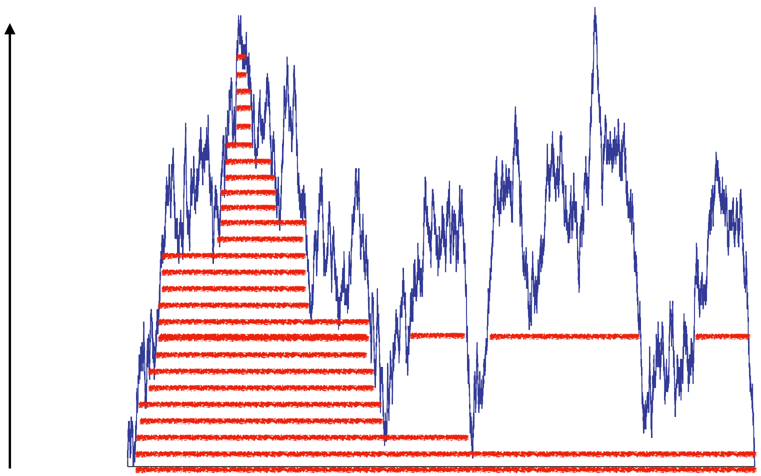
Credits: A. Caraceni

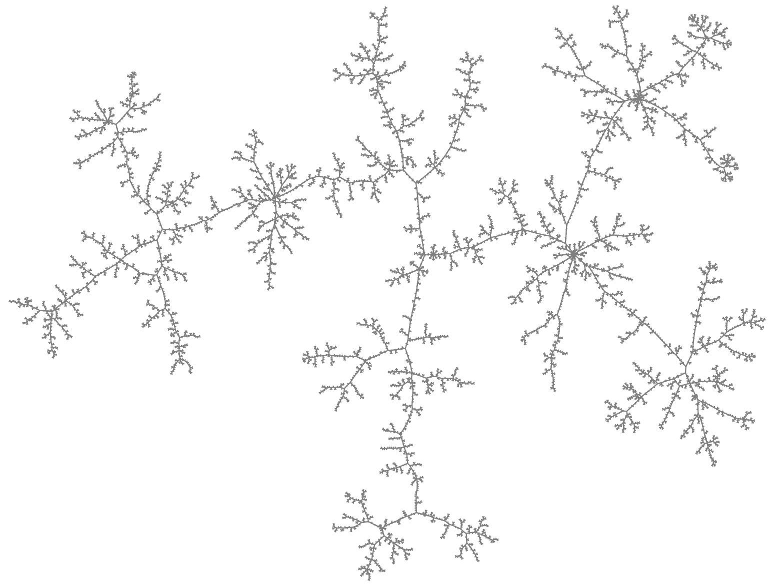
Elements of Proof


C'est dans les vieux pots
que l'on fait
les meilleures soupes
Isabelle C.

Back to the CRT

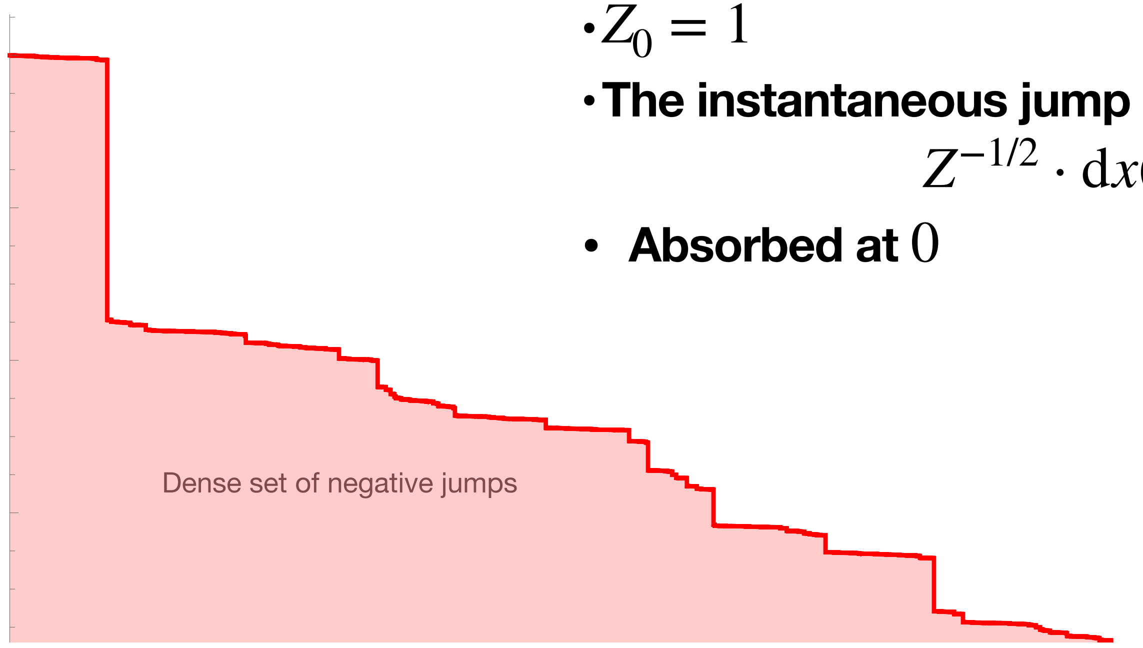




Bertoin 2002: $\mathscr{T}$ **can be seen as a self-similar Markov tree coding for the genealogy of particles undergoing fragmentation: A particle of mass** m **splits into two particles of mass** xm **and** $(1-x)m$ **at rate**

$$m^{-1/2} \cdot (x(1-x))^{-3/2}$$

Locally largest fragment

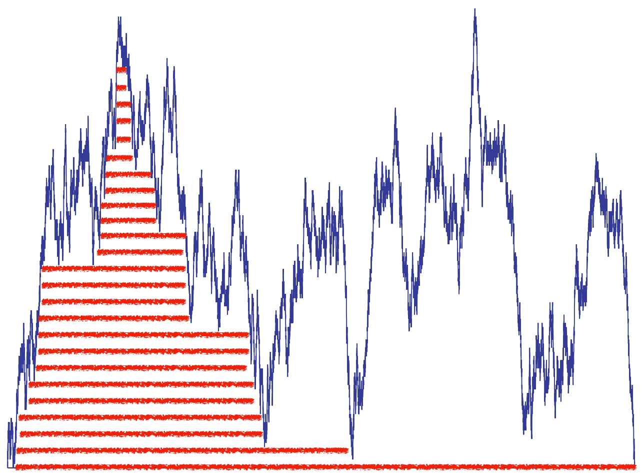


Locally largest fragment is the (pssMp) process Z such that

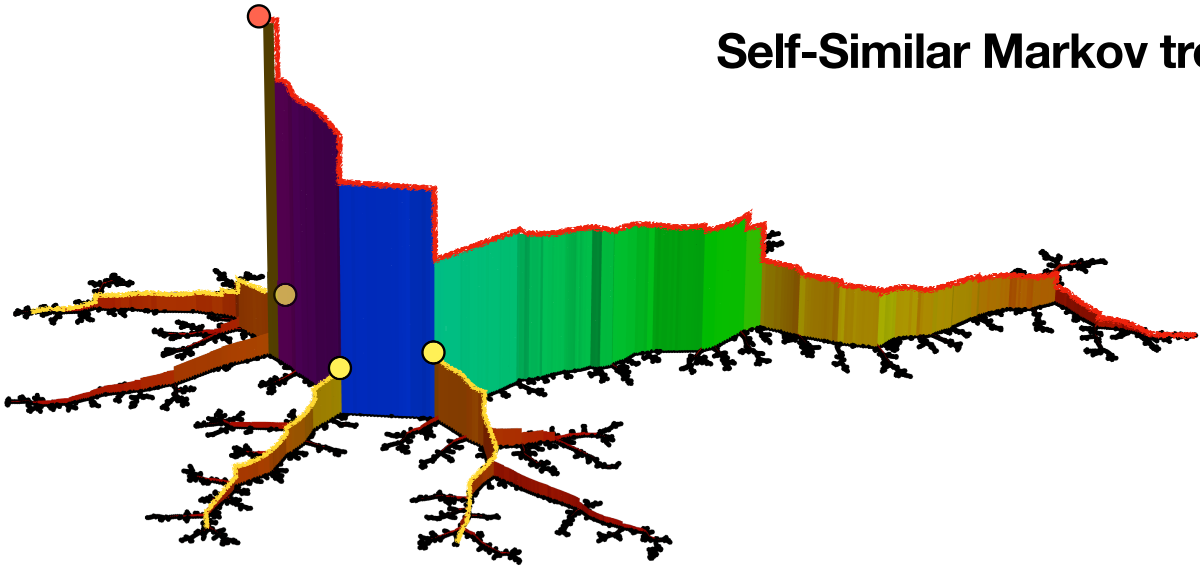
- $Z_0 = 1$
- The instantaneous jump measure is law of $-Zx$ under

$$Z^{-1/2} \cdot \mathrm{d}x(x(1-x))^{-3/2}\mathbf{1}_{x<1/2}$$

- Absorbed at 0



Back to the CRT



Self-Similar Markov trees, see Bertoin, C., Riera 2024

Locally largest fragment is the (pssMp) process Z such that

- $Z_0 = 1$
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$$Z^{-1/2} \cdot \mathrm{d}x(x(1-x))^{-3/2}\mathbf{1}_{x<1/2}$$

- Absorbed at 0

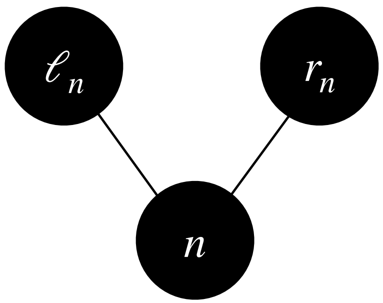
Reproduction : negative jumps of Z.
Each reproduction event starts an independent locally largest fragment. Iterate. Glue.

A particle of mass m splits into two particles of mass xm and $(1-x)m$ at rate

$$m^{-1/2} \cdot (x(1-x))^{-3/2}$$

The winner takes it all

$$C(a,b) = \frac{(2a+1)(2a+2)(2a+3+3(2b+1))}{(2a+2b+1)(2a+2b+2)(2a+2b+3)}$$



The process $(\ell_n, r_n)_{n\geq 1}$ is a Markov chain such that

- $\ell_n + r_n = n - 1$
- **one of ℓ or r is eventually constant, the other takes it all**
- **But sometimes, both processes grow large.**

When both ℓ_n and r_n are large, say

$$\ell_n = xn \quad \textbf{and} \quad r_n = (1-x)n$$

Then the history to reach those values is deterministic in the fluid limit

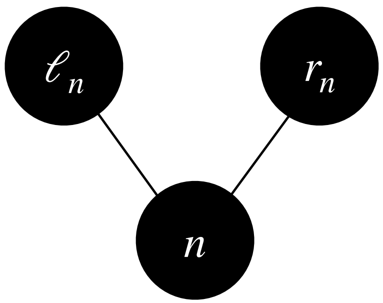
The winner takes it all

Fix $x > 0$. Conditionally on

$$\ell_n = xn \quad \textbf{and} \quad r_n = (1-x)n$$

Then we have

$$\left(n^{-1}\ell_{nt}, n^{-1}r_{nt}\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{(\mathbb{P})} \left(f(x,t), f(1-x,t)\right)_{t\in[0,1]}$$



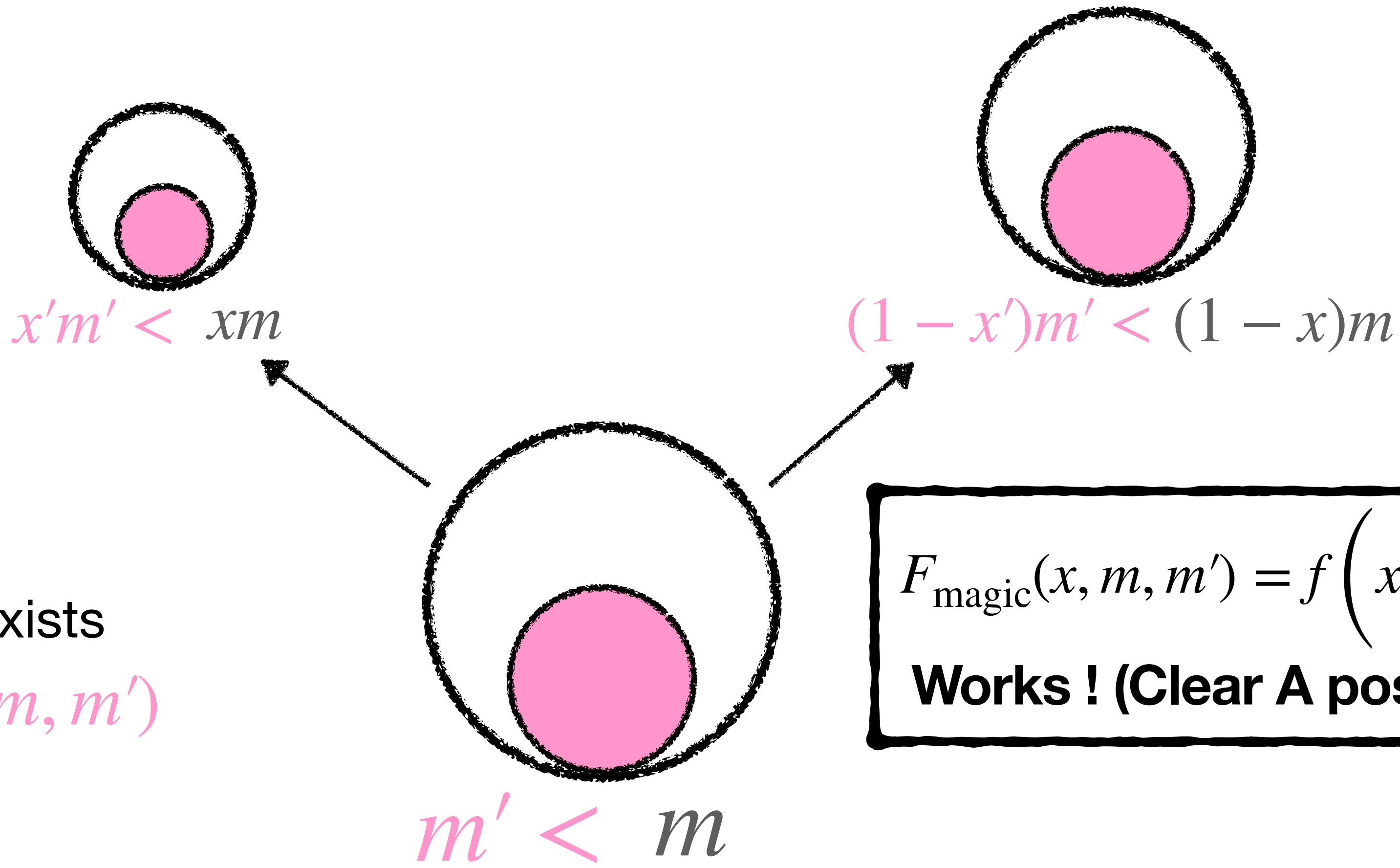
$$f(x,t) = \begin{cases} \dfrac{t\left(t(1-2x)^2 - \sqrt{t}\sqrt{t-1+\frac{1}{(1-2x)^2}}(1-2x)^2 + 4(1-x)x\right)}{2t(1-2x)^2+8(1-x)x} & \textit{if } x \leqslant \frac{1}{2}, \\ \\ \dfrac{t\left(t(1-2x)^2 + \sqrt{t}\sqrt{t-1+\frac{1}{(1-2x)^2}}(1-2x)^2 + 4(1-x)x\right)}{2t(1-2x)^2+8(1-x)x} & \textit{if } x \geqslant \frac{1}{2}, \end{cases}$$

is the unique solution to the system

$$\begin{cases} \frac{\partial f_x}{\partial t} = c(f(x,t)/t) & \textit{for all } t \in \mathbb{R}_+, \\ f_x(1) = x. \end{cases}$$

Particle m splits into 2 particles xm and $(1-x)m$ at rate $m^{-1/2} \cdot (x(1-x))^{-3/2}$

Pushforward of $m^{-1/2} \cdot (x(1-x))^{-3/2} \mathrm{d}x$ by F_{magic} is $(m')^{-1/2} \cdot (x'(1-x'))^{-3/2} \mathrm{d}x'$



Suppose there exists

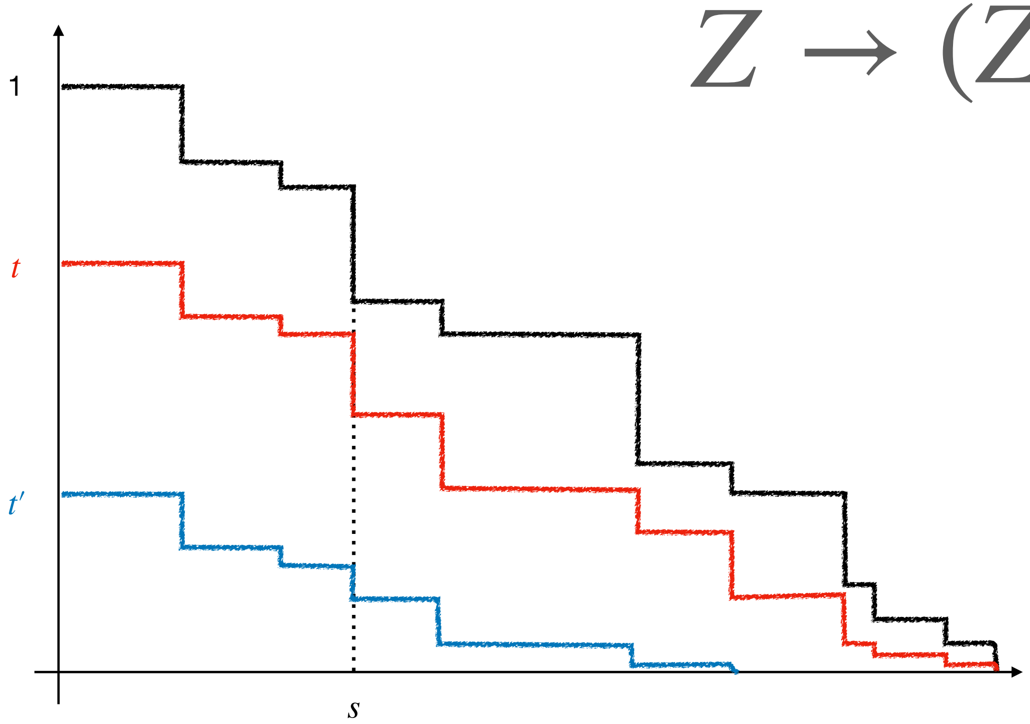
$x' = F_{\text{magic}}(x, m, m')$

$$F_{\text{magic}}(x, m, m') = f\left(x, \frac{m'}{m}\right)\frac{1}{m}$$

Works ! (Clear A posteriori)

Trimming the CRT

$$Z \to (Z^{(t)} : t \in [0,1])$$



If Z performs a jump of $xZ_{s\_}$ at time $s > 0$

then $Z_s^{(t)}$ performs a jump of

$$f\left(x, \frac{Z_{s-}^{(t)}}{Z_{s-}}\right)\frac{1}{Z_{s\_}} \cdot Z_{s\_}$$

(SDE pure jump).

Then for each $t \in [0,1]$ the process $Z^{(t)}$ has the law of a locally largest fragment but started from t instead of 1.

Iterate inside each branch. Glue -> $(\mathcal{T}_t)_{t \in [0,1]}$