A new estimate on the cost of fast controls for the Schrödinger equation

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Summary

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The controlled Schrödinger equation at one end

$T, L > 0.$

The equation under study

$$\begin{cases} iy_t + y_{xx} = 0 & \text{in } (0, T) \times (0, L), \\ y(0, \cdot) = y^0 & \text{in } (0, L), \\ y(\cdot, 0) = u(t) & \text{in } (0, T), \\ y(\cdot, L) = 0 & \text{in } (0, T). \end{cases} \qquad \text{(Cont-Sch)}$$

Functional setting and controllability

- $y^0 \in H^{-1}(0, L)$, $u \in L^2(0, T)$ the boundary control.
- (Cont-Sch) is well-posed linear control system:

$$||y(t, \cdot)||_{C^0([0,T],H^{-1}(0,L))} \leqslant C(T, L) \left(||y_0||_{H^{-1}(0,L)} + ||u||_{L^2(0,T)}\right).$$

- (Cont-Sch) well-known to be null-controllable at any time $T > 0$: for any $y^0 \in H^{-1}(0, L)$, $\exists u \in L^2(0, T)$ s.t. $y(T, \cdot) = 0$.

The cost of the control (1)

- For any $y^0 \in H^{-1}(0, L)$, unique optimal (for the $L^2(0, T)$-norm) control $u_{opt} \in L^2(0, T)$ and $y^0 \mapsto u_{opt}$ is linear continuous.
- The norm of this operator is called the the cost of the control, denoted $C_S(T, L)$.
- Another description: $C_S(T, L)$ smallest $C > 0$ such that for every $y^0 \in H^{-1}(0, L)$, $\exists$ control u driving y^0 to 0 at time T with

$$||u||_{L^2(0,T)} \leqslant C||y^0||_{H^{-1}(0,L)}.$$

- Important open problem is precise asymptotic estimates on $C_S(T, L)$ as T goes to 0 ("the cost of fast controls"). Optimal results only for LTI systems in finite dimension (Seidman'88 MCSS).
- Here, we prove a new upper bound for (Cont-Sch).

The cost of the control (2)

Why studying the cost of the control, in general ?

- Theoretical interest: natural question, statement simple, but is in reality a difficult question.
- Important to study problems with singular limit (vanishing viscosity/dispersion), see works by Coron, Glass, Guerrero, Laurent, Léautaud, Lissy...
- Important in order to attack nonlinear problems by fixed point method, see works by Fernandez-Cara, Zuazua...
- By duality, quantification of accuracy of reconstruction for some inverse problem. Here, reconstructing initial condition for free Schrödinger equation with Dirichlet boundary condition by one Neumann boundary observation.

State of the art

- For small T, $C_S(T, L)$ roughly of the form $\exp(\beta/T)$ for small T and some $\beta > 0$ depending on L.
- To catch precisely β, introduce

$$\beta_- = \liminf_{T\to 0^+} T\log(C_S(T, L)), \quad \beta_+ = \limsup_{T\to 0^+} T\log(C_S(T, L)).$$

- The constants $\beta_\pm$ verify

$$L^2/4 \leqslant \beta_- \leqslant \beta_+ \leqslant 3L^2/2.$$

L^2 comes from natural scaling. Upper bound: Tenenbaum-Tucsnak'07 JDE. Lower bound: Miller'04 ARMA (see also Lissy'15 JDE). Conjecture is that $\beta_- = \beta_+ = L^2/4$.

Theorem (Lissy'25)

β_+ verifies

$$\beta_+ \leqslant \frac{\sqrt[4]{27}}{4} L^2 < 0,56987677 L^2. \tag{Up-Beta}$$

Comparison with heat equation

A more popular problem is heat equation controlled at one end.

- With the same notations, $L^2/2 \leqslant \beta_- \leqslant \beta_+ \leqslant 0.6867L^2$. Lower bound: Lissy'15 JDE. Last upper bound: Darde-Ervedoza' 19 APDE.
- But for heat, we conjecture $\beta_- = \beta_+ = L^2/2$, whereas for (Cont-Sch) we conjecture $\beta_- = \beta_+ = L^2/4$. Makes this problem very intriguing: until now, worse upper bounds for (Cont-Sch) than for heat, but we expect in reality better bounds than for heat! Here, for the first time, upper bound "beats" heat.
- So problem seems more difficult for Schrödinger, might explain that it attracted less attention.

Summary

Some notations

Denote Fourier and inverse Fourier transform by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)dx, \ \mathcal{F}^{-1}g(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} g(\xi)d\xi.$$

$\omega > 0$ bounded measurable function. Introduce its logarithmic integral

$$\mathcal{L}(\omega) := \int_{\mathbb{R}} \frac{\log \omega(x)}{1+x^2} dx \in [-\infty, +\infty). \qquad \text{(Log-Int)}$$

$\mathcal{L}(\omega) > -\infty$ if and only if $\log(\omega) \in L^1(\mathbb{R}, \langle x \rangle^{-2} dx)$. "Decreases stronger than $^{-C|x|}$". Let

$$\mathrm{spec}(f) := \mathbb{R} \setminus \bigcup_{\Omega} \{\Omega \text{ open set}, \mathcal{F}f = 0 \text{ a.e. on } \Omega\}, \ f \in L^2(\mathbb{R}).$$

For any $\sigma > 0$, introduce (Paley-Wiener class)

$$PW_2(\sigma) = \{f \in L^2(\mathbb{R}) | \mathrm{spec}(f) \subset [-\sigma, \sigma]\} \quad .$$

A form of the Beurling-Malliavin multiplier Theorem

As given in Mashregi-Nazarov-Havin'05.

Theorem ((BM1), Beurling-Malliavin'62 Acta Mathematica)

If $\mathcal{L}(\omega) < -\infty$ and $\log(\omega)$ globally Lipschitz, then, for any $\sigma > 0$, there exists a nonzero $f \in PW_2(\sigma)$ such that $|f(x)| \leqslant \omega(x)$.

Some remarks

- $\mathcal{L}(\omega) < -\infty$ necessary: for a L^2-function f with semibounded spectrum, $\mathcal{L}(|f|) = -\infty \Rightarrow f = 0$ (an uncertainty principle...).
- Alone is not sufficient. But Lipschitz condition on $\log(\omega)$ can be replaced.
- The denomination "multiplier theorem" will be explained later on.

The seventh proof

The proof given in Mashregi-Nazarov-Havin'05 is very enlightening.

Two steps

1. Prove Theorem 2.1 when ω is "well-behaved", in the sense that its modified Hilbert Transform (see later on for a definition) has "small" global Lipschitz constant. Personal terminology: weight is *well-prepared*.
2. Secondly, for a given weight, find a modification of this weight that "resembles" the original weight and is well-prepared in the sense of the first point.

Our contribution

- Propose a quantitative version of this proof.
- First step was done in Jin-Zhang'20 MA, we need some refinements.
- Second step remains. Quite general strategy of the proof. Works as soon as the harmonic extension on the upper half plane of the weight "resembles" the original weight.

Our main Theorem

Theorem (Lissy'25)

If $\mathcal{L}(\omega) < +\infty$, *and* ω *is s.t.* $\exists \alpha > 0$, $K_0 > 0$ *s.t. for any* $x, y \in \mathbb{R}^2$,

$$|\log \omega(x) - \log \omega(y)| \leqslant K_0 |x - y|^{\alpha}.$$

Then, for any $0 < \sigma' < \sigma < 1/10$, $\exists \psi \in L^2(\mathbb{R}) \setminus \{0\}$ *s.t.*

$$\operatorname{supp} \psi \subset [0, \sigma], \quad |\mathcal{F}\psi| \leq C(K_0, \alpha, \sigma, \sigma')\omega,$$

and on one of the interval $(-1, -1/2)$ *or* $(1/2, 1)$, *for some* $C > 0$,

$$|\mathcal{F}\psi(x)| \geq C(\sigma - \sigma')^6 C(K_0, \alpha, \sigma, \sigma')\omega(x),$$

For an explicit $C(K_0, \alpha, \sigma, \sigma')$.

For the sake of simplicity, here, only present the method on a particular weight. But we believe that our proof might have further applications.

Summary

Hilbert transform

H_0 standard Hilbert transform: for $f \in L^1(\mathbb{R}, \langle x \rangle^{-1} dx)$,

$$H_0(f)(x) = f * \mathrm{p.v.}\,\frac{1}{\pi x} := \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{|t| \geq \varepsilon} \frac{f(x-t)}{t} dt.$$

In Fourier, $\mathcal{F}(H_0(f))(\xi) = -isg(\xi)\mathcal{F}(f)(\xi)$. We easily deduce

$$\text{Inversion formula } \; H_0(H_0(f)) = -f.$$

Interest of Hilbert transform

Is useful in complex analysis: if $f : \mathbb{R} \to \mathbb{R}$, we have an extension of f on upper half plane by harmonic function, still denoted f. Every harmonic function is the real part of an holomorphic function. Turns out that the imaginary part is $H_0(f)$.

Poisson and conjugate Poisson transforms

Poisson kernel $P_t(x) = \dfrac{t}{\pi(x^2+t^2)}$, $t>0$, $x\in\mathbb{R}$,

and Poisson transform for $f \in L^1(\mathbb{R}, \langle x\rangle^{-2}dx)$,

$$P_t f(x) = P_t * f(x) = \int_{\mathbb{R}} \frac{tf(y)}{\pi(t^2+(x-y)^2)}dy.$$

Conjugate Poisson kernel $Q_t(x) = \dfrac{x}{\pi(x^2+t^2)}$, $t>0$, $x\in\mathbb{R}$,

and conjugate Poisson transform: for $f \in L^1(\mathbb{R}, \langle x\rangle^{-1}dx)$,

$$Q_t f(x) = Q_t * f(x) = \int_{\mathbb{R}} \frac{(x-y)f(y)}{\pi(t^2+(x-y)^2)}dy.$$

Passing to Fourier gives easily $Q_s(P_t) = Q_{t+s}$, limit as $s\to 0$ gives $H_0(P_t) = Q_t$ (when makes sense).

Kober's modification of Hilbert transform (1943), and modified conjugate Poisson transform

Remind $\log(\omega) \in L^1(\mathbb{R}, \langle x \rangle^{-2} dx)$. Previous setting not adapted ! Need to change Hilbert and conjugate Poisson transform so that they decay as $|x|^{-2}$ as $|x| \to \infty$. Here $f \in L^1(\mathbb{R}, \langle x \rangle^{-2} dx)$.

New definitions of Hilbert/conjugate Poisson

$$H(f)(x) = \frac{1}{\pi} \,\mathrm{p.\,v.} \int_{\mathbb{R}} f(y) \left(\frac{1}{x-y} + \frac{y}{y^2+1} \right) dy,$$

$$\tilde{Q}_t(f)(x) = \frac{1}{\pi} \int_{\mathbb{R}} f(y) \left(\frac{(x-y)}{(x-y)^2+t^2} + \frac{y}{y^2+1} \right) dy.$$

For regular functions, differ from H_0 and Q only by a constant. Analogue of previous formula:

- $H(H(f)) = -f + Const(f)$ (constant not important).
- $H(P_t f) = \tilde{Q}_t f + Const(t, f)$ (same).

Outer functions and the crucial Lemma

For, $\Omega \in L^1(\mathbb{R}; \langle x \rangle^{-2} dx)$, the functions f of the form

$$f = ae^{-(\Omega + iH(\Omega))}, \quad |a| = 1$$

are called *outer functions*.

Lemma (MNH'05)

Assume that $\omega = e^{-\Omega} \in L^2$ and $\mathcal{L}(\omega) > -\infty$. In addition, we assume that $\omega^2 e^{2\pi i \sigma x}$ is an outer function. Then there exists $\psi \in L^2$ with $\operatorname{supp} \psi \subset [0, \sigma]$ *and* $|\mathcal{F}\psi| = \omega$.

The second step for well-prepared weights

Theorem (Lissy'25, refinement of Jin-Zhang'20 MA.)

Assume that ω is positive, essentially bounded, satisfies $\mathcal{L}(\omega) > -\infty$ and is such that $H(\Omega)$ is differentiable, $H(\Omega)'$ is essentially bounded and

$$\|H(\Omega)'\|_{L^\infty} \leq \pi\sigma', \qquad \text{(WP-Weight)}$$

where $0 < \sigma' < \sigma < 1/10$. Then there is $\psi \in L^2(\mathbb{R})$ with

$$\operatorname{supp}\psi \subset [0,\sigma], \quad |\mathcal{F}\psi| \leq \omega,$$

and on one of the interval $(-1,-1/2)$ or $(1/2,1)$, we have

$$|\mathcal{F}\psi(x)| \geq C(\sigma-\sigma')^6\omega(x),$$

for some numerical constant $C > 0$.

Idea of the proof (1)

Follows closely Jin-Zhang'20 MA.

- Goal is apply lemma: find $\tilde{\omega}$ s.t. $\tilde{\omega}^2 e^{2\pi i\sigma x}$ outer function, $0 < \tilde{\omega} \leqslant \omega$ and on $(-1,-1/2)$ or $(1/2,1)$, we have $\tilde{\omega} \geq C(\sigma-\sigma')^6\omega(x)$, then take Fourier transform.
- So look at $\tilde{\omega}$ under the form $\tilde{\omega} = m\omega$, with $0 < m \leqslant 1$ and some lower bound on m.
- Outer function condition implies that we need

$$H(2\log(\tilde{\omega})) = 2\pi\sigma x + c \mod 2\pi.$$

- So equivalently, find m such that

$$H(\log(m)) = \pi\sigma x + c/2 - H(\log(\omega)) \mod \pi.$$

Idea of the proof (2)

- Need to apply inverse Hilbert transform, possible for instance if bounded function (which is in any case needed to have $0 < m \leqslant 1$).
- For example, if $\omega = 1$, take the sawtooth function

$$s(x) = \pi\sigma x - \pi\lfloor \sigma x \rfloor - \frac{\pi}{2},$$

- Then $\log m = -H(s)$ solves our problem.

How to modify this proof if $\omega \neq 1$?

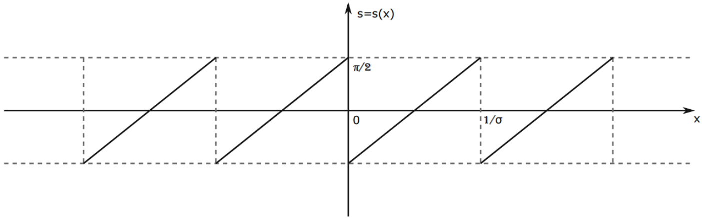

Figure: Sawtooth function (JZ'20)

Elements of proof (1)

- Increasing σ' to σ and modifying a little bit ω, we can assume wlog that $\omega \in L^2(\mathbb{R})$ and $\|H(\Omega_0)'\|_{L^\infty} \leq \pi\sigma$.
- Let $m = e^{-M}$, where $M = H(s)$ and

$$s(x) = s_0(x) - \pi k(x) - \frac{\pi}{2},$$

where

$$s_0(x) = \pi\sigma x + H(\Omega)(x), \quad k(x) = \lfloor \pi^{-1} s_0(x) \rfloor.$$

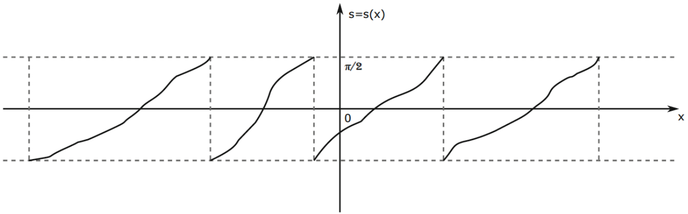

Figure: Modified sawtooth function (JZ'20)

Elements of proof (2)

- s_0 increasing and so is k, which takes integer values. We then set $\tilde{\omega} = m\omega_0$.
- Applying a result by Jun and Zhang, we obtain good upper and good local lower bounds on M (remind $m = e^{-M}$, leading eventually to the desired estimates.
- From the construction of $M = H(s)$ and inversion formula,

$$H(-2M - 2\Omega_0) = 2\pi\sigma x - 2\pi k(x) - \pi - 2c(M),$$

where $e^{2\pi i k(x)} = 1$. So for some a with $|a| = 1$,

$$\widetilde{\omega}^2 e^{2\pi i\sigma x} = e^{-2M-2\Omega_\mathbf{0}+2\pi i\sigma x} = ae^{-2M-2\Omega_\mathbf{0}+iH(-2M-2\Omega_\mathbf{0})}.$$

- $\widetilde{\omega}^2 e^{2\pi i\sigma x}$ outer function, apply previous Lemma.

A particular weight

From now on,

$$\omega(x) = \begin{cases} e^{-\sqrt{2\pi x}}, & x \geqslant 0, \\ 1, & x < 0. \end{cases}$$

- By definition,

$$\Omega(x) = \sqrt{2\pi x}\,\mathbb{1}_{(0,+\infty)}(x).$$

- By luck, $H(\Omega)$ explicitly computable:

$$H(\Omega(x)) = -\sqrt{\pi}, x > 0,$$

$$H(\Omega(x)) = -\sqrt{\pi} + \sqrt{2\pi|x|}, x < 0$$

- So $H(\Omega(x))$ not Lipschitz !!
  Not a well-prepared weight in the sense of (WP-Weight).

Transforming a weight into a well-prepared weight

Theorem

For any $T > 0$, *for any* $\varepsilon \in (0,1)$, *there exists* $\psi \in L^2(\mathbb{R})$ *such that*

$$\operatorname{supp}\psi \subset [0,T], \quad |\mathcal{F}\psi| \le \omega,$$

and on one of the interval $(-1,-1/2)$ *or* $(1/2,1)$, *we have*

$$|\mathcal{F}\psi(x)| \ge CT^6(1-\varepsilon)^6\omega(x)e^{-\frac{3^{\frac{3}{4}}}{4T(1-\varepsilon)}},$$

for some numerical constant $C > 0$.

Idea is that Poisson "renders smooth" the function, with a lower Lipschitz constant, but the global "shape" is quite similar to the original weight.

Idea of the proof (1)

$$P_t\Omega(x) = \sqrt{\pi\left(\sqrt{x^2+t^2}+x\right)},\ \forall x \in \mathbb{R},\ \forall t > 0$$

Using previous formula, $H(P_t\Omega) = \tilde{Q}_t\Omega + C_t(\Omega)$.

Explicit computation gives

$$\tilde{Q}_t\Omega(x) = -\sqrt{\pi} + \sqrt{\pi\left(\sqrt{x^2+t^2}-x\right)}.$$

So

$$H(P_t\Omega)'(x) = -\frac{\sqrt{\pi\left(\sqrt{t^2+x^2}-x\right)}}{2\sqrt{t^2+x^2}}.$$

Then,

$$||H(P_t\Omega)'||_\infty = \frac{\sqrt{\pi}3^{3/4}}{4\sqrt{t}},$$

Proof (2)

Let $\sigma = T$ and $\sigma' = T(1-\varepsilon)$. Choose t s.t.

$$\frac{\sqrt{\pi}3^{3/4}}{4\sqrt{t}} = \pi T(1-\varepsilon).$$

We have $P_t\Omega(x)$ and Ω leads to the fact that we always have

$$P_t\Omega(x) - \Omega(x) \geqslant 0,$$

with max reached for $x = 0$. So

$$\Omega(x) \leqslant P_t(x) \leqslant \Omega(x) + \sqrt{\pi t}.$$

Using previous result enables to conclude.

Summary

The moment method (1)

$-\Delta_D : D(\Delta_D) \to H^{-1}(0,1)$ positive definite operator with compact resolvent. k-th eigenvalue is

$$\lambda_k = k^2\pi^2,$$

with (normalized in H^{-1}-norm) eigenvector

$$e_k(x) := \sqrt{2}k\pi \sin\left(k\pi x\right).$$

Control operator

$$b := \delta_0' \circ \Delta^{-1},$$

so that (Cont-Sch) can be rewritten in an abstract way as

$$\begin{cases} iy_t + \Delta y = bu & \text{in } (0,T)\times(0,1), \\ y(0,\cdot) = y^0 & \text{in } (0,1). \end{cases}$$

The moment method (2)

Write

$$y^0(x) = \sum_{k=1}^{\infty} a_k e_k(x),\ (a_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*).$$

and $b_k = \langle b, e_k\rangle_{H^{-1}(0,1)}$. Then, by projection, for $k \in \mathbb{N}^*$ and $t \in [0, T]$,

$$\langle y(t), e_k\rangle_H = a_k e^{-i\lambda_k t} + b_k \int_0^t e^{-i\lambda_k(t-s)} u(s)ds.$$

$y(T, \cdot) = 0$ equivalent to: for $k \in \mathbb{N}^*$,

$$\int_0^T e^{i\lambda_k s} u(s)ds = -\frac{a_k}{b_k},$$

the right-hand side being well-defined and in $l^2(\mathbb{N}^*)$ since

$$|b_k| \geqslant C,$$

.

The moment method (3)

Assume bi-orthogonal family to $\{t \mapsto e^{i\lambda_k t}\}$ in $L^2(0,T)$ (dual basis) $\{\psi_m\}_{m\in\mathbb{N}^*}$ such that for every $(k,l) \in (\mathbb{N}^*)^2$,

$$\langle e^{i\lambda_k t}, \psi_l \rangle_{L^2(0,T)} = \delta_{kl}.$$

Then formal control

$$u(t) := -\sum_{l\in\mathbb{N}^*} \frac{a_l}{b_l}\psi_l(t),$$

that is a real control if in $L^2(0,T)$.
Above condition can be rewritten as

$$\int_0^T e^{2i\pi\mu_k t}\psi_l(t)dt = \delta_{kl}, \quad \mu_k = \frac{\pi k^2}{2}.$$

Can be rewritten as

$$\mathcal{F}(\psi_l)(-\mu_k) = \delta_{kl}, \ \mathrm{supp}(\mathcal{F}(\psi_l)) \subset [0,T]. \qquad \text{(Zer-Four)}$$

Proof (1)

Firstly construct $\mathcal{F}(\psi_l)$, and then, deduce ψ_l by Fourier inversion, so that (Zer-Four) is verified. As usual, we introduce the canonical product

$$Q(z) = \prod_{k=1}^{+\infty} \left(1 + \frac{z}{\mu_k}\right),$$

and

$$P_l(z) = \frac{Q(z)}{(z+\mu_l)Q'(-\mu_l)}.$$

Of course, P_l is constructed so that

$$P_l\left(-\mu_k\right) = \delta_{kl}.$$

Easy estimates give (link with previous part)

$$|P_l(x)| \leqslant 4\omega^{-1}(x),\ x \in \mathbb{R},$$

where ω is our particular weight.

Proof (2)

Moreover, $|P_I|$ explodes exponentially in $\sqrt{x}$ as $x \to +\infty$, so $\mathcal{F}^{-1}(P_I)$ will not necessarily make sense as a function. Hence, in order to apply $\mathcal{F}^{-1}$, use a multiplier that compensates bad behaviour. **All the game and difficulty is to construct appropriate, and if possible, as optimal as possible multipliers**.

Let $\varepsilon \in (0,1)$. Change ω as

$$\omega_\varepsilon(x) = \begin{cases} e^{-\sqrt{2\pi x}(1+\varepsilon)}, & x \geqslant 0, \\ 1, & x < 0, \end{cases}$$

Then existence of ψ such that

$$\operatorname{supp}\psi \subset [0, T(1-\varepsilon)], \quad |\mathcal{F}\psi| \leq \omega_\varepsilon, \tag{Up-We-Sch}$$

and on one of the interval $(-1,-1/2)$ or $(1/2,1)$, we have

$$|\mathcal{F}\psi(x)| \geq C_\varepsilon T^6 \omega_\varepsilon(x) e^{-\frac{3^{\frac{3}{4}}(1+\varepsilon)}{4T(1-\varepsilon)^2}}, \tag{Low-We-Sch}$$

We call $m = \pm 1/4$, depending on when previous lower bound is true.

Proof (3)

$$\text{Let } g_l(z) = \frac{\mathcal{F}\psi(z - m + \mu_l)}{\mathcal{F}\psi(m)} \left(\frac{i(e^{-2i\pi\varepsilon T(z+\mu_l)} - 1)}{2\pi\varepsilon T(z + \mu_l)} \right).$$

By construction, $g_l P_l$ is an entire function that verifies $g_l P_l(-\mu_k) = \delta_{kl}$ by (32). Moreover, After some tedious computations,

$$|g_l P_l(x)| \leqslant \frac{R_\varepsilon(T)}{1 + |x + \mu_l|} e^{\frac{3^{\frac{3}{4}}(1+\varepsilon)}{4T(1-\varepsilon)^2}}, \forall x \in \mathbb{R},$$

where $R_\varepsilon(T)$ is from now on a fractional function of T. Uses both (Up-We-Sch) and (Low-We-Sch). Then construct u with previous formula, and make basic estimations to prove that $u \in L^2(0, T)$ with the good estimate.

Open problems

- **Fractional Schrödinger equations.** Let $\alpha > 1$ and let us call $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. Try to improve results in Lissy'17 MCRF by the same method.
- **Heat equation.** ω has to be changed into $e^{-\sqrt{\pi|x|}}$ for $x \in \mathbb{R}$. A good part of the strategy works, problem comes from that at some point, since the λ_k are now changed into $-i\lambda_k$, it will be mandatory to understand well the behaviour of $\mathcal{F}(g_l)$ on the imaginary axis. Work in progress, by slightly different method.
- **Other constructions.** Here, exploit the link between the Hilbert transform and the Poisson and conjugate Poisson transforms. Other possible kernels ?

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Reference

Pierre Lissy, *Effective multipliers for weights whose log are Hölder continuous. Application to the cost of fast boundary controls for the 1D Schrödinger equation*, preprint available on HAL and arxiv.

Thank you for you attention.