Control theory and splitting methods

Karine Beauchard
ENS Rennes, IRMAR
joint work with

Frédéric Marbach

Adrien Laurent

institut universitaire de France
FONDATION SIMONE ET CINO DEL DUCA INSTITUT DE FRANCE 1975
anr agence nationale de la recherche

Goal of this talk

- **highlight the deep links between numerical splitting methods and control theory**

- provide (control-inspired) proofs of conjectures of the order theory of splitting methods

Control theory: the infinitesimal question

Let $m \in \mathbb{N}^*$ and $f_0, \dots, f_m : \mathbb{R}^d \to \mathbb{R}^d$ smooth such that $f_0(0) = 0$.

$$\dot{x}(t) = f_0(x(t)) + u_1(t) f_1(x(t)) + \cdots + u_m(t) f_m(x(t))$$

(Small-state) Small-time local controllability (STLC):

$$\forall T, \delta > 0, \quad \exists r > 0, \quad \forall x^* \in B_{\mathbb{R}^d}(0, r), \quad \exists u \in L^1((0,T); \mathbb{R}^m)$$
$$\text{such that} \quad x(T; u, 0) = x^* \text{ and } x([0,T]) \subset B(0, \delta).$$

- For linear systems $\dot{x} = Ax + Bu$: NSC = Kalman rank condition. [1960]
- If $f_0 = 0$, NSC = LARC: Lie$(f_1, \dots, f_m)(0) = \mathbb{R}^d$. [Chow, Rashevski 1930']
- If $f_0 \neq 0$, the iterated Lie brackets at 0 still have the answer, [Krener1973] Lie$(f_0, \dots, f_m)(0) = \mathbb{R}^d$ is necessary, [Herman 1963, Nagano 1966] but not sufficient: bad Lie brackets $\mathrm{ad}_{f_1}^{2k}(f_0)(0)$. [Sussmann 1983, Stefani 1986]

Rudolf Kalman
1930-2016

Sophus Lie
1842-1899

Splitting methods

We want to compute the flow of $f_0 + f_1$ from the flows of f_0 and f_1.

Splitting method of order N: (α, β) such that $\forall f_0, f_1$, $\forall x_0$

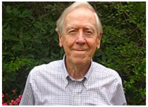
$$e^{T(f_0+f_1)}x_0 = e^{\alpha_1 T f_0} e^{\beta_1 T f_1} \dots e^{\alpha_k T f_0} e^{\beta_k T f_1} x_0 + \underset{T \to 0}{O}(T^{N+1}). \qquad (1)$$

Ex: Lie - Trotter : $e^{T(f_0+f_1)}x_0 = e^{T f_0} e^{T f_1} x_0 + O(T^2)$
Strang : $e^{T(f_0+f_1)}x_0 = e^{\frac{1}{2} T f_0} e^{T f_1} e^{\frac{1}{2} T f_0} x_0 + O(T^3)$

- Without constraint i.e. $(\alpha, \beta) \in \mathbb{R}^k \times \mathbb{R}^k$: existence $\forall N$.
- With only forward flows of f_0 i.e. $(\alpha, \beta) \in \mathbb{R}_+^k \times \mathbb{R}^k$: order ≤ 2.
[Goldman Kaper 1996, Blanes Casas 2005]
- If $(\alpha, \beta) \in \mathbb{R}_+^k \times \mathbb{C}_+^k$ order 6 OK, if $(\alpha, \beta) \in \mathbb{C}_+^k \times \mathbb{C}_+^k$ order 44 OK
[Castella Chartier Descombes Vilmart 2009, Hansen Ostermann 2009, Blanes Casas Chartier Murua 2016]
- With additionnal commutator flows: with $[f_1, [f_1, f_0]]$ order 4 OK
[Takahashi Imada 1984, Auzinger Hofstatter Koch 2019]

Sophus Lie
1842-1899

Hale Trotter
1931-2022



Gilbert Strang
1934-

Splitting methods with $+f_0$ as control systems

- **An** $(\mathbb{R}_+, \mathbb{K})$**-splitting method** $\quad e^{\alpha_k T f_0} e^{\beta_k T f_1} \dots e^{\alpha_1 T f_0} e^{\beta_1 T f_1}$
(i.e. with $(\alpha, \beta) \in \mathbb{R}_+^k \times \mathbb{K}^k$) **is a trajectory of the control system**

$$\dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t))$$

where $\quad u = \sum_j \beta_j T \delta_{t=\tau_j}, \quad \beta_j \in \mathbb{K}, \quad \tau_j = \sum_{j' \leq j} \alpha_{j'} T.$

Ex: Lie - Trotter $\quad e^{Tf_0} e^{Tf_1} \quad \longrightarrow \quad u(t) = \delta_0(t)$
Strang $\quad e^{\frac{1}{2} T f_0} e^{T f_1} e^{\frac{1}{2} T f_0} \quad \longrightarrow \quad u(t) = \delta_{T/2}(t)$

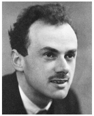
- **A splitting method involving** $f_2 := [f_1, [f_1, f_0]]$

$$e^{\alpha_k T f_0} e^{\beta_k T f_1} e^{\alpha'_k T f_0} e^{\gamma_k T f_2} \dots e^{\alpha_1 T f_0} e^{\beta_1 T f_1} e^{\alpha'_1 T f_0} e^{\gamma_1 T f_2}$$

(where $(\alpha, \beta, \gamma) \in \mathbb{R}_+^{2k} \times \mathbb{R}^k \times \mathbb{R}^k$) **is a trajectory of the control syst**

$$\dot{x}(t) = f_0(x(t)) + u_1(t) f_1(x(t)) + u_2(t) f_2(x(t))$$

where u_1, u_2 are finite sums of Dirac masses with disjoint supports.



Paul Dirac
1902-1984

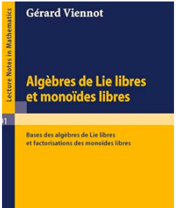
Common tools: (1) Free algebras

- Let $X := \{X_0, X_1\}$ be non-commutative **unknowns**.
- Let $\mathcal{A}(X)$ be the **free algebra** over X, i.e. the vector space of non-commutative polynomials. e.g. $7X_0^2 + 3X_1X_0 + 2X_0X_1$
- Let $\mathcal{L}(X)$ the **free Lie algebra** over X, i.e. the smallest vector subspace of $\mathcal{A}(X)$ containing X_0, X_1, and stable by the Lie bracket (commutator) operation $[a, b] := ab - ba$.
- Let $\widehat{\mathcal{A}}(X)$ be the algebra of **formal power series**. e.g. $\sum_{k=0}^{\infty} k! X_0^k X_1$
$$\mathcal{L}(X) \subset \mathcal{A}(X) \subset \widehat{\mathcal{A}}(X)$$
- One can "**evaluate**" (although not injective)

$$b \in \mathcal{L}(X) \hookrightarrow f_b \in C^{\omega}(\mathbb{R}^d; \mathbb{R}^d) \hookrightarrow f_b(0) \in \mathbb{R}^d$$

e.g. $[X_1, X_0] = X_1X_0 - X_0X_1 \to [f_1, f_0] = (Df_0)f_1 - (Df_1)f_0$

$\to [f_1, f_0](0) =: f_{(X_1, X_0)}(0)$

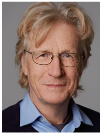


Lecture Notes in Mathematics

Gérard Viennot

Algèbres de Lie libres et monoïdes libres

Bases des algèbres de Lie libres
et factorisations des monoïdes libres



C. Reutenauer

Common tools: (2) Formal linear differential equation

Let $u \in L^1(0,T)$. $\begin{cases} \dot{S}(t) = S(t)(X_0 + u(t)X_1 + \sum_{b\in A} u_b(t) b) \\ S(0) = 1 \end{cases}$

Definition

The solution is the formal-series valued function $S : [0,T] \to \widehat{\mathcal{A}}(X)$ whose components S_n of degree n satisfy $S_0(t) = 1$

$$S_{n+1}(t) = \int_0^t \Big(S_n(\tau)(X_0 + u(\tau)X_1) + \sum_{b\in A} u_b(\tau) S_{n+1-|b|}(\tau) b \Big) \mathrm{d}\tau.$$

Chen-Fliess expansion: $S(t) = \sum_{n\in\mathbb{N}} \sum_{\sigma\in\{0,1\}^n} c_\sigma(t,u) X_{\sigma_1} \cdots X_{\sigma_n}$.

If $u = \beta\delta_\tau$, take the limit $\epsilon \to 0$ on the solutions for $u = \frac{\beta}{\epsilon} 1_{(\tau-\epsilon,\tau+\epsilon)}$.

Kuo Tsai Chen

1923-1987

Michel Fliess

1945-

Linearization principle (converse characteristics method)

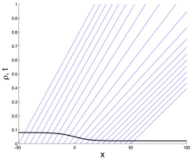
One transforms the **nonlinear ODE in** $\mathbb{R}^d$**:** $\dot{x} = f_0(x) + u(t) f_1(x)$, into a **linear equation in Op**$(C^\infty(\mathbb{R}^d, \mathbb{R}))$**:** $\frac{\mathrm{d}}{\mathrm{d}t} L(t) = L(t)(\underbrace{f_0 \cdot \nabla}_{X_0} + u(t) \underbrace{f_1 \cdot \nabla}_{X_1})$ on which we use the algebra developed for the formal differential equation $\dot{S}(t) = S(t)(X_0 + u(t) X_1)$.

The zero-order operator $\left| \begin{array}{llll} L(t): & C^\infty(\mathbb{R}^d, \mathbb{R}) & \to & C^\infty(\mathbb{R}^d, \mathbb{R}) \\ & \varphi & \mapsto & (p \mapsto \varphi(x(t; u, p))) \end{array} \right.$

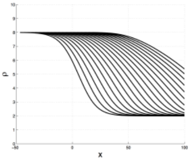
satisfies $\forall \varphi \in C^\infty(\mathbb{R}^d, \mathbb{R})$, $\forall p \in \mathbb{R}^d$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \big(L(t)\varphi\big)(p) &= D\varphi\Big(x(t; f, p)\Big).(f_0 + u(t) f_1)(x(t; f, p)) \\ &= \Big(L(t)(f_0 \cdot \nabla + u(t) f_1 \cdot \nabla)\varphi\Big)(p) \end{aligned}$$

a)



b)



Truncation and approximation

$$\begin{cases} \dot{S}(t) = S(t)(X_0 + u(t)X_1 + \sum_{b\in A} u_b(t) b) & \text{in } \widehat{\mathcal{A}}(X) \\ S(0) = 1 \end{cases}$$

If $S_N(t; X, u)$ is the component of $S(t)$ on polynomials of degree $\leq N$ then, $\forall f_0, f_1, u, x_0$, the solution of

$$\begin{cases} \dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t)) + \sum_{b\in A} u_b(t) f_b(x(t)) \\ x(0) = x_0 \end{cases}$$

satisfies

$$x(1; Tf, u, x_0) = S_N(1; Tf, u)(Id)(x_0) + \underset{T\to 0}{O}(T^{N+1})$$

Splitting has to do with the reachability of $e^{X_0+X_1}$ for $S_N(t; X, u)$.

Splitting method = free trajectory reaching $e^{X_0+X_1}$

$\mathcal{A}^N(X)$: Free nilpotent algebra of index N $\{\text{polynom. of deg.} \leq N\}$
$\mathcal{L}^N(X)$: Free nilpotent Lie algebra of index N $\mathcal{L}(X) \cap \mathcal{A}^N(x)$
$G^N(X) = \exp(\mathcal{L}^N(X))$: Lie group analytic submanifold

$$\begin{cases} \dot{S}_N(t) = S_N(t)\left(X_0 + u(t)X_1 + \sum_{b\in A} u_b(t) b\right) & \text{in } G^N(X) \\ S_N(0) = 1 \end{cases} \quad (2)$$

Splitting method of order N involving the elements of A

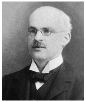
⇔ **Trajectory of (2) with controls** u, u_b **= finite sums of Dirac masses** with disjoint supports **such that** $S_N(1; X, u) = e^{X_0+X_1}$

⇐ **Controllability of the system (2)**

$u(t) = a\underline{u}(\frac{t}{T}) \Rightarrow S_N(t; u) = \lambda S_N(\frac{t}{T}, \underline{u})$ where $\left| \begin{array}{ll} \lambda : & \mathcal{A}^N(X) \to \mathcal{A}^N(X) \\ & X_0 \mapsto T X_0 \\ & X_1 \mapsto a X_1 \end{array} \right.$

$e^A e^B = e^Z$ where $Z = A + B + \frac{1}{2}[A,B] + \frac{1}{12}([A,[A,B]] + [B,[B,A]]) + \ldots$

Henry F. Baker
1866-1956



John E. Campbell
1862-1924

Felix Hausdorff
1868-1942

Wilhelm Magnus
1907-1990

The simplest case

Theorem

For every $N \in \mathbb{N}^*$, there exists an $(\mathbb{R}, \mathbb{R})$-splitting method of order N.

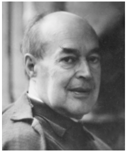
Theorem (Chow 1939, Rashevski 1938)

If $f_0, f_1 : \mathbb{R}^d \to \mathbb{R}^d$ are smooth vector fields s.t. Lie$(f_0, f_1)(x_0) = \mathbb{R}^d$ then the system $\dot{x}(t) = u_0(t) f_0(x(t)) + u_1(t) f_1(x(t))$ is STLC at x_0.

$$\begin{cases} \dot{S}_N(t) = S_N(t)(u_0(t)X_0 + u_1(t)X_1) & \text{in } G^N(X) \\ S_N(0) = 1 \end{cases}$$

For $f_0(S) = SX_0$ and $f_1(S) = SX_1$ we have $f_b(S) = Sb$ thus Lie$(f_0, f_1)(1) = \mathcal{L}^N(X)$.

Wei Liang Chow
1911-1995



Petr Rashevskii
1907-1983

Complex systems with a complex control $u : (0,T) \to \mathbb{C}$

Theorem (B, Laurent, Marbach 2024)

For every $N \in \mathbb{N}^*$, there exists an $(\mathbb{R}_+, \mathbb{C})$-splitting method of order N.

Theorem (B, Laurent, Marbach 2024, (Sussmann 1987))

If $f_0, f_1 : \mathbb{C}^d \to \mathbb{C}^d$ are holomorphic, $f_0(0) = 0$ and Lie$(f_0, f_1)(0) = \mathbb{C}^d$ then the system $\dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t))$ is STLC.

$$\begin{cases} \dot{S}_N(t) = S_N(t)(X_0 + u(t)X_1) \quad \text{on } G^N(X) \\ S_N(0) = 1 \end{cases}$$

We build a trajectory from 1 to e^{TX_0} with a controllable linearized system
[Coron 's return method]

- LARC provides u with a controllable linearized system.
- Then replacing u by $u \diamond e^{i\frac{2\pi}{N}} u \diamond \cdots \diamond e^{i\frac{2(N-1)\pi}{N}} u : (0, NT) \to \mathbb{C}$ increases the valuation of the polynomial $e^{-TX_0} S_N(T; X, u) - 1$. And iterate.

Why does the valuation of $\mathcal{Z}_N$ increase?

$$\begin{cases} \dot{S}_N(t) = S_N(t)(X_0 + u(t)X_1) & \text{in } G^N(X) \\ S_N(0) = 1 \end{cases}$$

$$S_N(t,u) = e^{tX_0} e^{\mathcal{Z}_N(t,u)} \text{ where } \mathcal{Z}_N(t,u) \in \mathcal{L}^N_{\oslash}(X)$$

- $\mathcal{Z}_N(T_1 + T_2, u_1 \diamond u_2) = \mathcal{Z}_N(T_1, u_1) + \mathcal{Z}_N(T_2, u_2) +$ higher valuation

$$e^{(T_1+T_2)X_0} e^{\mathcal{Z}(T_1+T_2, u_1 \diamond u_2)} = e^{T_2 X_0} e^{\mathcal{Z}(T_2,u_2)} e^{T_1 X_0} e^{\mathcal{Z}(T_1,u_1)}$$

thus $\quad e^{\mathcal{Z}(T_1+T_2, u_1 \diamond u_2)} = e^{-T_1 X_0} e^{\mathcal{Z}(T_2,u_2)} e^{T_1 X_0} e^{\mathcal{Z}(T_1,u_1)} \quad +$ CBH

- $\mathcal{Z}_N(T,u) = \sum_{j=1}^{N-1} \underbrace{Z_j}_{\deg_{X_1} = j} \quad \Rightarrow \quad \mathcal{Z}_N(T, e^{i\theta}u) = \sum_{j=1}^{N-1} e^{ij\theta} Z_j.$

Finally with $\quad v := u \diamond e^{i\frac{2\pi}{N}} u \diamond \cdots \diamond e^{i\frac{2(N-1)\pi}{N}} u \quad$ we obtain

$$\mathcal{Z}_N(NT, v) = \sum_{k=0}^{N-1} \sum_{j=1}^{N-1} e^{i\frac{2k\pi}{N} j} Z_j + \text{higher valuation}$$

Obstructions to splitting methods of order N

- The maximal order of an $(\mathbb{R}_+, \mathbb{R})$-splitting method is 2.
  [Goldman-Kaper 1996, Blanes, Casas 2005]
- The max order of an $(\mathbb{R}_+, \mathbb{R})$ splitting method involving $\mathrm{ad}^2_{X_1}(X_0)$ is 4
  [Takahashi Imada 1984, Auzinger Hofstatter Koch 2019]

What happens next?

$\forall k \in \mathbb{N}^*$, $W_k := \mathrm{ad}^2_{\mathrm{ad}^{k-1}_{X_0}(X_1)}(X_0)$ is an obstruction to STLC [B. Marbach 2022]

Theorem (B, Laurent, Marbach 2024)

The maximal order of an $(\mathbb{R}_+, \mathbb{R})$*-splitting method involving* $W_1, \dots, W_{N-1}$ *is* $2N$. *(*W_{2N} *is an obstruction to order* $2N+1$*)*

Sussmann's product, coordinates of the 2nd kind [1986]:

if $\mathcal{B}$ is a Hall basis of $\mathcal{L}(X)$ then $\quad S(t; X, u) = \overleftarrow{\prod_{b \in \mathcal{B}}} \exp\left(\xi_b(t,u) b\right)$

Marshall Hall Jr
1910-1990

Hector Sussmann
1946-

Easy computation of the $\xi_b(t,u)$: Lazard elimination

$$\begin{cases} \dot{S}(t) = S(t)(X_0 + u(t)X_1) \\ S(0) = 1 \end{cases} \qquad \text{Goal : } S(t) = \overleftarrow{\prod_{b\in\mathcal{B}}} e^{\xi_b(t;u)b}$$

Step 1: Eliminate X_1 by looking for $S(t)$ of the form $S_1(t)e^{\xi(t)X_1}$.

$$\begin{aligned} \dot{S}_1(t) &= S(t)(X_0 + u(t)X_1)e^{-\xi_{X_1}(t)X_1} + S(t)(-\dot{\xi}(t)X_1)e^{-\xi(t)X_1} \\ &= S_1(t)e^{\xi(t)X_1}X_0e^{-\xi(t)X_1} \qquad \text{because } \dot{\xi}(t) = u(t) \\ &= S_1(t)\sum_{m\in\mathbb{N}} \underbrace{\frac{\xi(t)^m}{m!}\cdot\dot{\xi}_{X_0}(t)}_{=\dot{\xi}_{\mathrm{ad}^m_{X_1}(X_0)}(t)} \mathrm{ad}^m_{X_1}(X_0) \qquad \to \text{does not involve } X_1 \end{aligned}$$

Step $(j+1)$: $\dot{S}_j(t) = \sum_{b\in H_j} \dot{\xi}_b(t)b$

Choose a bracket $a \in H_j$ and look for $S_j(t)$ of the form $S_{j+1}(t)e^{\xi_a(t)a}$.

$$\dot{S}_{j+1}(t) = S_{j+1}(t)\sum_{m\in\mathbb{N}}\sum_{b\in H_y\setminus\{a\}} \underbrace{\frac{(\xi_a(t))^m}{m!}\dot{\xi}_b(t)}_{\dot{\xi}_{\mathrm{ad}^m_a(b)}(t)} \mathrm{ad}^m_a(b)$$

The concept of "Hall basis" of $\mathcal{L}(X)$ is adapted to this algorithm.
[Viennot 1978, Sussmann 1986]

And reciprocally...

Theorem (B, Laurent, Marbach 2024)

There exists an $(\mathbb{R}_+,\mathbb{R})$*-splitting method of order* $2N$ *involving* $W_1,\dots,W_{N-1}$.

- If $\operatorname{ad}^2_{f_1}(f_0)=f_j$ then the controllability of

$$\dot{x}=f_0(x)+u_1(t)f_1(x)+\cdots+u_m(t)f_m(x)$$

  is equivalent to the controllability of

$$\dot{x}=f_0(x)+u_1(t)f_1(x)+\cdots+u_m(t)f_m(x)+u_{m+1}(t)[f_1,f_0](x)$$

- Let $M_0:=X_1$ and $M_{j+1}=[M_j,X_0]$. Then $W_j=\operatorname{ad}^2_{M_{j-1}}(X_0)$.

$$\begin{cases}\dot{S}_N(t)=S_N(t)(X_0+\sum_{b\in A}u_b(t)b) & \text{in } G^N(X)\\ S_N(0)=1\end{cases}$$

$$\begin{aligned}
& A_0:=\{X_1,W_1,\dots,W_{N-1}\}\\
\Leftrightarrow\quad & A_1=A_0\cup\{M_1\} \quad \text{because } \operatorname{ad}^2_{X_1}(X_0)=W_1\in A_0\\
\Leftrightarrow\quad & A_2=A_1\cup\{M_2\} \quad \text{because } \operatorname{ad}^2_{M_1}(X_0)=W_2\in A_1, \quad \text{etc...}\\
& A_N\supset\{X_1,M_1,\dots,M_{2N}\} \text{ and } \operatorname{Lie}(M_1,\dots,M_{2N})=\mathcal{L}^N_{\oslash}(X)
\end{aligned}$$

Conclusion

- Splitting methods $\sim$ trajectories of (free) control systems.
- Control theory has tools to prove obstructions and to prove abstract existence results.
- They formalise the order theory of splitting methods with sign constraints on the coefficients.
- Splitting methods with $(\mathbb{R}^+, \mathbb{C})$ coefficients exist for any order.
- There exists an $(\mathbb{R}_+, \mathbb{R})$ splitting method of order $2N$ involving $W_1, \ldots, W_{N-1}$. (a 100% quadratic theory)

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Quiz

Who are they?