# Determination of discontinuous diffusion coefficients for the heat equation on a tree-shaped network

Emmanuelle Crépeau, Lionel Rosier and Julie Valein

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# Coefficient inverse problem in the heat equation

In a smooth bounded domain $\Omega \subset \mathbb{R}^n$, it writes for instance,

$$\begin{cases} \partial_t y(x,t) - \partial_x(c^*(x)\partial_x y(x,t)) = g(x,t), & (x,t) \in \Omega \times (0,T), \\ y(x,t) = h(x,t), & (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases}$$

- Given data: Source terms $g, h$ ; initial data: $y^0$;
- Unknown: the diffusion coefficient $c = c^*(x)$;
- Additional measurement:
    - internal measurement: $\partial_t y(t,x)$, $\partial_{tt} y(t,x)$ on $\omega \times (0,T)$
    - or boundary measurement: $\partial_\nu \partial_t y(t,x)$, $\partial_\nu \partial_{tt} y(t,x)$ on $\Gamma_0 \times (0,T)$
    - at a time $T' \in (0,T)$.

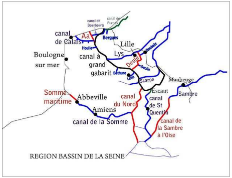
# Determination of the coefficient in the heat equation

$$\begin{cases} \partial_t y - \partial_x(c^* \, \partial_x y) = g, & \Omega \times (0,T), \\ y = h, & \partial\Omega \times (0,T) \\ y(0) = y^0, & \Omega. \end{cases}$$

*Is it possible to retrieve the coefficient* $c^* = c^*(x), \; x \in \Omega$ *from internal or boundary measurements* $d^*$ *on* $\omega \times (0,T)$ *or* $\Gamma_0 \times (0,T)$ *and on* $\Omega \times \{T'\}$*?*

- Uniqueness: Given $c_1 \neq c_2$, can we guarantee $d_1 \neq d_2$ ?
- Stability: If $d_1 \simeq d_2$, can we guarantee that $c_1 \simeq c_2$ ?

# PDE on networks



Applications :

- control or stabilize the vibrations of elastic structures (as bridges, cranes,...),
- regulate the height of water in networks of irrigation canals,
- find the topography of the bottom in a network of irrigation canals,
- detect water losses by measurements in nodes,
- control gas flow in pipelines through compressors,
- determine the blood pressure leaving the heart with a finger pressure measurement,
- control road traffic on a network of roads or the flow of blood in a network of arteries,...

# PDE on networks

On networks, the state is represented by several components

$$Z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_N(t) \end{bmatrix}$$

and the components are coupled together by boundary conditions.
If $p < N$ is the number of controls/observations, it is therefore necessary to pass the information on the remaining $N - p$ branches.
Goals:

- minimize the number of observations, feedbacks or controls,
- choice of placement of observations, feedback mechanisms or controls based on network topology and branch lengths.

# An inverse problem on network

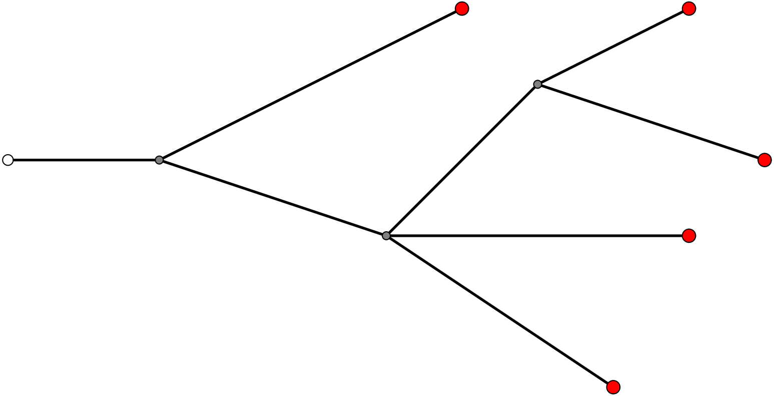


Figure: An $8$ branches tree-shaped network $\mathcal{R}$, with an unobserved root node and $5$ observed leaf nodes ●.

# A star-shaped network with internal observations

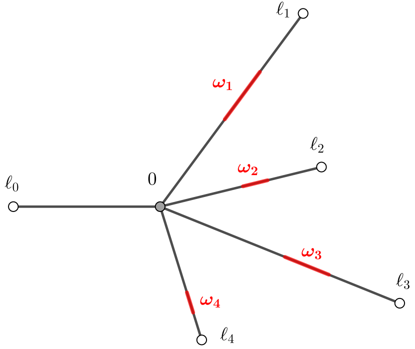


Figure: An $5$ branches star-shaped network $\mathcal{R}$, with $4$ observed open sets.

# Notations

Let us thus consider a finite star-shaped network $\mathcal{R}$.

- $n+1$ edges $e_j$ all connected at a single vertex which is the origin of all edges.
- $\ell_j$: the length of the branch $e_j$ ($j \in \{0, \cdots, n\}$).
- $f_j$: the restriction of the function $f$ on $\mathcal{R}$ to the branche $e_j$.
- $[f]_0 := \displaystyle\sum_{j=0}^{n} f_j(0)$.
- $$L^2(\mathcal{R}) = \left\{f : \mathcal{R} \to \mathbb{R},\ f_j \in L^2(0, l_j),\ \forall j \in \{0, \cdots, n\}\right\},$$
  $$H_0^1(\mathcal{R}) = \left\{f : \mathcal{R} \to \mathbb{R},\ f_j \in H^1(0, l_j),\ f_j(l_j) = 0,\ f_j(0) = f_i(0), \forall i, j\right\}.$$
- $\displaystyle\int_{\mathcal{R}} f(x)dx = \sum_{j=0}^{n} \int_0^{\ell_j} f_j(x)dx.$
- $\displaystyle\|f\|^2_{L^2(\mathcal{R})} = \int_{\mathcal{R}} |f|^2\, dx, \qquad \|f\|^2_{H_0^1(\mathcal{R})} = \int_{\mathcal{R}} |f_x|^2\, dx.$
- $n$ open sets $\omega_i \subset (0, l_i)$, $\omega_i \neq \emptyset$ ($i \in \{1, \cdots, n\}$), $\omega = \displaystyle\bigcup_{i \in \{1, \cdots, n\}} \omega_i$.
- $Q = \mathcal{R} \times (0, T)$.

# Inverse problem on a network

On each branch $e_j$ of the network, we consider the one-dimensional heat equation system

$$\begin{cases} \partial_t y_j(x,t) - \partial_x(c_j(x)\partial_x y_j(x,t)) = g_j(x,t), & (x,t) \in (0,\ell_j) \times (0,T), \\ y_j(x,0) = y_j^0(x), & x \in (0,\ell_j), \end{cases}$$

with

$$\begin{cases} y_j(\ell_j,t) = h_j(t),\ \forall j \in \{0,\cdots,n\}, \\ y_j(0,t) = y_i(0,t),\ \forall i,j \in \{0,\cdots,n\}, & (C^0) \\ [c\partial_x y(t)]_0 = 0 & (K). \end{cases}$$

## Assumptions on the diffusion coefficient

The diffusion coefficient $c$ is assumed to be piecewise $C^1$, i.e. $c_i \in C^1([0,l_i])$, $i \in \{0,\cdots,n\}$, and

$$0 < c_{min} \leq c(x) \leq c_{max}, \quad x \in \mathcal{R}. \tag{1}$$

# Inverse problems on a network

## Two inverse problems

- Is it possible to retrieve the diffusion coefficient $c(x)$, for $x \in \mathcal{R}$ from internal measurements of the solution on a subdomain on all but one edges of the network and a measurement of the solution at a time $T'$?
- Is it possible to retrieve the diffusion coefficient $c(x)$, for $x \in \mathcal{R}$ from boundary measurements of the solution on all but one of the external edges of the tree and a measurement of the solution at a time $T'$?

# Stability estimates with internal measurements [Crépeau, Rosier, V. 2025]

## Theorem

*Let* $T > 0$, $T' = T/2$, $\tau \in (0, T/2)$ *and* $\omega \subset \mathcal{R}$ *be an open set with* $\omega \cap (0, l_j) \neq \emptyset$ *for* $j \in \{1, \cdots, n\}$. *Let* $\tilde{c}$ *be a piecewise* $C^1$ *function satisfying* (1) *and let* $h_j = 0$ *for* $j \in \{0, \cdots, n\}$.
*Then for any* $y^0 \in L^2(\mathcal{R})$, *there exists* $g \in C_c^\infty(\omega \times (0, T))$ *such that for any piecewise* $C^1$ *function* $c$ *satisfying* (1),

$$c_i(0) = \tilde{c}_i(0) \quad \forall i \in \{0, \cdots, n\}, \text{ and } \max_{j \in \{0, \cdots, n\}} \|c_j\|_{W^{1,\infty}(0,l_j)} \leq c_{max}^{1,\infty},$$

*if* $y$ *and* $\tilde{y}$ *denote the solutions associated with* $c$ *and* $\tilde{c}$ *respectively, with* $g$ *and* $\tilde{y}_0 = y_0$, *then we have*

$$\|c - \tilde{c}\|_{L^2(R)} \leq C \Big( \|y_t - \tilde{y}_t\|_{L^2(\omega \times (\tau, T))} + \|y_{tt} - \tilde{y}_{tt}\|_{L^2(\omega \times (\tau, T))}$$
$$+ \|y_t(T') - \tilde{y}_t(T')\|_{L^2(\omega)} + \|y(T') - \tilde{y}(T')\|_{H^1(\mathcal{R})} + \|y(T') - \tilde{y}(T')\|_{H^2(\omega)} \Big).$$

# Stability estimates with boundary measurements [Crépeau, Rosier, V. 2025]

## Theorem

*Let* $T > 0$, $T' = T/2$, $\tau \in (0, T/2)$ *and* $\tilde{c}$ *be a piecewise* $C^1$ *function satisfying* (1). *Then for any* $y^0 \in L^2(\mathcal{R})$, *there exist some functions* $h_j \in C_c^1(0,T)$, $j \in \{0, \cdots, n\}$ *with* $h_0 \equiv 0$, *such that for any* $c$ *piecewise* $C^1$ *function satisfying* (1),

$$\begin{gathered}
c_i(0) = \tilde{c}_i(0) \qquad \forall i \in \{0, \cdots, n\}, \\
c_i(l_i) = \tilde{c}_i(l_i) \qquad \forall i \in \{1, \cdots, n\}, \\
\max_{j \in \{0, \cdots, n\}} \|c_j\|_{W^{1,\infty}(0,l_j)} \leq c^{1,\infty}_{max},
\end{gathered}$$

*if* $y$ *and* $\tilde{y}$ *denote the respective solutions associated with* $c$ *and* $\tilde{c}$ *respectively, with* $g \equiv 0$ *and* $\tilde{y}^0 = y^0$, *we have*

$$\begin{aligned}
\|c - \tilde{c}\|_{L^2(\mathcal{R})} \leq C \Bigg( \sum_{j=1}^{n} \Bigg[ & \|y_{tx}(l_j) - \tilde{y}_{tx}(l_j)\|_{L^2(\tau,T)} + \|y_{ttx}(l_j) - \tilde{y}_{ttx}(l_j)\|_{L^2(\tau,T)} \\
& + \left| y_{j,x}(l_j, T') - \tilde{y}_{j,x}(l_j, T') \right| \Bigg] + \left\| y(T') - \tilde{y}(T') \right\|_{H^1(\mathcal{R})} \Bigg).
\end{aligned}$$

# Related works

## Heat equation on a domain

- Benabdallah, Gaitan, Le Rousseau 2007
- Benabdallah, Dermenjian, Le Rousseau 2007
- Le Rousseau, Robbiano 2011
- Yuan, Yamamoto 2009
- Cristofol, Gaitan, Niinimaki, Poisson 2013
- Dinakar, Barani Balan, Balachandran 2017
- Poisson 2008

## Inverse problem on a network

- Baudouin, Crépeau, V. 2011, Baudouin, de Buhan, Crépeau, V. 25
- Ignat, Pazoto, Rosier 2012
- Apraiz, Barcena-Petisco 2023
- Avdonin, Kurasov 2008, Belishev 2004, Belishev, Vakulenko 2006,...
- Avdonin, Bell, Mikhaylov, Nurtazina 2019, Avdonin, Mikhaylov, Nurtazina 2017, Avdonin, Zhao 2021,...

# Outline

1. Carleman estimates

2. Stability results

# Outline

1. Carleman estimates
    - Carleman estimates in $L^2(\mathbb{R})$ with interior observations
    - Carleman estimates with one less derivative (interior observations)
    - Carleman estimates with one less derivative (boundary observations)
2. Stability results
    - Stability results with interior observations
    - Stability results with boundary observations

# Carleman weight function $\beta$

$$\beta_0(x) = a_0 x + d, \quad x \in (0, \ell_0)$$
$$\forall j \in \{1, \cdots, n\},\ \beta_j(x) = a_j x^2 + b_j x + d, \quad x \in (0, \ell_j).$$

There exist $(a_0, \dots, a_n, b_1, \dots, b_n, d) \in (\mathbb{R}_-)^{n+1} \times (\mathbb{R}_+)^{n+1}$ and $\bar{\beta} > 0$ such that for all $i, j \in \{0, \cdots, n\}$ and all $x \in [0, l_j]$, we have

(i) $\beta_j(0) = \beta_j(0)$

(ii) $0 < \frac{\bar{\beta}}{2} < \beta_j(x) < \bar{\beta}$

(iii) $\beta_j'(l_j) < 0$

(iv) $\beta'(x) \neq 0$ on $\mathcal{R} \backslash \omega$

(v) the matrix $A_\beta$ is positive definite:

$$\exists \alpha > 0,\ \forall \xi \in \mathbb{R}^{n+1},\ (A_\beta \xi, \xi) \geq \alpha \|\xi\|^2$$

where $A_\beta$ is $(n+1) \times (n+1)$ symmetric matrix defined by

$$A_\beta := \begin{pmatrix} \beta_0'(0) + \beta_1'(0) & \beta_0'(0) & \cdots & \beta_0'(0) & -\beta_0'(0)[c\beta']_0 \\ & \beta_0'(0) + \beta_2'(0) & \ddots & \vdots & -\beta_0'(0)[c\beta']_0 \\ & & \ddots & \beta_0'(0) & \vdots \\ & & & \beta_0'(0) + \beta_n'(0) & -\beta_0'(0)[c\beta']_0 \\ & & & & \beta_0'(0)([c\beta']_0)^2 + [c^2(\beta')^3]_0 \end{pmatrix}.$$

# Carleman estimate in $L^2(\mathbb{R})$ with interior observations

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{t(T-t)}, \qquad \eta(x,t) = \frac{e^{\lambda\bar{\beta}} - e^{\lambda\beta(x)}}{t(T-t)}, \qquad (x,t) \in \mathcal{R} \times (0,T)$$

## Theorem

*There exist $\lambda_0 > 0$, $s_0 > 0$ and $C > 0$ such that for $s \geq s_0$, $\lambda \geq \lambda_0$*

$$\begin{aligned} s\lambda^2 \iint_Q \varphi\, |u_x|^2\, e^{-2s\eta} dxdt + s^3\lambda^4 \iint_Q \varphi^3\, |u|^2\, e^{-2s\eta} dxdt \\ + s^3\lambda^3 \int_0^T |\varphi(0,t)|^3 |u(0,t)|^2 e^{-2s\eta(0,t)} dt \\ \leq C \left[ s^3\lambda^4 \int_0^T \int_{\widetilde{\omega}} \varphi^3\, |u|^2\, e^{-2s\eta} dxdt + \iint_Q |u_t \pm (cu_x)_x|^2\, e^{-2s\eta} dxdt \right. \\ \left. + s\lambda \int_0^T \varphi(0,t)\, |k(t)|^2\, e^{-2s\eta(0,t)}\, dt + \frac{1}{s^2} \int_0^T |k_t(t)|^2\, e^{-2s\eta(0,t)}\, dt \right], \end{aligned}$$

*for $u \in D(L) = \{u \in L^2(0,T; H_0^1(\mathcal{R})), Lu = u_t - (cu_x)_x \in L^2(Q)\}$ if $\pm = -$, or $u \in D(L^*) = \{u \in L^2(0,T; H_0^1(\mathcal{R})), L^*u = -u_t - (cu_x)_x \in L^2(Q)\}$ if $\pm = +$, where $k \in H^1(0,T)$ is as*

$$[cy_x(t)]_0 =: k(t).$$

# Some ideas of the proof

We set $w = e^{-s\eta}u$ and the conjugate operator $Pw = -e^{-s\eta}L^*(e^{s\eta}w)$. Easy calculations bring

$$Pw = \underbrace{(cw_x)_x + s^2\lambda^2\varphi^2(\beta')^2cw + s\eta_t w}_{P_1w}$$

$$+\underbrace{w_t - 2s\lambda\varphi c\beta' w_x - 2s\lambda^2\varphi c(\beta')^2 w}_{P_2w} + \underbrace{-s\lambda\varphi(c\beta')_x w + s\lambda^2\varphi c(\beta')^2 w}_{Rw},$$

and

$$\iint_Q |Pw - Rw|^2 dxdt$$

$$= \iint_Q (|P_1w|^2 + |P_2w|^2)dxdt + 2\iint_Q P_1wP_2wdxdt.$$

# Some ideas of the proof

The main work of the proof consists in the computation and bound from below of the cross-term

$$I = \iint_Q P_1 w \, P_2 w dx dt, \qquad \text{by:}$$

- positive and dominant terms as

$$s\lambda^2 \iint_Q \varphi \left|w_x\right|^2 dxdt + s^3\lambda^4 \iint_Q \varphi^3 \left|w\right|^2 dxdt + s^3\lambda^3 \int_0^T |\varphi(0,t)|^3 |w(0,t)|^2 dt$$

- negative boundary terms (measured)

$$-s^3\lambda^4 \int_0^T \int_\omega \varphi^3 \left|w\right|^2 dxdt - s\lambda^2 \int_0^T \int_\omega \varphi \left|w_x\right|^2 dxdt$$

- negative central node terms

$$-s\lambda \int_0^T \varphi(0,t) e^{-2s\eta(0,t)} \left|k(t)\right|^2 \, dt - \frac{1}{s^2} \int_0^T e^{-2s\eta(0,t)} \left|k_t(t)\right|^2 \, dt$$

# Terms at the central node

The terms at the central nodes are

$$B = s\lambda \int_0^T \varphi(0,t)(A_\beta W(t), W(t))dt$$
$$+ \text{ terms in } k(t) + \text{ dominated terms,}$$

where $W(t) \in \mathbb{R}^{n+1}$ is defined by

$$W(t) = \big((c_j(0) w_{j,x}(0,t))_{j=1\dots n}, s\lambda\varphi(0,t)w(0,t)\big)$$

By assumption on $A_\beta$, we have

$$s\lambda \int_0^T \varphi(0,t)(A_\beta W(t), W(t))dt \geq \alpha s^3\lambda^3 \int_0^T |\varphi(0,t)|^3 \, |w(0,t)|^2 \, dt$$
$$+ \alpha s\lambda \sum_{j=1}^{n} \int_0^T \varphi(0,t) \, |c_j(0) w_{j,x}(0,t)|^2 \, dt.$$

# Carleman estimates with one less derivative (interior observations)

## Theorem

*Let us consider the following problem:*

$$\begin{cases} q_t - (cq_x)_x = f_x, & \text{in } Q, \\ q_j(l_j, t) = 0, & \forall j \in \{0, \cdots, n\}, \forall t \in (0,T), \qquad (D) \\ q_i(0,t) = q_j(0,t), & \forall i, j \in \{0, \cdots, n\}, \forall t \in (0,T), \qquad (C^0) \\ [cq_x(t)]_0 = 0, & \forall t \in (0,T), \qquad (K) \\ q(x,0) = q_0(x), & \forall x \in \mathcal{R}. \end{cases}$$

*Assume that $q_0 \in L^2(\mathcal{R})$, $f \in L^2(0,T,H^1(\mathcal{R}))$ with $f(0,t) = 0$ for a.e. $t \in (0,T)$. Pick $\lambda \geq \lambda_0$, then there exist $\bar{s} \geq s_0 > 0$ and $C > 0$ such that for $s \geq \bar{s}$*

$$\iint_Q s^3\lambda^4\varphi^3 e^{-2s\eta} |q|^2 \, dxdt \\ \leq C \left( \iint_Q s^2\lambda^2\varphi^2 e^{-2s\eta} |f|^2 \, dxdt + \int_0^T \int_\omega s^3\lambda^4\varphi^3 e^{-2s\eta} |q|^2 \, dxdt \right).$$

# Some ideas of the proof ([Imanuvilov, Yamamoto 2003], [Fernandez-Cara, Guerrero 2006], [Poisson 2008])

The function $q$ is solution in $L^2(Q)$ if and only if for any $g \in L^2(Q)$

$$\iint_Q qg\,dxdt = -\iint_Q f z_x\,dxdt + \int_{\mathcal{R}} q_0 z(0)\,dx,$$

where $z \in D(L^*)$ is the solution of

$$\begin{cases} -z_t - (cz_x)_x = g, & \text{in } Q, \\ (D), (C^0), (K) & \\ z(x,T) = 0, & \forall x \in \mathcal{R}. \end{cases}$$

We consider $L := \partial_t - \partial_x(c\partial_x)$, $L^* := -\partial_t - \partial_x(c\partial_x)$ in $Q$ and

$$\begin{cases} L^*(e^{-2s\eta}Lp) + s^3\lambda^4\varphi^3 e^{-2s\eta} p \mathbb{1}_\omega = s^3\lambda^4\varphi^3 e^{-2s\eta} q, & \text{in } Q, \\ (D),\ (C^0), (K) \text{ for } e^{-2s\eta}Lp & \\ (e^{-2s\eta}Lp)(x,0) = (e^{-2s\eta}Lp)(x,T) = 0, & \forall x \in \mathcal{R}, \end{cases}$$

with unknown $p \in P_0 := \{u \in C^2(\bar{Q});\ (D), (C^0), (K)\}$.

# Some ideas of the proof

The variational problem has a unique solution $p \in P := \bar{P_0}^{\|\cdot\|_a}$:

$$\underbrace{\iint_Q e^{-2s\eta} Lp Lp' dxdt + \int_0^T \int_\omega s^3 \lambda^4 \varphi^3 e^{-2s\eta} pp' dxdt}_{=:a(p,p')} = \underbrace{\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} qp' dxdt,}_{=:b_q(p')} \quad \forall p' \in P.$$

Let us set $\hat{u} := -s^3 \lambda^4 \varphi^3 e^{-2s\eta} \mathbb{1}_\omega p$, $\hat{z} = e^{-2s\eta} Lp$. We see that $\hat{z} \in D(L^*)$ and $\hat{z}$ is solution of the null controllability problem

$$\begin{cases} -\hat{z}_t - (c\hat{z}_x)_x = \hat{u} + s^3 \lambda^4 \varphi^3 e^{-2s\eta} q, & \text{in } Q, \\ (D), (C^0), (K) & \\ \hat{z}(x,T) = 0, \ \hat{z}(x,0) = 0, & \forall x \in \mathcal{R}, \end{cases}$$

and

$$\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 \, dxdt = -\iint_Q f \hat{z}_x dxdt - \int_0^T \int_\omega q\hat{u} dxdt.$$

# Some ideas of the proof

We can prove

$$\iint_Q e^{2s\eta}\left|\hat{z}\right|^2 dxdt + \int_0^T \int_\omega s^{-3}\lambda^{-4}\varphi^{-3}e^{2s\eta}\left|\hat{u}\right|^2 dxdt \leq C \iint_Q s^3\lambda^4\varphi^3 e^{-2s\eta}\left|q\right|^2 dxdt$$

and

$$\iint_Q s^{-2}\lambda^{-2}\varphi^{-2}e^{2s\eta}c\left|\hat{z}_x\right|^2 dxdt \leq C \iint_Q s^3\lambda^4\varphi^3 e^{-2s\eta}\left|q\right|^2 dxdt.$$

Therefore

$$\begin{aligned}\iint_Q s^3\lambda^4\varphi^3 e^{-2s\eta}\left|q\right|^2 dxdt &= -\iint_Q f\hat{z}_x dxdt - \int_0^T \int_\omega q\hat{u} dxdt \\ &\leq \left(\iint_Q s^{-2}\lambda^{-2}\varphi^{-2}e^{2s\eta}\left|\hat{z}_x\right|^2\right)^{1/2}\left(\iint_Q s^2\lambda^2\varphi^2 e^{-2s\eta}\left|f\right|^2\right)^{1/2} \\ &+ \left(\int_0^T\int_\omega s^{-3}\lambda^{-4}\varphi^{-3}e^{2s\eta}\left|\hat{u}\right|^2\right)^{1/2}\left(\int_0^T\int_\omega s^3\lambda^4\varphi^3 e^{-2s\eta}\left|q\right|^2\right)^{1/2} \\ &\leq C\left(\iint_Q s^3\lambda^4\varphi^3 e^{-2s\eta}\left|q\right|^2\right)^{1/2}\left(\left(\iint_Q s^2\lambda^2\varphi^2 e^{-2s\eta}\left|f\right|^2\right)^{1/2}\right. \\ &\qquad \left. + \left(\int_0^T\int_\omega s^3\lambda^4\varphi^3 e^{-2s\eta}\left|q\right|^2\right)^{1/2}\right).\end{aligned}$$

# Carleman estimates with one less derivative (boundary observations)

## Theorem

*Let us consider the following problem:*

$$\begin{cases} q_t - (cq_x)_x = f_x, & \text{in } Q, \\ q_j(l_j,t) = 0, & \forall j \in \{0,\cdots,n\}, \\ q_i(0,t) = q_j(0,t), & \forall i,j \in \{0,\cdots,n\}, \forall t \in (0,T), \\ [cq_x(t)]_0 = 0, & \forall t \in (0,T), \\ q(x,0) = q_0(x), & \forall x \in \mathcal{R}. \end{cases}$$

*Assume that* $q_0 \in L^2(\mathcal{R})$, $f \in L^2(0,T,H^1(\mathcal{R}))$ *with* $f(0,t) = 0$ *for a.e.* $t \in (0,T)$. *Pick* $\lambda \geq \lambda_0$. *Let* $V_\gamma := \cup_{1\leq i\leq n}(l_i - \epsilon, l_i)$ *for some* $\epsilon \in (0, \inf_{1\leq i\leq n} l_i)$. *Then there exist* $\bar{s} \geq s_0 > 0$ *and* $C > 0$ *such that if* $q \in L^2(Q) \cap L^2(0,T,H^2(V_\gamma))$, *then for* $s \geq \bar{s}$

$$\iint_Q s^3\lambda^4\varphi^3 e^{-2s\eta}|q|^2\,dxdt \leq C\left(\iint_Q s^2\lambda^2\varphi^2 e^{-2s\eta}|f|^2\,dxdt \right.$$
$$\left. + \int_0^T \sum_{j=1}^n s\lambda\varphi_j(l_j,t)e^{-2s\eta_j(l_j,t)}\left|f_j(l_j,t) + c_j(l_j)q_{j,x}(l_j,t)\right|^2 dt\right).$$

# Outline

1. Carleman estimates
    - Carleman estimates in $L^2(\mathbb{R})$ with interior observations
    - Carleman estimates with one less derivative (interior observations)
    - Carleman estimates with one less derivative (boundary observations)

2. Stability results
    - Stability results with interior observations
    - Stability results with boundary observations

# Stability

Let $y$ and $\tilde{y}$ be the solutions of

$$\begin{cases} y_t - (cy_x)_x = g\mathbb{1}_\omega, \\ y_j(l_j,t) = h_j(t), \\ y_j(0,t) = y_i(0,t), \\ [cy_x(t)]_0 = 0, \\ y(x,0) = y^0(x), \end{cases} \qquad \text{and} \qquad \begin{cases} \tilde{y}_t - (\tilde{c}\tilde{y}_x)_x = g\mathbb{1}_\omega, \\ \tilde{y}_j(l_j,t) = h_j(t), \\ \tilde{y}_j(0,t) = \tilde{y}_i(0,t), \\ [\tilde{c}\tilde{y}_x(t)]_0 = 0, \\ \tilde{y}(x,0) = \tilde{y}^0(x), \end{cases}$$

respectively. Let

$$\xi := c - \tilde{c} \quad \text{and} \quad v := (y - \tilde{y})_t.$$

Then $v$ satisfies the following problem

$$\begin{cases} v_t - (cv_x)_x = (\xi\tilde{y}_{tx})_x, & \forall (x,t) \in \mathcal{R} \times (0,T), \\ v_j(l_j,t) = 0, & \forall j \in \{0,\cdots,n\}, t \in (0,T), \\ v_j(0,t) = v_i(0,t), & \forall i,j \in \{0,\cdots,n\},\ t \in (0,T), \\ [cv_x(t)]_0 = -[\xi\tilde{y}_{tx}(t)]_0\,, & \forall t \in (0,T), \\ v(x,0) = (cy^0_x)_x - (\tilde{c}\tilde{y}^0_x)_x, & \forall x \in \mathcal{R}. \end{cases}$$

# Stability for a stationary problem

We first investigate the $L^2(\mathcal{R})$-stability of the solution $c(x)$ of the following stationary problem

$$\begin{cases} -(cu_x)_x = f, & \text{in } \mathcal{R}, \\ u_j(0) = u_i(0), & \forall i, j \in \{0, \cdots, n\}, \\ [cu_x]_0 = 0, \end{cases}$$

where $u \in H^1(\mathcal{R})$ and $f \in L^2(\mathcal{R})$.
Let $\tilde{c}(x)$ solve

$$\begin{cases} -(\tilde{c}\tilde{u}_x)_x = \tilde{f}, & \text{in } \mathcal{R}, \\ \tilde{u}_j(0) = \tilde{u}_i(0), & \forall i, j \in \{0, \cdots, n\}, \\ [\tilde{c}\tilde{u}_x]_0 = 0, \end{cases}$$

where $\tilde{u} \in H^1(\mathcal{R})$ and $\tilde{f} \in L^2(\mathcal{R})$.

# Stability for a stationary problem

## Lemma

*Let* $\delta > 0$ *and* $T' \in (0,T)$. *Assume that the functions* $c$ *and* $\tilde{c}$ *are piecewise* $C^1$, *that they satisfy* (1), *and that they are such that*

$$(H_c) \qquad c_i(0) = \tilde{c}_i(0) \qquad \forall i \in \{0, \cdots, n\}.$$

*Assume further that*

$$\begin{array}{ll} (H_1) & |\beta' \tilde{u}_x| \geq \delta > 0 \quad \text{in} \quad \mathcal{R} \setminus \omega^0, \\ (H_2) & \forall j \in \{0, \cdots, n\},\ u_j \in C^2([0, l_j]) \text{ and } \tilde{u}_j \in C^2([0, l_j]), \\ (H_3) & \|\tilde{u}\|_{L^\infty(\mathcal{R})} + \|\tilde{u}_x\|_{L^\infty(\mathcal{R})} \leq \delta^{-1}, \end{array}$$

*for some open set* $\omega^0$ *with* $\omega^0 \subset \omega$.
*Let* $\lambda \geq \lambda_0$. *Then there exists* $s_1 \geq s_0$ *such that for any* $s \geq s_1$

$$s^2 \delta^2 \int_{\mathcal{R} \setminus \omega^0} |c - \tilde{c}|^2 e^{-2s\eta} dx \leq C \int_{\mathcal{R}} \left| f - \tilde{f} \right|^2 e^{-2s\eta} dx + Cs \int_{\omega^0} |c - \tilde{c}|^2 e^{-2s\eta} dx + C(s, c_{max}) \|u - \tilde{u}\|^2_{H^1(\mathcal{R})}.$$

# Stability for a stationary problem

**Lemma**

*Assume further that*

$(H_4)$ *There exist some open intervals* $\omega_i^1 \subset (0, l_i)$ *such that:*

1. $\omega^0 \subset \omega^1 := \cup_{i\in\{1,\cdots,n\}}\omega_i^1 \subset \omega$,
2. $\tilde{u}_{|\partial\omega^1} = 0$,
3. $\tilde{u} \in C^2(\omega^1)$ *and* $\tilde{u}_x^2 - \tilde{u}\tilde{u}_{xx} \geq \delta > 0$ *in* $\omega^1$.

*Assume in addition that* $\|c\|_{W^{1,\infty}(\omega_1)} \leq c_{max}^{1,\infty}$. *Then there exist* $C > 0$ *and* $C(c_{max}^{1,\infty}) > 0$ *such that*

$$\int_{\omega^1} |c-\tilde{c}|^2 \, dx \leq C \left\| f - \tilde{f} \right\|_{L^2(\omega^1)}^2 + C(c_{max}^{1,\infty}) \left\| u - \tilde{u} \right\|_{H^2(\omega^1)}^2 .$$

# Stability for a stationary problem

### Proposition

*Under the previous assumptions, there exist* $C > 0$, $s_0 > 0$ *such that for all* $s \geq s_0$, *there exist* $C(s) > 0$ *and* $C(s, c_{max}, c^{1,\infty}_{max}) > 0$ *with*

$$s^2 \int_{\mathcal{R}} |c - \tilde{c}|^2 \, e^{-2s\eta} dx \leq C \int_{\mathcal{R}} \left| f - \tilde{f} \right|^2 e^{-2s\eta} dx + C(s) \left\| f - \tilde{f} \right\|^2_{L^2(\omega^1)} + C(s, c_{max}, c^{1,\infty}_{max}) \left( \|u - \tilde{u}\|^2_{H^1(\mathcal{R})} + \|u - \tilde{u}\|^2_{H^2(\omega^1)} \right).$$

# Stability for the evolution problem

Let $y$ and $\tilde{y}$ be the solutions of

$$\begin{cases} y_t - (cy_x)_x = g\mathbb{1}_\omega, \\ y_j(l_j, t) = 0, \\ y_j(0,t) = y_i(0,t), \\ [cy_x(t)]_0 = 0, \\ y(x,0) = y^0(x), \end{cases} \quad \text{and} \quad \begin{cases} \tilde{y}_t - (\tilde{c}\tilde{y}_x)_x = g\mathbb{1}_\omega, \\ \tilde{y}_j(l_j, t) = 0, \\ \tilde{y}_j(0,t) = \tilde{y}_i(0,t), \\ [\tilde{c}\tilde{y}_x(t)]_0 = 0, \\ \tilde{y}(x,0) = y^0(x), \end{cases}$$

respectively, where $y^0 \in L^2(\mathcal{R})$ and $g \in C_c^\infty(\omega \times (0,T))$ (conveniently chosen).

## Aim

To derive stability estimates for $c - \tilde{c}$ from measurements of derivatives of $y - \tilde{y}$ for $(x,t) \in \omega \times (0,T)$ and for $(x,t) \in \mathcal{R} \times \{T/2\}$.

As we shall use the stability estimates in the stationary case, we need first to prove the existence of some control input $g$ such that $(H_1), (H_2), (H_3)$ and $(H_4)$ are satisfied for $\tilde{u} = \tilde{y}(., T/2)$.

# Existence of $g$ satisfying $(H_1) - (H_4)$

## Proposition

*Let $\tilde{c}$ be a piecewise $C^1$ function satisfying* (1) *and* $(H_c)$. *Let* $T > 0$ *and* $\tilde{y}^0 \in L^2(\mathcal{R})$. *Pick* $T' = T/2$. *Then there exist two open sets* $\omega^0 \subset \omega^1 \subset \omega$, *a function* $g \in C_c^\infty(\omega \times (0,T))$ *such that the solution* $\tilde{y}$ *associated with* $\tilde{y}_2^0(x) := \tilde{y}^0(x) - 2$ *and* $h_j(t) = -2$ *for* $j \in \{0, \cdots, n\}$ *and* $t \in (0,T)$ *satisfies for some* $\delta > 0$

$$|\beta' \tilde{y}_x(.,T')| \geq \delta > 0 \qquad in\ \mathcal{R} \setminus \omega^0; \tag{2}$$

$$\tilde{y}_j(.,T') \in C^2([0,l_j]) \qquad \forall j \in \{0, \cdots, n\}; \tag{3}$$

$$\|\tilde{y}(.,T')\|_{L^\infty(\mathcal{R})} + \|\tilde{y}_x(.,T')\|_{L^\infty(\mathcal{R})} \leq \delta^{-1}; \tag{4}$$

$$\tilde{y}(.,T')_{|\partial\omega^1} = 0; \tag{5}$$

$$|\tilde{y}_x(.,T')|^2 - \tilde{y}(.,T')\tilde{y}_{xx}(.,T') \geq \delta > 0\ in\ \omega^1. \tag{6}$$

Proof: We proceed as in [Cristofol, Gaitan, Niinimaki, Poisson 2013]:

- we construct a function $\rho(x)$ satisfying conditions very similar to (2)-(6),
- we show that a control input $g$ can be designed so that the solution $\tilde{y}$ is close enough to $\rho$ for the norm $\|u\|_{L^\infty(\mathcal{R})} + \|u_x\|_{L^\infty(\mathcal{R})} + \|u_{xx}\|_{L^\infty(\mathcal{R})}$ at time $t = T'$ (approximate controllability).

# Proof of stability results with interior observations

Let

$$\xi := c - \tilde{c} \quad \text{and} \quad v := (y - \tilde{y})_t .$$

Then $v$ satisfies the following problem

$$\begin{cases} v_t - (c v_x)_x = (\xi \tilde{y}_{tx})_x, & \forall (x,t) \in \mathcal{R} \times (0,T), \\ v_j(l_j, t) = 0, & \forall j \in \{0, \cdots, n\}, t \in (0,T), \\ v_j(0,t) = v_i(0,t), & \forall i, j \in \{0, \cdots, n\},\ t \in (0,T), \\ \left[ c v_x(t) \right]_0 = - \left[ \xi \tilde{y}_{tx}(t) \right]_0 = 0, & \forall t \in (0,T), \\ v(x,0) = (\xi y_x^0)_x, & \forall x \in \mathcal{R}, \end{cases}$$

where we can prove that $y_{tx}, y_{ttx}, \tilde{y}_{tx}, \tilde{y}_{ttx} \in C([0,T], L^\infty(\mathcal{R}))$, $\xi \tilde{y}_{ttx} \in C([0,T], H^1(\mathcal{R}))$, and we have

$$\begin{aligned} \underbrace{\int_{\mathcal{R}} \left| v(T') \right|^2 e^{-2s\eta(T')} dx}_{=:A_1} &= \int_0^{T'} \int_{\mathcal{R}} 2 v v_t e^{-2s\eta} dx dt + \int_0^{T'} \int_{\mathcal{R}} |v|^2 \frac{\partial}{\partial t} \left( e^{-2s\eta} \right) dx dt \\ &= \underbrace{\int_0^{T'} \int_{\mathcal{R}} 2 v v_t e^{-2s\eta} dx dt}_{=:A_2} \underbrace{- 2s \int_0^{T'} \int_{\mathcal{R}} |v|^2 \eta_t e^{-2s\eta} dx dt}_{=:A_3} . \end{aligned}$$

# Upper bounds for $A_1$

We have

$$|A_3| \leq Cs \iint_Q |v|^2 \varphi^3 e^{-2s\eta} dxdt$$

and we use the Carleman estimate with one less derivative with $f = \xi \tilde{y}_{tx} \in L^2(0, T, H^1(\mathcal{R}))$, $f(0, t) = 0$ (since $\xi(0) = 0$):

$$\begin{aligned} |A_3| &\leq C\left(\iint_Q |\xi \tilde{y}_{tx}|^2 \varphi^2 e^{-2s\eta} dxdt + s\int_0^T \int_\omega |v|^2 \varphi^3 e^{-2s\eta} dxdt\right) \\ &\leq C\left(\int_{\mathcal{R}} |\varphi(T')|^2 e^{-2s\eta(T')} |\xi|^2 dx + s\int_0^T \int_\omega |v|^2 \varphi^3 e^{-2s\eta} dxdt\right). \end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$|A_2| \leq C\left(\iint_Q s|v|^2 \varphi^3 e^{-2s\eta} dxdt + \iint_Q s^{-1}\varphi^{-1} e^{-2s\eta} |v_t|^2 dxdt\right).$$

# Upper bounds for $A_1$

Observe that $v_t$ satisfies

$$\begin{cases} (v_t)_t - (cv_{xt})_x = (\xi \tilde{y}_{ttx})_x, & \forall (x,t) \in \mathcal{R} \times (0,T), \\ v_{j,t}(l_j,t) = 0, & \forall j \in \{0, \cdots, n\}, t \in (0,T), \\ v_{j,t}(0,t) = v_{i,t}(0,t), & \forall i,j \in \{0, \cdots, n\}, t \in (0,T), \\ [cv_{tx}(t)]_0 = -[\xi \tilde{y}_{ttx}(t)]_0 = 0, & \forall t \in (0,T). \end{cases}$$

Applying Carleman estimate with one less derivative to $v_t$, we have

$$\begin{aligned} \iint_Q s^{-1}\varphi^{-1}e^{-2s\eta}|v_t|^2\,dxdt &\le Cs^{-4}\iint_Q (s\varphi)^3 e^{-2s\eta}|v_t|^2\,dxdt \\ &\le Cs^{-4}\left[\int_0^T\int_\omega s^3|v_t|^2\varphi^3 e^{-2s\eta}dxdt + \iint_Q |\xi\tilde{y}_{ttx}|^2(s\varphi)^2 e^{-2s\eta}dxdt\right] \\ &\le C\left[\int_0^T\int_\omega s^{-1}|v_t|^2\varphi^3 e^{-2s\eta}dxdt + \int_{\mathcal{R}} |\xi|^2 s^{-2}\varphi^2(T')e^{-2s\eta(T')}dx\right]. \end{aligned}$$

Thus, we obtain the following upper bounds for $A_1$ for $s$ sufficiently large:

$$A_1 \le C\left[\int_{\mathcal{R}} |\varphi(T')|^2 e^{-2s\eta(T')}|\xi|^2\,dx + \int_0^T\int_\omega (s|v|^2 + s^{-1}|v_t|^2)\varphi^3 e^{-2s\eta}dxdt\right].$$

# Lower bounds for $A_1$

First we have

$$v(T') = (cy_x - \tilde{c}\tilde{y}_x)_x(T').$$

Thanks to the study of the stationary case, we can write

$$\begin{aligned} A_1 &= \int_{\mathcal{R}} \left[(cy_x - \tilde{c}\tilde{y}_x)_x(T')\right|^2 e^{-2s\eta(T')} dx \\ &\geq Cs^2 \int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 \, dx - C(s) \Big( \left\| y_t(T') - \tilde{y}_t(T') \right\|^2_{L^2(\omega^1)} \\ &\quad + \left\| y(T') - \tilde{y}(T') \right\|^2_{H^1(\mathcal{R})} + \left\| y(T') - \tilde{y}(T') \right\|^2_{H^2(\omega^1)} \Big) . \end{aligned}$$

# End of the proof of stability results with interior observations

Thus, we obtain

$$\begin{aligned} s^2 \int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 \, dx - C(s)(\|y_t(T') - \tilde{y}_t(T')\|^2_{L^2(\omega^1)} \\ + \|y(T') - \tilde{y}(T')\|^2_{H^1(\mathcal{R})} + \|y(T') - \tilde{y}(T')\|^2_{H^2(\omega^1)}) \\ \leq C \left[ \int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 \, dx + \int_0^T \int_\omega (s\,|v|^2 + s^{-1}\,|v_t|^2) \varphi^3 e^{-2s\eta} dx dt \right]. \end{aligned}$$

Thus, for $s$ sufficiently large, we have the stability estimate

$$\begin{aligned} s^2 \int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 \, dx \leq C \int_0^T \int_\omega \varphi^3 (s\,|y_t - \tilde{y}_t|^2 + s^{-1}\,|y_{tt} - \tilde{y}_{tt}|^2) e^{-2s\eta} dx dt \\ + C(s)(\|y_t(T') - \tilde{y}_t(T')\|^2_{L^2(\omega^1)} + \|y(T') - \tilde{y}(T')\|^2_{H^1(\mathcal{R})} + \|y(T') - \tilde{y}(T')\|^2_{H^2(\omega^1)}). \end{aligned}$$

# Stability results with boundary observations

As for the proof of the Carleman result with one less derivative, we extend the network $\mathcal{R}$ in another network $\hat{\mathcal{R}}$ and we follow the previous case. We make the following assumptions, $(H_c)$ and

$$(H_{Ext})\ \ c_i(l_i) = \tilde{c}_i(l_i), \quad \forall i \in \{1, \cdots, n\}.$$

# Stability for the stationary problem

## Proposition

*Let* $c, \tilde{c}$ *piecewise* $C^1$ *satisfying* (1) *and* $\max_{j\in\{0,\cdots,n\}} \|c_j\|_{W^{1,\infty}(0,l_j)} \leq c_{max}^{1,\infty}$. *Under hypotheses* $(H_1)$ *(with* $\omega^0 = \emptyset$*),* $(H_2)$, $(H_3)$, $(H_c)$ *and* $(H_{Ext})$*, there exist* $C > 0$, $s_0 > 0$ *such that for all* $s \geq s_0$*, there exist* $C(s)$, $C(s, c_{max}) > 0$,

$$s^2 \int_{\mathcal{R}} \varphi^2 |\xi|^2 e^{-2s\eta} dx \leq C \int_{\mathcal{R}} \left| f - \tilde{f} \right|^2 e^{-2s\eta} dx$$

$$+ C(s, c_{max}) \left( \|u - \tilde{u}\|_{H^1(\mathcal{R})}^2 + \sum_{j=1}^{n} e^{-2s\eta(l_j,T')} |u_{j,x}(l_j) - \tilde{u}_{j,x}(l_j)|^2 \right).$$

Proof: We only need to take care of the boundary term. Using the fact that $\beta_0'(l_0) < 0$ and that $\beta_j'(l_j) > 0$ for $j \in \{1, \cdots, n\}$ (since $\beta$ is given for $\hat{\mathcal{R}}$ and $\omega \subset \hat{\mathcal{R}} \setminus \mathcal{R}$), we can obtain the result.

# Stability for the evolution problem

Let $y$ and $\tilde{y}$ be the solutions of

$$\begin{cases} y_t - (cy_x)_x = 0, \\ y_0(l_0,t) = 0, \\ y_j(l_j,t) = h_j(t), \\ y_j(0,t) = y_i(0,t), \\ [cy_x(t)]_0 = 0, \\ y(x,0) = y^0(x), \end{cases} \qquad \text{and} \qquad \begin{cases} \tilde{y}_t - (\tilde{c}\tilde{y}_x)_x = 0, \\ \tilde{y}_0(l_0,t) = 0, \\ \tilde{y}_j(l_j,t) = h_j(t), \\ \tilde{y}_j(0,t) = \tilde{y}_i(0,t), \\ [\tilde{c}\tilde{y}_x(t)]_0 = 0, \\ \tilde{y}(x,0) = y^0(x), \end{cases}$$

respectively, where $y^0 \in L^2(\mathcal{R})$ and $h_j \in C^1_c(0,T)$ (conveniently chosen).
We extend the network $\mathcal{R}$ in another network $\hat{\mathcal{R}}$, we fix an open set $\omega \subset \cup_{j=1}^n (l_j, \hat{l}_j)$ and we also extend $c$ and $\tilde{c}$ smoothly in $\hat{\mathcal{R}}$.
Applying a previous result to $\hat{\mathcal{R}}$ and $\tilde{y}(\tau)$, we can find an input $g \in C^\infty_c(\omega \times (\tau,T))$ such that $(H_1), (H_2), (H_3)$ are satisfied for $\tilde{u} = \tilde{y}(.,T/2)$.
We then construct $h_j \in C^1_c([0,T])$ for $j \in \{1,\cdots,n\}$ such that $h_j(t) = 0$ for $t \in (0,\tau)$ and $h_j(t) = \psi(t)\tilde{y}_j(l_j,t)$ for $t \in [\tau,T]$.
We follow the proof of the case of interior observations.

# Proof of stability results with boundary observations

Let

$$\xi := c - \tilde{c} \quad \text{and} \quad v := (y - \tilde{y})_t .$$

Then $v$ satisfies the following problem

$$\begin{cases} v_t - (c v_x)_x = (\xi \tilde{y}_{tx})_x , & \forall (x,t) \in \mathcal{R} \times (0,T), \\ v_j(l_j, t) = 0, & \forall j \in \{0, \cdots, n\}, t \in (0,T), \\ v_j(0,t) = v_i(0,t), & \forall i,j \in \{0, \cdots, n\},\ t \in (0,T), \\ [c v_x(t)]_0 = -\left[\xi \tilde{y}_{tx}(t)\right]_0 = 0, & \forall t \in (0,T), \\ v(x,0) = (\xi y_x^0)_x , & \forall x \in \mathcal{R}. \end{cases}$$

We have

$$\begin{aligned} \underbrace{\int_{\mathcal{R}} \left| v(T') \right|^2 e^{-2s\eta(T')} dx}_{=:A_1} &= \int_0^{T'} \int_{\mathcal{R}} 2 v v_t e^{-2s\eta} dx dt + \int_0^{T'} \int_{\mathcal{R}} |v|^2 \frac{\partial}{\partial t} \left( e^{-2s\eta} \right) dx dt \\ &= \underbrace{\int_0^{T'} \int_{\mathcal{R}} 2 v v_t e^{-2s\eta} dx dt}_{=:A_2} - \underbrace{2s \int_0^{T'} \int_{\mathcal{R}} |v|^2 \eta_t e^{-2s\eta} dx dt}_{=:A_3} . \end{aligned}$$

# Upper bounds for $A_1$

We have

$$|A_3| \leq Cs \iint_Q |v|^2 \varphi^3 e^{-2s\eta} dxdt$$

and we use the Carleman estimate with one less derivative with $f = \xi \tilde{y}_{tx} \in L^2(0, T, H^1(\mathcal{R}))$, $f(0, t) = 0$ (since $\xi(0) = 0$):

$$\begin{aligned}
|A_3| \quad &\leq \quad C \left( \iint_Q |\xi \tilde{y}_{tx}|^2 \varphi^2 e^{-2s\eta} dxdt \right. \\
&\qquad \left. + \int_0^T \sum_{j=1}^n s^{-1} \varphi_j(l_j, t) e^{-2s\eta_j(l_j,t)} \xi(l_j) \tilde{y}_{tx}(l_j, t) + c(l_j) v_x(l_j, t)^2 dt \right) \\
&\leq \quad C \left( \int_{\mathcal{R}} \left|\varphi(T')\right|^2 e^{-2s\eta(T')} |\xi|^2 dx \right. \\
&\qquad \left. + \int_0^T \sum_{j=1}^n s^{-1} \varphi_j(l_j, t) e^{-2s\eta_j(l_j,t)} \xi(l_j) \tilde{y}_{tx}(l_j, t) + c(l_j) v_x(l_j, t)^2 dt \right).
\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$|A_2| \leq C \left( \iint_Q s |v|^2 \varphi^3 e^{-2s\eta} dxdt + \iint_Q s^{-1} \varphi^{-1} e^{-2s\eta} |v_t|^2 dxdt \right).$$

# Upper bounds for $A_1$

Observe that $v_t$ satisfies

$$\begin{cases} (v_t)_t - (cv_{xt})_x = (\xi \tilde{y}_{ttx})_x, & \forall (x,t) \in \mathcal{R} \times (0,T), \\ v_{j,t}(l_j,t) = 0, & \forall j \in \{0,\cdots,n\}, t \in (0,T), \\ v_{j,t}(0,t) = v_{i,t}(0,t), & \forall i,j \in \{0,\cdots,n\}, t \in (0,T), \\ \left[cv_{tx}(t)\right]_0 = -\left[\xi \tilde{y}_{ttx}(t)\right]_0 = 0, & \forall t \in (0,T). \end{cases}$$

Applying Carleman estimate with one less derivative to $v_t$, we have

$$\iint_Q s^{-1}\varphi^{-1}e^{-2s\eta}\left|v_t\right|^2 dxdt \leq Cs^{-4}\iint_Q (s\varphi)^3 e^{-2s\eta}\left|v_t\right|^2 dxdt$$

$$\leq C\left[s^{-2}\int_{\mathcal{R}} \left|\varphi(T')\right|^2 e^{-2s\eta(T')}\left|\xi\right|^2 dx\right.$$

$$\left. + s^{-3}\int_0^T \sum_{j=1}^{n} \varphi_j(l_j,t)e^{-2s\eta_j(l_j,t)}\left|\xi(l_j)\tilde{y}_{ttx}(l_j,t) + c(l_j)v_{xt}(l_j,t)\right|^2 dt\right].$$

# Upper bounds for $A_1$

Thus, we obtain the following upper bounds for $A_1$ for $s$ sufficiently large:

$$\begin{aligned} A_1 \quad \leq \quad & C \Bigg[ \int_{\mathcal{R}} \left|\varphi(T')\right|^2 e^{-2s\eta(T')} \left|\xi\right|^2 dx \\ & + s^{-3} \int_0^T \sum_{j=1}^{n} \varphi_j(l_j,t) e^{-2s\eta_j(l_j,t)} \left|\xi(l_j)\tilde{y}_{ttx}(l_j,t) + c(l_j) v_{xt}(l_j,t)\right|^2 dt \\ & + \int_0^T \sum_{j=1}^{n} s^{-1} \varphi_j(l_j,t) e^{-2s\eta_j(l_j,t)} \left|\xi(l_j)\tilde{y}_{tx}(l_j,t) + c(l_j) v_x(l_j,t)\right|^2 dt \Bigg]. \end{aligned}$$

# Lower bounds for $A_1$

First we have

$$v(T') = (cy_x - \tilde{c}\tilde{y}_x)_x(T').$$

Thanks to the study of the stationary case, we can write

$$\begin{aligned} A_1 &= \int_{\mathcal{R}} \left[(cy_x - \tilde{c}\tilde{y}_x)_x(T')\right|^2 e^{-2s\eta(T')} dx \\ &\geq Cs^2 \int_{\mathcal{R}} e^{-2s\eta(T')} \left|\xi\right|^2 dx - C(s) \Bigg( \|y(T') - \tilde{y}(T')\|^2_{H^1(\mathcal{R})} \\ &\quad + \sum_{j=1}^{n} e^{-2s\eta(l_j,T')} \left|y_{j,x}(l_j,T') - \tilde{y}_{j,x}(l_j,T')\right|^2 \Bigg). \end{aligned}$$

# End of the proof of stability results with boundary observations

Thus with $(H_{Ext})$, we get for $s$ sufficiently large

$$
\begin{aligned}
& s^2 \int_{\mathcal{R}} e^{-2s\eta(T')} \left|\xi\right|^2 dx \\
& \leq C(s) \Bigg[ \|y(T') - (T')\|^2_{H^1(\mathcal{R})} + \sum_{j=1}^{n} e^{-2s\eta(l_j,T')} \left|y_{j,x}(l_j,T') - \tilde{y}_{j,x}(l_j,T')\right|^2 \\
& \quad + s^{-3} \int_0^T \sum_{j=1}^{n} \varphi_j(l_j,t) e^{-2s\eta_j(l_j,t)} \left|\xi(l_j)\tilde{y}_{ttx}(l_j,t) + c(l_j)v_{xt}(l_j,t)\right|^2 dt \\
& \quad + \int_0^T \sum_{j=1}^{n} s^{-1} \varphi_j(l_j,t) e^{-2s\eta_j(l_j,t)} \left|\xi(l_j)\tilde{y}_{tx}(l_j,t) + c(l_j)v_x(l_j,t)\right|^2 dt \Bigg].
\end{aligned}
$$

This completes the proof, since $\xi(l_j)\tilde{y}_{tx}(l_j) = 0$ for $j \in \{1, \cdots, n\}$ (see $(H_{Ext})$).

# Conclusion

- Stability estimates for the determination of discontinuous diffusion coefficients on star-shaped networks of heat equations by internal or boundary measurements.
- Extension to tree-shaped network.
- Other examples of network?
- Recontruction of the coefficients?
- Other equations?