

Control of the half-heat equation

Joint work with Andreas Hartmann

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Control in PDEs in Hauts-de-France

Introduction

Definition (Half-heat equation)

$$|D_x| \left(\sum_{n \in \mathbb{Z}} a_n e^{inx} \right) := \sum_{n \in \mathbb{Z}} |n| a_n e^{inx}. \text{ System: } (\partial_t + |D_x|)f = \mathbf{1}_\omega u \text{ on } \mathbb{T} := \mathbb{R} \setminus \mathbb{Z}$$

Question

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- Is this equation null-controllable? Answer: **no**.

Proof (K. 2015).

$$\text{Solutions of } (\partial_t + |D_x|)g = 0: g(t, x) = \sum_n a_n e^{-|n|t} e^{inx}.$$

$$\text{Null-controllability} \implies \sum_{n>0} |a_n|^2 e^{-2nT} \leq c \int_{[0, T] \times \omega} \left| \sum_{n>0} a_n e^{-nt} e^{inx} \right|^2 dt dx$$

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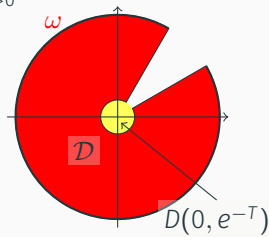
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- Change of variables: $z = e^{-t+ix}$
- Null-controllability \implies for every polynomials $p \in \mathbb{C}[X]$, $\|p\|_{L^2(D(0, e^{-T}))} \leq C \|p\|_{L^2(\mathcal{D})}$
- This inequality does not hold.



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Plan

Results

Necessary/sufficient conditions for the control of positive frequencies

The space of null-controllable data does not depend on time

Control of all frequencies

Results

Definition (Riesz projection)

$$\text{if } f(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}, P_+ f(x) = \sum_{n \geq 0} a_n e^{inx}.$$

Definition (The H^2 control system)

$$(\partial_t + |D_x|)f(t, x) = \mathbf{1}_\omega u(t, x), \quad f(0, \cdot) \in L^2(\mathbb{T}) \quad (E_{L^2})$$

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Annoyance: $H^2(\mathbb{T}) = P_+(L^2(\mathbb{T}))$ is the Hardy space

Definition

$$\mathcal{NC}_{L^2}(\omega, T) = \{f_0 \in L^2(\mathbb{T}), \exists u \in L^2([0, T] \times \omega), \text{ solution } f \text{ of } (E_{L^2}) \text{ s.t. } f(T, \cdot) = 0\}$$

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Theorem (Hartmann-K 2024)

$\mathcal{NC}_{H^2}(\omega, T)$ and $\mathcal{NC}_{L^2}(\omega, T)$ do not depend on T .

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If $f_0 \in H^2(\mathbb{T})$ is nonzero and analytic on \mathbb{T} , $f_0 \notin \mathcal{NC}_{H^2}(\omega)$.

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Let $f_0 \in H^2(\mathbb{T}) \cap W^{1/2,2}(\mathbb{T})$. Then $f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $h \in W_{00}^{1/2,2}(\omega)$ such that $f_0 = P_+h$.

$W_{00}^{1/2,2}(\omega) \subset \mathcal{NC}_{L^2}(\omega)$.

	Internal controls	Shaped controls [Micu-Zuazua 2006] $\forall \epsilon > 0, \hat{h}(n) \geq ce^{-\epsilon n}$ $\exists \epsilon > 0, \hat{h}(n) \leq Ce^{-\epsilon n}$
Control system	$(\partial_t + D_x)f = \mathbf{1}_\omega u$	$(\partial_t + D_x)f = h(x)u(t)$
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Set of null-control- lable states	dense subspace Independent of time	$\{0\}$	$\neq \{0\}$ depends on time?
Regularity of null- controllable states	cannot be analytical		some analytical states

Necessary/sufficient conditions
for the control of positive
frequencies

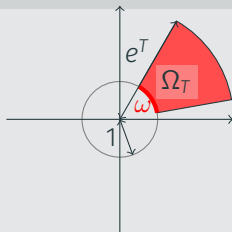
Further annoyances

$$\mathbb{R}/2\pi\mathbb{Z} = \mathbb{T} \approx \{z \in \mathbb{C}, |z| = 1\}, \quad H^2(\mathbb{T}) \approx \left\{ \sum_{n \geq 0} a_n z^n, (a_n) \in \ell^2 \right\}, \quad f(x) \approx f(e^{ix})$$

Proposition (observability inequality)

Let $f_0 \in H^2(\mathbb{T})$. $f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $C > 0$ such that

$$\forall p \in \mathbb{C}[X], \quad \left| \int_0^{2\pi} p(e^{it}) \overline{f_0(e^{it})} dt \right| \leq C \|p\|_{L^2(\Omega_T)}.$$



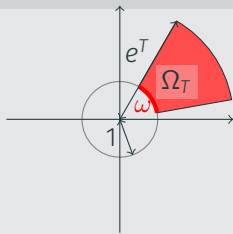
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Proof.

$f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $C > 0$ such that for every $g_0 \in H^2(\mathbb{T})$,

$$\left| \langle f_0, e^{-T|D_x|} g_0 \rangle_{H^2(\mathbb{T})} \right| \leq C \int_0^T \int_{\omega} |e^{-t|D_x|} g_0(e^{ix})|^2 dt dy.$$

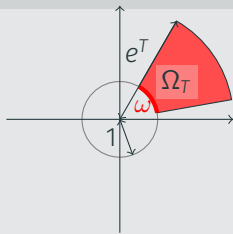
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$$e^{-t|D_x|} g_0(e^{ix}) = \sum_{n \geq 0} \widehat{g_0}(n) e^{iny-nt} = \sum_{n \geq 0} \widehat{g_0}(n) (e^{iy-t})^n.$$

Change of variables $z = e^{-t+ix}$.

Theorem (Hartmann-K 2024)

If $h \in W_{00}^{1/2,2}(\omega)$, $P_+h \in \mathcal{NC}_{H^2}(\omega)$.

Proof.

- extends h by 0 on $\partial\Omega_T$
- $h \in W^{1/2,2}(\partial\Omega_T)$
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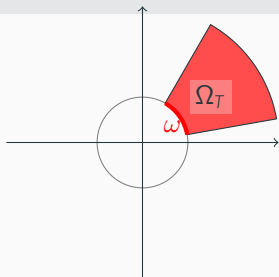
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- Left-hand side: $\approx \langle p, h \rangle_{L^2} = \langle P_+p, h \rangle_{L^2} = \langle P_+h, p \rangle_{L^2}$.
- Right-hand side: $\int_{\Omega_T} \partial_{\bar{z}}(\bar{h}p)(z) dA(z) = \int_{\Omega_T} p(z) \underbrace{\partial_{\bar{z}}\bar{h}(z)}_{\in L^2(\Omega_T)} dA(z)$



□

Theorem (Hartmann-K 2024)

Let $f_0 \in H^2(\mathbb{T})$. If $f_0 \in \mathcal{NC}_{H^2}(\omega)$, f_0 extends as a holomorphic function to $\mathbb{C} \setminus \text{closure}(\omega)$, and $f_0(z) \xrightarrow{|z| \rightarrow \infty} 0$. Moreover $\int_{|z|>1} |f_0'(z)|^2 dA(z) < +\infty$.

Proof.

- let $k_u(z) = \frac{1}{2\pi(1 - \bar{u}z)}$. If $f_0 \in H^2(\mathbb{T})$, $f_0(u) = \langle f_0, k_u \rangle_{H^2}$.
- for all $p \in \mathbb{C}[X]$, $|\langle f_0, p \rangle_{H^2}| \leq C \|p\|_{L^2(\Omega_T)}$
- extend $\ell(p) := \langle f_0, p \rangle_{H^2}$ by density to $A^2(\Omega_T)$ (space of holomorphic functions on Ω_T which are $L^2(\Omega_T)$)

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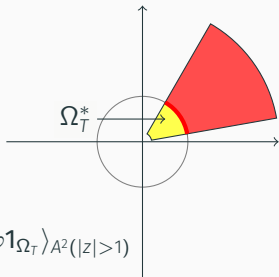
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We can define $\ell(k_u)$ if $u \notin \Omega_T^*$.

- $\phi(u) = \begin{cases} f_0(u) & \text{if } |u| < 1 \\ \ell(k_u) & \text{if } u \notin \Omega_T^* \end{cases}$ extends f_0 .

- $\|k_u\|_{L^2(\Omega_T)} = O(1/|u|)$: $f_0(z) \xrightarrow{|z| \rightarrow \infty} 0$

- $\ell(p) = \langle \varphi, p \rangle_{A^2(\Omega_T)}$ $f_0'(u) = \ell(\partial_{\bar{u}} k_u) = \langle k_u^{\text{Bergman}}, \varphi \mathbf{1}_{\Omega_T} \rangle_{A^2(|z|>1)}$



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If $f_0 \in H^2(\mathbb{T}) \cap W^{1/2,2}(\mathbb{T})$, and $f_0 \in \mathcal{NC}_{H^2}(\omega)$, there exists $h \in W_{00}^{1/2,2}(\omega)$ such that $f_0 = P_+h$.

Proof.

- Extend f_0 holomorphically to $\mathbb{C} \setminus \text{closure}(\omega)$ as above

- Write $f_0 = \begin{cases} \sum_{n \geq 0} h_n z^n & |z| < 1 \\ -\sum_{n > 0} h_{-n} \frac{1}{z^n} & |z| > 1 \end{cases}$

- Set $h(e^{i\theta}) = \sum_{n \in \mathbb{Z}} h_n e^{in\theta}$

- Fatou's jump theorem:

$$h(e^{i\theta}) = \lim_{r \rightarrow 1^-} (f_0(re^{i\theta}) - f_0(r^{-1}e^{i\theta})) = 0 \text{ if } f_0 \text{ is holomorphic at } e^{i\theta} \quad \square$$

Theorem (Hartmann-K 2024)

Let $f_0 \in \mathcal{NC}_{H^2}(\omega)$. There exists $h \in L^2(\mathbb{T})$ such that

- $f_0 = P_+ h$
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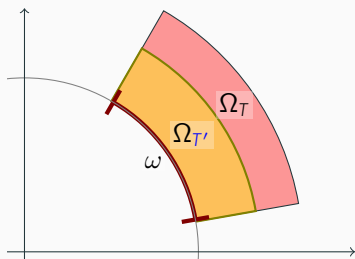
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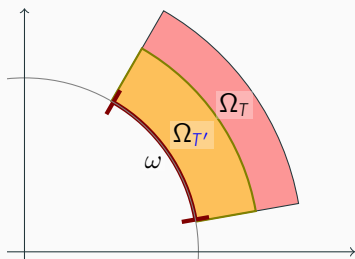
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- $f_0 = P_+ h$
- $\text{Supp}(h) \subset \text{closure}(\omega)$
- The solution u of
$$\begin{cases} \Delta u(x) = 0, & x \in \Omega_T \\ u(x) = h(x), & x \in \omega \\ u(x) = 0, & x \in \partial\Omega_T \setminus \omega \end{cases}$$
 satisfies $\partial_z u \in L^2(\Omega_T)$

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- Let $T' < T$
- Assume $|\langle p, f_0 \rangle_{H^2}| \lesssim \|p\|_{L^2(\Omega_T)}$
- Prove $|\langle p, f_0 \rangle_{H^2}| \lesssim \|p\|_{L^2(\Omega_{T'})}$



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Theorem (Orsoni, Hartmann Orsoni)

Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ open bounded. Assume that $d(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) > 0$. Then any $g \in A^2(\Omega_1 \cap \Omega_2)$ can be written as $g = g_1 + g_2$ with $g_i \in A^2(\Omega_i)$.

Control of all frequencies

- We know when the projection on positive frequencies is null-controllable
- If $f_0 \in \mathcal{NC}_{L^2}(\omega, T)$, $P_+ f_0 \in \mathcal{NC}_{H^2}(\omega, T)$
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Proof

- Solution of half-heat: sum of holomorphic and anti-holomorphic functions in e^{-t+ix} .
- Need: recover the holomorphic part with only information on ω
- Need: continuous projection $\{f \in L^2(\Omega_T), \text{harmonic}\} \rightarrow A^2(\Omega_T)$.
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Translate properties of the H^2 control system to the L^2 control system

That's all folks!

